ORIGINAL PAPER

NUMERICAL INVESTIGATION OF SOME FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS VIA A NEW DERIVATIVE APPROACH

JAMSHAD AHMAD[1], IRSA GHAZAL[1], MUHAMMAD SHAKEEL[2],
QAZI MAHMOOD UL HASSAN[2]

Manuscript received: 27.03.2019; Accepted paper: 21.07.2019;
Published online: 30.09.2019.

Abstract. *In the present work, some mathematical models of non-linear types for first order integro-differential equation and system of fractional order are analytically tackled by the fractional VIM. The fractional derivative is considered in Caputo Fabrizio sense involving an exponential function and has interesting properties. The illustrative numerical examples show that the solution approach to the classical solutions as the order of the fractional derivative increases. The graphs obtained by generalized results are also plotted by using symbolic software Maple. Comparison of our approximate solution with the exact numerical results of the problems is presented to illustrate the reliability and efficiency of the method.*

Keywords: *variational iteration method, fractional integro-differential equation, Caputo- Fabrizio fractional derivative, approximate solution.*

1. INTRODUCTION

Fractional integro-differential equations are the mathematical models of many impediments from various engineering and sciences applications [9]. In fact, IDEs rise in several physical methods, i.e. nano hydrodynamics [11], formation of glass [13], wind ripple in the desert [11] and [10] drop wise condensation. The study of these equations has become hot topic. To improve the analysis of these systems and equations, it is required to obtain the solution of these equations..In literature, fractional derivative types; Riemann-Liouville, Caputo derivatives and Jumarie derivative ([1], [6], [8], [7]) are most used. Fabrizio and Caputo introduced a different fractional derivation without singular kernel in 2015. Nieto and Losada developed some properties of this new derivative definition and these results were published in the paper [5]. The solution of generalized second grade fluid over an oscillating vertical plate in terms of CF fractional derivative [9] is studied by Khan and Shah.

Many analytical methods have been introduced and used for the getting of approximate and exact solutions of FIDEs. Calculations of exact solutions of most of the FIDEs are difficult. In 2017, work on the non-integer order derivative, CF and AB fractional derivatives was done by, A. Khan, K. Ali Abro, A. Tassaddiq and I. Khan [4]. They examined the relative analysis of CF and Atangana–Balean (AB) fractional derivatives for heat and mass transfer of a second grade fluid.In 2017, D. Baleanu et al. worked on two higher-order CF FDEs to find the approximate solutions. While modeling any real world phenomena by FIDEs, RL derivative showed certain drawbacks. [13] Therefore, we will use a modified

[1] University of Gujrat, Faculty of Science, Department of Mathematics, 50700 Gujrat, Pakistan.
E-mail: jamshadahmadm@gmail.com.
[2] University of Wah, Faculty of Science, Department of Mathematics, 47040 Wah, Pakistan.

fractional differential operator proposed by Caputo's work. Also in literature, any numerical and approximate methods were used to tackle the solution of FIDEs. He [15] was the first to intend VIM. This method has been successfully implemented by several researchers on diverse problems in determination of exact and approximate solutions which arise in engineering and scientific hitches. VIM is very effective method to find approximate solutions of different problems.

2. FRACTIONAL CALCULUS

2.1. Definition. The CF fractional derivative of $r(t)$ is define as [4] is defined as

$$ {}^{CF}D^{\omega}r(t)=\frac{N(\omega)}{1-\omega}\int_0^t \exp\left[\frac{-\omega}{1-\omega}(t-s)\right]r'(s)ds, where, t\geq 0, 0<\omega\leq 1 \quad (1)$$

$$N(0)=N(1)=1.$$

2.2. Definition. Caputo-Fabrizio (CF) fractional integral of $r(x)$ is studied by Nieto and Losada, [2] is given by

$$ {}^{CF}I^{\omega}r(x)=\frac{2(1-\omega)}{(2-\omega)N(\omega)}r(x)+\frac{2}{(2-\omega)N(\omega)}\int_0^x u(s)ds, 0<\omega\leq 1$$
$$N(\omega)=\frac{2}{(2-\omega)}, 0<\omega\leq 1 \quad (2)$$

