ORIGINAL PAPER

SOLUTION OF FRACTIONAL FOKKER PLANCK EQUATION USING FRACTIONAL POWER SERIES METHOD

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Abstract. *The aim of this paper is to present solution of the fractional Fokker Plank equation (FFPE) using fractional power series (FPS) method. The fractional derivatives are considered in Caputo sense, via MATLAB for providing the numerical and graphical solutions.*

Keywords: *Fractional Fokker Plank equation, Fractional power series method, Caputo derivatives, Fractional derivatives.*

1. INTRODUCTION

The FFPE is significant in differential equation that provide solutions to number of problems formulated in terms of fractional order differential and difference equations; therefore, it has recently become a subject of interest for many authors in the field of fractional Fokker Planck and its applications, physical laws are also expressed more accurately in terms of differential equations of arbitrary order [1-18]. The Fokker-Planck equation was introduced by Adriaan Fokker and Max Planck to describe the brownian motion of particles and the diffusion mode of chemical reactions by Risken [30]. Further, FPE has important applications in various areas such as plasma physics, population dynamic, engineering, neurosciences, nonlinear hydrodynamics, laser physics, pattern formation, psychology and marketing [19-30]. For the two -variable case, to which attention is restricted here, the equation defined by Mohamed et al. [31] in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}A_i(x,t,u) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_i \partial x_j}B_i(x,t,u)\right]u \tag{1.1}$$

Definition 1. The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{ds} \tag{1.2}$$
$$(\alpha > 0, x > 0, m-1 < \alpha \le m, m \in \mathbb{N}).$$

In similar manner of integer-order differentiation, Caputo fractional derivative operator is a linear operation as

$$D^\alpha(\beta p(x) + \gamma q(x)) = \beta D^\alpha p(x) + \gamma D^\alpha q(x), \tag{1.3}$$

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where α and γ are constants. For the Caputo's derivative, we have $D^{\alpha}k = 0$, if k is constant

$$D^{\alpha}x^{n} = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil\alpha\rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha} & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil\alpha\rceil \end{cases} \tag{1.4}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0,1,2,...\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

Definition 2. For the variable t and coefficients $c_n(n = 0,1,...,\infty)$, if $t > t_0$ the fractional power series (FPS) about t_0 is defined as:

$$\sum_{n=0}^{\infty} c_n(t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + ... \tag{1.5}$$

where, $0 \leq m-1 < \alpha \leq m, m \in \mathbb{N}^{+}$.

In the present paper, we will use the Fractional Power Series Method (FPSM) [32, 33] for solving the time fractional Fokker-Planck equation (TFFPE). The fractional derivatives described here are in the sense of Caputo, for more details on Caputo derivatives [4, 12, 16, 34, 35].

2. MAIN RESULTS

Theorem. Let the radius of convergence (ROC) for the function with FPS representation $f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$, $0 \leq t < R$, is greater than zero (i.e, $R > 0$.), then for $m \in \mathbb{N}^{+}$ and $m-1 < \alpha \leq m$, the following expression holds true.

$$D^{\alpha}(f(t)) = \sum_{n=1}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}. \tag{1.6}$$

Proof: From the linearity concept of Caputo derivative and the idea of power series derivative, we have:

$$\begin{aligned} D^{\alpha}(f(t)) &= D^{\alpha}\sum_{n=0}^{\infty} c_n t^{n\alpha} = D^{\alpha}(c_0 + c_1 t^{\alpha} + c_2 t^{2\alpha} + ...) \\ &= D^{\alpha}c_0 + c_1 D^{\alpha}t^{\alpha} + c_2 D^{\alpha}t^{2\alpha} + ... \\ &= \sum_{n=0}^{\infty} c_n D^{\alpha}t^{n\alpha} \end{aligned} \tag{1.7}$$

$$c_1 D^{\alpha}t^{\alpha} + c_2 D^{\alpha}t^{2\alpha} + ... = \sum_{n=1}^{\infty} c_n D^{\alpha}(t^{n\alpha}) \tag{1.8}$$

From the power rule of Caputo derivative, we have

$$D^{\alpha}t^{n\alpha} = \begin{cases} 0, & n\alpha \in \mathbb{N}_0, n\alpha < \lceil\alpha\rceil \\ \frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)} t^{n\alpha-\alpha}, & n\alpha \in \mathbb{N}_0, n\alpha \geq \lceil\alpha\rceil \end{cases} \tag{1.9}$$

$$D^{\alpha}t^{n\alpha} = \begin{cases} 0, & n\alpha \in \mathbb{N}_0, n\alpha < \lceil \alpha \rceil \\ \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}, & n\alpha \in \mathbb{N}_0, n\alpha \geq \lceil \alpha \rceil \end{cases} \tag{1.10}$$

$\mathbb{N}_0 = \{1,2,...\}$, $\lceil \alpha \rceil \in \mathbb{Z}$, $\lceil \alpha \rceil \geq \alpha$.