3. NUMERICAL SCHEME

To explain the essential procedure of the FVIM, we consider

$$ {}^{CF}D^{\omega}y(x)=g(x)+\lambda\int_a^b H(x,s)y(s)ds. \quad (3)$$

According to VIM, first the correction functional for Eq. (3)

$$y_{k+1}(x)=y_k(x)+{}^{CF}I^{\alpha}\left[\mu(s)\left(\sum_{k=0}^{\infty}{}^{CF}D^{\alpha}y_k(x)-\tilde{g}(x)-\lambda\int_a^b H(x,s)\tilde{y}_k(s)ds\right)\right]$$

and now we apply the definition of CF fractional integral,

$$ {}^{CF}I^{\omega}u(t)=\frac{2(1-\omega)}{(2-\omega)N(\omega)}u(t)+\frac{2}{(2-\omega)N(\omega)}\int_0^t u(s)ds, 0<\omega\leq 1$$
$$N(\omega)=\frac{2}{(2-\omega)}, 0<\omega\leq 1$$

$$y_{k+1}(x) = y_k(x) + \frac{2(1-\omega)}{(2-\omega)N(\omega)}\mu(s) + \frac{2}{(2-\omega)N(\omega)}$$
$$\int_0^x \mu(s)\left[(\sum_{k=0}^{\infty} {}^{CF}D^{\omega} y_k(x) - \tilde{g}(x) - \lambda\int_a^b H(x,p)\tilde{y}_k(p)dp \right]ds \tag{4}$$

Here $\mu(s)$ is a general Lagrange multiplier and for 1st order $\mu = -1$ and the general formula is

$$\mu(s) = (-1)^n \frac{(s-x)^{n-1}}{(n-1)!} \tag{5}$$

4. NUMERICAL PROBLEMS

In this section, the study of numerical solution of some problems of Fredholm and Volterra FIDEs with the aid of fractional variational iteration method is carried out.

Problem 4.1 Suppose the non-linear Fredholm FIDEs [12]

$${}^{CF}D^{\omega} f(p) = g(p) + \int_0^1 pt[f(t)]^3 dt,\ 0 < \omega \le 1 \tag{6}$$

$$g(p) = \frac{8}{5\Gamma\left(\frac{5}{4}\right)} p^{5/4} + \frac{15}{8} p, \tag{7}$$

with subject to the condition $f(0) = 1$ with the exact solution

$$\mathrm{f(p)} = \mathrm{p}^2 + 1. \tag{8}$$

The correction functional (CF) according to the above described procedure is

$$f_{k+1}(p) = f_k(p) + {}^{CF}I^{\omega}\left[\mu(s)({}^{CF}D^{\omega} f_k(p) - 10.59 p^{\frac{5}{4}} - \frac{15}{8}p - \int_0^1 pt[f_k(t)]^3 dt \right]$$