From equations (1.4) and (1.6), we arrive at

$$D^{\alpha}(f(t)) = \sum_{n=1}^{\infty} c_n D^{\alpha}(t^{n\alpha}) = \sum_{n=1}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha} \tag{1.11}$$

where $f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$.

3. APPLICATIONS OF POWER SERIES METHOD

Example 1. We consider the time fractional Fokker-Planck equation (TFFPE) for $A_1 = -1, B_1 = 1$

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

with the initial condition

$$u(x,0) = x. \tag{2.2}$$

The exact solution to equation (2.1) for the non-fractional case at $\alpha = 1$ is

$$u(x,t) = x + t, \quad (x, t \geq 0).$$

Rewriting equation (2.1) in the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \tag{2.3}$$

Now, we apply fractional Power series method, we suppose

$$u(x,t) = \sum_{k=0}^{\infty} a_k(x) t^{\alpha k} = a_0(x) + a_1(x)t^{\alpha} + a_2(x)t^{2\alpha} + ... \tag{2.4}$$

Using Theorem 1,

$$D^{\alpha}(u(x,t)) = D^{\alpha}\left(\sum_{k=0}^{\infty} a_k(x) t^{\alpha k}\right) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha}, \tag{2.5}$$

$$\frac{\partial u(x,t)}{\partial x} = \sum_{k=0}^{\infty} \frac{\partial a_k(x)}{\partial x} t^{\alpha k} = \frac{\partial a_0(x)}{\partial x} + \frac{\partial a_1(x)}{\partial x} t^{\alpha} + \frac{\partial a_2(x)}{\partial x} t^{2\alpha} + ... \tag{2.6}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{k=0}^{\infty} \frac{\partial^2 a_k(x)}{\partial x^2} t^{\alpha k} = \frac{\partial^2 a_0(x)}{\partial x^2} + \frac{\partial^2 a_1(x)}{\partial x^2} t^{\alpha} + \frac{\partial^2 a_2(x)}{\partial x^2} t^{2\alpha} + ... \tag{2.7}$$

Applying equations (2.5)-(2.7) into (2.3) and looking at the coefficients of t^{α}.

$$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha} = \sum_{k=0}^{\infty} \frac{\partial a_k(x)}{\partial x} t^{\alpha k} + \sum_{k=0}^{\infty} \frac{\partial^2 a_k(x)}{\partial x^2} t^{\alpha k}, \tag{2.8}$$

$$a_1(x)\Gamma(\alpha+1)+a_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^{\alpha}+\ldots=\left(\frac{\partial a_0(x)}{\partial x}+\frac{\partial a_1(x)}{\partial x}t^{\alpha}+\frac{\partial a_2(x)}{\partial x}t^{2\alpha}+\ldots\right)$$
$$+\left(\frac{\partial^2 a_0(x)}{\partial x^2}+\frac{\partial^2 a_1(x)}{\partial x^2}t^{\alpha}+\frac{\partial^2 a_2(x)}{\partial x^2}t^{2\alpha}+\ldots\right) \quad (2.9)$$

Hence, we obtain

$$a_1(x)\Gamma(\alpha+1)=\frac{\partial a_0(x)}{\partial x}+\frac{\partial^2 a_0(x)}{\partial x^2}, \quad (2.10)$$

$$a_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}=\frac{\partial a_1(x)}{\partial x}+\frac{\partial^2 a_1(x)}{\partial x^2}. \quad (2.11)$$

From the initial condition $u(x,0)=x$, we have

$$u(x,0)=a_0(x)=x=u_0(x,t) \quad (2.12)$$

Therefore;

$$\frac{\partial a_0(x)}{\partial x}=1;\quad \frac{\partial^2 a_0(x)}{\partial x^2}=0. \quad (2.13)$$