Here $\mu(s)$ is Lagrange multiplier, for 1st order FIDEs, we have

$$\mu(s) = -1$$

For $k = 0$

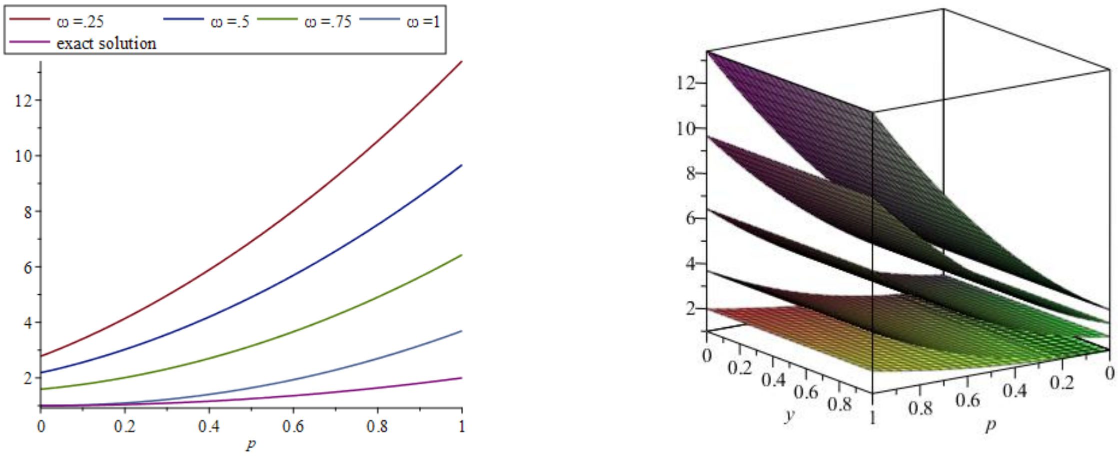
$$f_1(p) = f_0(p) + {}^{CF}I^{\omega}\left[(-1)({}^{CF}D^{\omega} f_0(p) - 10.59 p^{\frac{5}{4}} - \frac{15}{8}p - \int_0^1 pt[f_0(t)]^3 dt \right]$$

$$f_1(p)=1+10.59(1-\omega)p^{\frac{5}{4}}+(10.59)\frac{4}{9}p^{\frac{9}{4}}+\frac{19}{16}p^2+\frac{19}{8}(1-\omega)p$$

$$f_2(p)=1+4(1-\omega)^2p+(1-\omega)\left[2.375+3.818p+1.765p^{5/4}+2p^2\right]$$
$$+1.91p^2+0.784p^{9/4} \tag{9}$$

and hence we have

$$f(p)=1+4(1-\omega)^2p+(1-\omega)\left[2.375+3.818p+1.765p^{5/4}+2p^2\right]$$
$$+1.91p^2+0.784p^{9/4} \tag{10}$$



(i) (ii)

Figure 1. Displays the 2-Dim (i) and 3-Dim (ii) multiple graphs of the approximate solution (10), for the different values of ω. As we assign different values of ω there comes a slight change in the graph and we found that the solution of fractional IDE is approximately equal to the exact solution for $\omega=1$. The accuracy can be enhanced by taking higher level of approximate solutions.

Problem 4.2 Suppose the non-linear Fredholm FIDEs [16]

$$^{CF}D^{\omega}f(p)=g(p)+\int_0^1 pt[f(t)]^2dt,\ 0<\omega\leq 1, \tag{11}$$
$$g(p)=1-\frac{p}{4},$$

with subject to the condition $f(0)=0$ with the exact solution is $f(p)=p$.

The CF is given by

$$f_{k+1}(p)=f_k(p)+{}^{CF}I^{\omega}\left[(-1)({}^{CF}D^{\omega}f_k(p)-1+\frac{1}{4}x-\int_0^1 pt[f_k(t)]^2dt\right]$$

Here $\mu(s)$ is Lagrange multiplier, for 1st order FIDEs, we have

$$\mu(s)=-1, \text{ and for } k=0$$

$$f_1(p)=f_0(p)+{}^{CF}I^{\omega}\left[\mu(s)({}^{CF}D^{\omega}f_0(p)-1+\frac{1}{4}p-\int_0^1 pt[f_0(t)]^2\,dt\right]$$

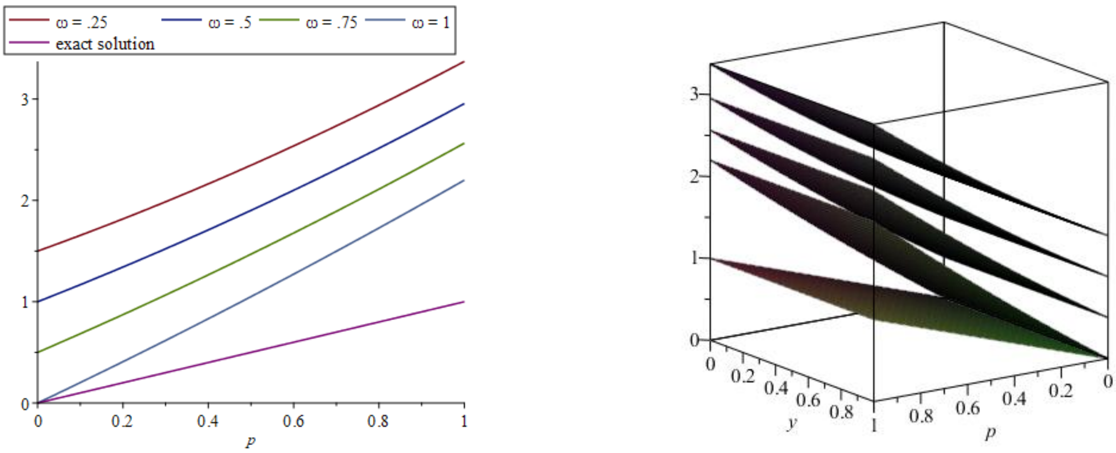
$$f_1(p)=0+{}^{CF}I^{\alpha}\left(+1-\frac{1}{4}p\right)$$

$$f_1(p)=(1-\omega)+p-(1-\omega)\frac{p}{4}-\frac{p^2}{8} \tag{12}$$

$$f_2(p)=2p+(1-\omega)\left[2-\frac{77p}{96}+\frac{5p^2}{24}\right]-0.2p^2+(1-\omega)^2\left[\frac{5p}{24}\right] \tag{13}$$

and hence we have

$$f(p)=2p+(1-\omega)\left[2-\frac{77p}{96}+\frac{5p^2}{24}\right]-0.2p^2+(1-\omega)^2\left[\frac{5p}{24}\right] \tag{14}$$



(i) (ii)

Figure 2. Displays the 2-Dim (i) and 3-Dim (ii) multiple graphs of the approximate solution (14), for the different values of ω. As we assign different values of ω there comes a slight change in the graph and we found that the solution of fractional IDE is approximately equal to the exact solution for $\omega=1$. The accuracy can be enhanced by taking higher level of approximate solutions.

Problem 4.3 Suppose the non-linear Volterra FIDEs [17]

$${}^{CF}D^{\omega}f(p)=1+\int_0^p e^{-t}[f(t)]^2\,dt,\ 0<\omega\le 1 \tag{15}$$

with condition

$$f(0)=1,\ f_0(p)=f(0),\ f_0(p)=1.$$

Now we apply fractional VIM, the CF is

$$f_{k+1}(p)=f_k(p)+{}^{CF}I^{\omega}\left[\mu(s){}^{CF}D^{\omega}f_k(p)-1-\int_0^p e^{-t}[f_k(t)]^2\,dt\right]$$

Here $\mu(s)$ is Lagrange multiplier, for 1st order FIDEs, we have

$$\mu(s) = -1$$

for k = 0 we have

$$f_1(p) = f_0(p) + {}^{CF}I^{\omega}\left[(-1)^{CF}D^{\omega}f_0(p) - 1 - \int_0^p e^{-t}[f_0(t)]^2\,dt\right]$$

First we solve $D^{\omega} f_0(x)$

$$f_1(p) = 1 - {}^{CF}I^{\omega}\left[-1 - \int_0^p e^{-t}\,dt\right]$$

Now we use CF integral (${}^{CF}I^{\omega}$) on above equation and obtain the following result