From equations (2.10)-(2.13), we obtain

$$a_1(x)=\frac{\frac{\partial a_0(x)}{\partial x}+\frac{\partial^2 a_0(x)}{\partial x^2}}{\Gamma(\alpha+1)}=\frac{1}{\Gamma(\alpha+1)},$$

Also, we get

$$\frac{\partial a_1(x)}{\partial x}=0=\frac{\partial^2 a_1(x)}{\partial x^2} \quad (2.14)$$

$$a_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}=\frac{\partial a_1(x)}{\partial x}+\frac{\partial^2 a_1(x)}{\partial x^2}=0+0=0. \quad (2.15)$$

Thus

$$u(x,t)=\sum_{k=0}^{\infty}a_k(x)t^{\alpha k}=a_0(x)+a_1(x)t^{\alpha}+a_2(x)t^{2\alpha}+\ldots=x+\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+0+0+\ldots$$
$$=x+\frac{1}{\Gamma(\alpha+1)}t^{\alpha}. \quad (2.16)$$

The exact solution for non-fractional case $\alpha=1$, we get

$$u(x,t)=x+\frac{1}{\Gamma(1+1)}t=x+\frac{1}{\Gamma(2)}t=x+t.$$

Table 1. Numerical values of equation (2.16) for different values of x, t.

x	t	$\alpha=1$	$\alpha=0.8$	$\alpha=0.6$	$\alpha=0.4$	$\alpha=0.2$
0.2	0.25	0.45	0.5542	0.6871	0.8473	1.0254
	0.5	0.7	0.8167	0.9384	1.0542	1.1481
	0.75	0.95	1.0529	1.1417	1.2046	1.2282
	1	1.2	1.2737	1.3192	1.3271	1.2891
0.5	0.25	0.65	0.7542	0.8871	1.0473	1.2254
	0.5	0.9	1.0167	1.1384	1.2542	1.3481
	0.75	1.15	1.2529	1.3417	1.4046	1.4282
	1	1.4	1.4737	1.5192	1.5271	1.4891
0.8	0.25	0.85	0.9542	1.0871	1.2473	1.4254
	0.5	1.1	1.2167	1.3384	1.4542	1.5481
	0.75	1.35	1.4529	1.5417	1.6046	1.6282
	1	1.6	1.6737	1.7192	1.7271	1.6891

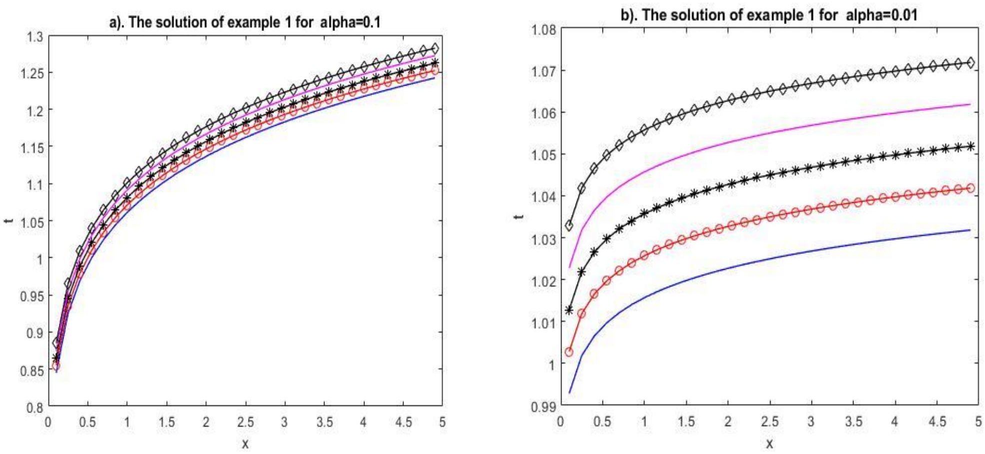

Figure 1. Two dimensional graphical solution of equation (2.1).

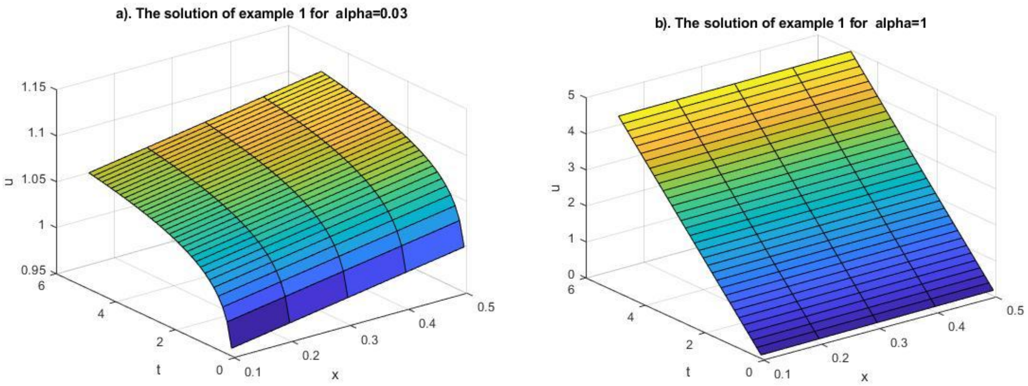

Figure 2. Three dimensional graphical solution of equation (2.1).

Example 2. The time fractional Fokker-Planck equation with the following modifications in equation (1.1) is $f(x) = x + 1, x \in R$. Let in equation (1.1) $n = 1, x_1 = x, A_1(x,t) = -(x + 1), B_1(x,t) = e^t x^2$, in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (1 + x)\frac{\partial u}{\partial x} - (e^t x^2)\frac{\partial^2 u}{\partial x} = 0 \tag{2.17}$$

Subject to the initial condition

$$u(x,0) = 1 + x \tag{2.18}$$

where $u(x,t) = (1 + x)e^t$ is the exact solution for $\alpha = 1$.

To apply FPSM, suppose that the solution of equation (2.17) takes the form

$$u(x,t) = \sum_{k=0}^{\infty} a_k(x) t^{\alpha k} \tag{2.19}$$

From the initial condition given on (2.18), we have $a_0(x) = 1 + x$.

With the help of theorem 1, we have

$$D^\alpha u(x,t) = \sum_{k=1}^{\infty} a_k(x) \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)\alpha} \tag{2.20}$$

Substituting (2.19) and (2.20) in to (2.17) and comparing the coefficients of t^{α}

$$\sum_{k=1}^{\infty} a_k(x)\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)}t^{(k-1)\alpha} = (1+x)\sum_{k=0}^{\infty}\frac{\partial a_k}{\partial x}t^{\alpha k} + (e^t x^2)\sum_{k=0}^{\infty}\frac{\partial^2 a_k}{\partial x^2}t^{\alpha k} \tag{2.21}$$

$$a_1(x)\Gamma(\alpha+1) = (1+x)\frac{\partial a_0}{\partial x} + e^t x^2\frac{\partial^2 a_0}{\partial x^2} \tag{2.22}$$

$$a_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} = (1+x)\frac{\partial a_1}{\partial x} + e^t x^2\frac{\partial^2 a_1}{\partial x^2} \tag{2.23}$$

$$a_3(x)\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} = (1+x)\frac{\partial a_2}{\partial x} + e^t x^2\frac{\partial^2 a_2}{\partial x^2} \tag{2.24}$$

Since

$$a_0(x) = 1+x, \frac{\partial a_0}{\partial x} = 1 \text{ and } \frac{\partial^2 a_0}{\partial x^2} = 0 \tag{2.25}$$

$$a_1(x) = \frac{1+x}{\Gamma(\alpha+1)}, a_2(x) = \frac{1+x}{\Gamma(2\alpha+1)}, a_1(x) = \frac{1+x}{\Gamma(3\alpha+1)}. \tag{2.26}$$

Thus

$$u(x,t) = a_0(x) + a_1(x)t^{\alpha} + a_2(x)t^{\alpha} + a_3(x)t^{3\alpha} + \ldots$$

$$= 1 + x + \frac{1+x}{\Gamma(\alpha+1)}t^{\alpha} + \frac{1+x}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{1+x}{\Gamma(3\alpha+1)}t^{3\alpha} + \ldots \tag{2.27}$$