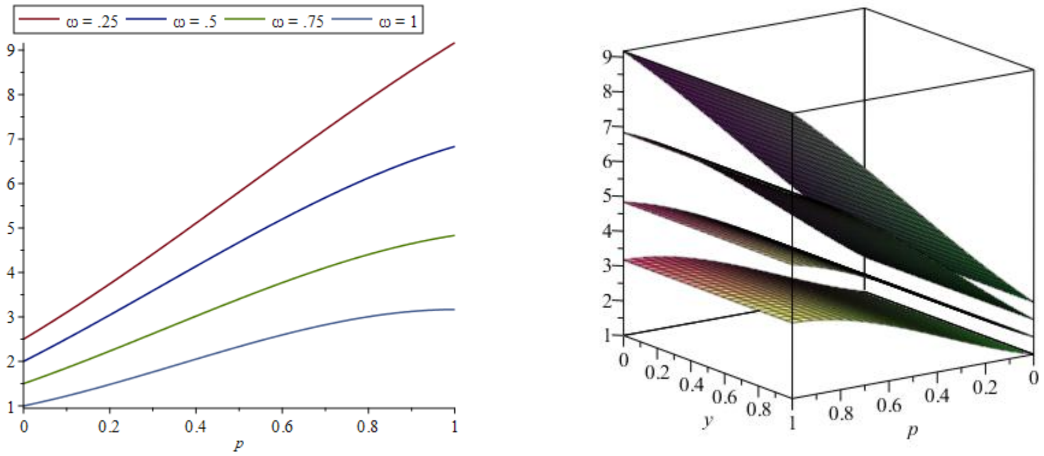
$$f_1(p) = 1 - {}^{CF}I^{\omega}\left[-1 + e^{-p} - 1\right]$$
$$f_1(p) = 1 - {}^{CF}I^{\omega}\left[-2 + e^{-p}\right]$$
$$f_1(p) = 1 + 2(1-\omega) + 2p - (1-\omega)e^{-p} + e^{-p} - 1$$

$$f_1(p) = 2(1-\omega) + 2p - (1-\omega)e^{-p} + e^{-p} \tag{16}$$

$$\begin{aligned} f_2(p) &= 1 + 2p + \frac{5}{2}p^2 - \frac{7}{2}p^3 + (1-\omega)^3\left[p + \frac{1}{2}p^2 - \frac{3}{2}p^3\right] \\ &+ (1-\omega)^2\left[-2p + \frac{11}{2}p^2 - \frac{5}{6}p^3\right] + (1-\omega)\left[2 + 6p - \frac{9}{2}p^2 + \frac{15}{6}p^3\right] \end{aligned} \tag{17}$$

and hence we have

$$\begin{aligned} f(p) &= 1 + 2p + \frac{5}{2}p^2 - \frac{7}{2}p^3 + (1-\omega)^3\left[p + \frac{1}{2}p^2 - \frac{3}{2}p^3\right] \\ &+ (1-\omega)^2\left[-2p + \frac{11}{2}p^2 - \frac{5}{6}p^3\right] + (1-\omega)\left[2 + 6p - \frac{9}{2}p^2 + \frac{15}{6}p^3\right] \end{aligned} \tag{18}$$



(i) (ii)

Figure 3. Displays the 2-Dim (i) and 3-Dim (ii) multiple graphs of the approximate solution (18), for the different values of ω. As we assign different values of ω there comes a slight change in the graph and we found that the solution of fractional IDE is approximately equal to the exact solution for $\omega = 1$. The accuracy can be enhanced by taking higher level of approximate solutions.

Problem 4.4 Suppose the system of linear Volterra FIDEs [4]

$$D^{\omega}u(p)=1-p^2+e^p+\int_0^p[u(t)+v(t)]dt\,,0<\omega\le 1 \tag{19}$$

$$D^{\omega}v(p)=3-3e^p+\int_0^p[u(t)-v(t)]dt\,,0<\omega\le 1 \tag{20}$$

with subject to the condition $u(0)=1,\quad v(0)=-1$. The exact solution is

$$(u,v)=\left(p+e^p\,,\,p-e^p\right).$$

The CF is as follows

$$u_{k+1}(p)=u_k(p)+{}^{CF}I^{\omega}\left[\mu(s)[D^{\omega}u_k(p)-1+p^2-e^p-\int_0^p[u_k(t)+v_k(t)]dt\right]$$