For non-fractional case $\alpha = 1$

$$u(x,t) = 1 + x + \frac{1+x}{\Gamma(2)}t + \frac{1+x}{\Gamma(3)}t^2 + \frac{1+x}{\Gamma(4)}t^3 + \ldots \tag{2.28}$$

$$= (1+x)\sum_{n=0}^{\infty}\frac{t^n}{n!} = (1+x)e^t. \tag{2.29}$$

Table 2. Numerical values of equation (2.28) for different values of x, t

x	t	$\alpha = 1$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$
0.2	0.25	3.9406	4.1308	4.6899	7.0188	5.9519
	0.5	4.375	4.6932	5.4766	7.1632	6.6488
	0.75	4.9219	5.3551	6.2590	7.2509	7.1694
	1	5.6	6.1303	7.0568	7.3147	7.6024
0.5	0.25	4.9258	5.1635	5.8623	8.7736	7.4398
	0.5	5.4688	5.8664	6.8458	8.9540	8.3110
	0.75	6.1523	6.6938	7.8237	9.0637	8.9618
	1	7	7.6629	8.8209	9.1434	9.5030
0.8	0.25	5.9109	6.1962	7.0348	10.5283	8.9278
	0.5	6.5625	7.0397	8.2149	10.7448	9.9732
	0.75	7.3828	8.0326	9.3885	10.8764	10.7541
	1	8.4	9.1955	10.5851	10.9721	11.4036

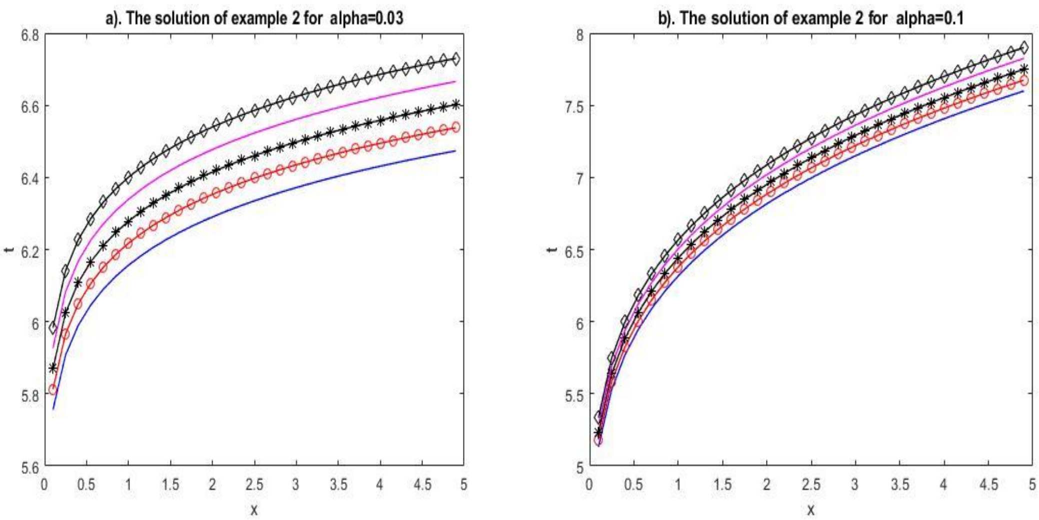

Figure 3. Two dimensional graphical solution of equation (2.17).

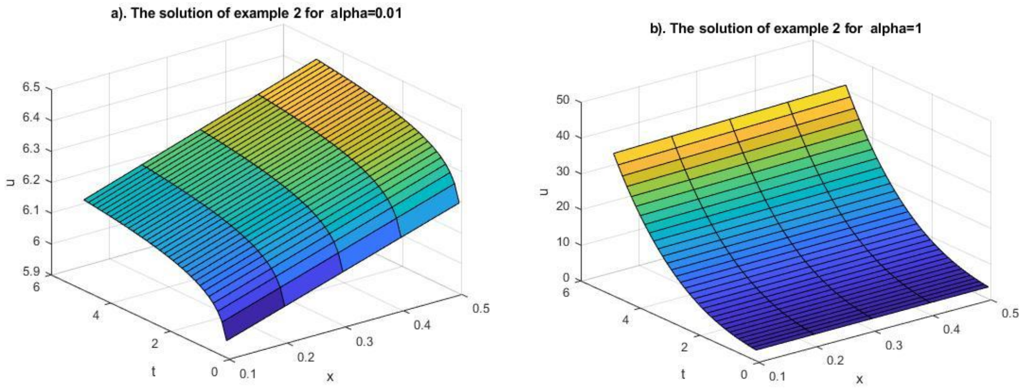

Figure 4. Three dimensional graphical solution of equation (2.17).

4. CONCLUSION

In this paper, we presented the power series method for finding the fractional Fokker Plank equation with comparably easy implementation. On the tables and graphical presentations, reasonable values are presented for fractional and non-fractional cases. Therefore, it is time to assure that, the proposed method is quit promising on reducing the computational cost of solving such a problem to a great extent.

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