Here $\mu(s)$ is Lagrange multiplier, for 1st order FIDEs, we have

$$\mu(s)=-1$$

For k = 0,

$$u_1(p)=u_0(p)-{}^{CF}I^{\omega}\left[D^{\omega}u_0(p)-1+p^2-e^p-\int_0^p[u_0(t)+v_0(t)]dt\right]$$

$$u_1(p)=1-{}^{CF}I^{\omega}\left[1-p^2+e^p\right]$$

$$u_1(p)=1+(1-\omega)+p-(1-\omega)p^2-\frac{1}{3}p^3+e^p(1-\omega)+e^p-1$$

$$u_1(p)=(1-\omega)+p-(1-\omega)p^2-\frac{1}{3}p^3+e^p(1-\omega)+e^p \tag{21}$$

$$\begin{aligned}u_2(p)&=1+4p+\frac{p^2}{2}+\frac{p^3}{6}+\frac{p^4}{24}-(1-\omega)^2\left[2+4p+p^2-\frac{p^3}{3}-2e^p\right]\\&+(1-\omega)\left[6+4p+\frac{5}{2}p^2+\frac{p^3}{2}-\frac{p^4}{12}-2e^p\right]\end{aligned} \tag{22}$$

and the general solution is

$$\begin{aligned}u(p)&=1+4p+\frac{p^2}{2}+\frac{p^3}{6}+\frac{p^4}{24}-(1-\omega)^2\left[2+4p+p^2-\frac{p^3}{3}-2e^p\right]\\&+(1-\omega)\left[6+4p+\frac{5}{2}p^2+\frac{p^3}{2}-\frac{p^4}{12}-2e^p\right]\end{aligned} \tag{23}$$

Now for $v(p)$

$$D^{\omega}v(p) = -3e^{p} + 3 + \int_{0}^{p}[u(t) - v(t)]dt$$

$$v_{k+1}(p) = v_k(p) + {}^{CF}I^{\omega}\left[\mu(s)D^{\omega}v_k(p) - 3 + 3e^{p} - \int_{0}^{p}[u_k(t) - v_k(t)]dt\right]$$

For k = 0 we have

$$v_1(p) = v_0(p) - {}^{CF}I^{\omega}\left[-3 + 3e^{p} - \int_{0}^{p}[u_0(t) - v_0(t)]dt\right]$$

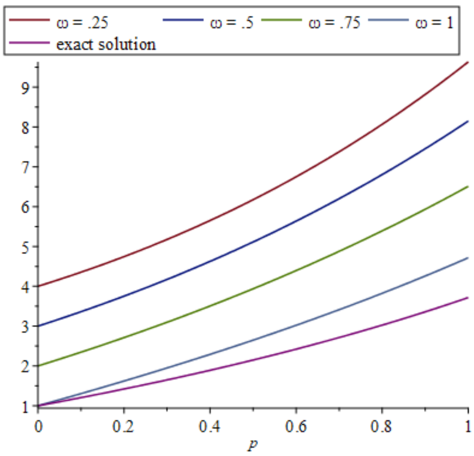
$$v_1(p) = -1 - {}^{CF}I^{\omega}\left[-3 + 3e^{p} - 2p\right]$$

$$v_1(p) = 2 + 3p - 3e^{p} + p^2 + (1-\omega)\left[3 + 2p - 3e^{p}\right] \tag{24}$$

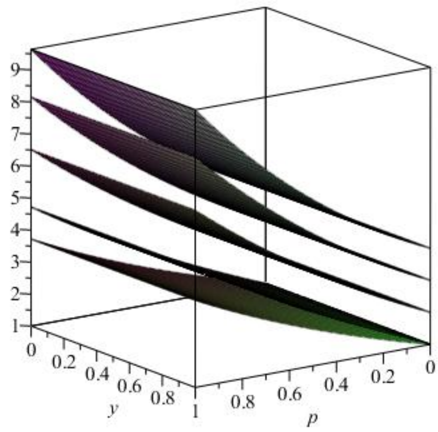
$$v_2(p) = -1 - 4p - \frac{p^2}{2} - \frac{p^3}{6} + (1-\omega)\left[2 - 3p - p^2 - \frac{p^3}{3} - 5e^{p}\right] - (1-\omega)^2\left[2 + 2p + p^2 + 2e^{p}\right] \tag{25}$$

and the general solution is

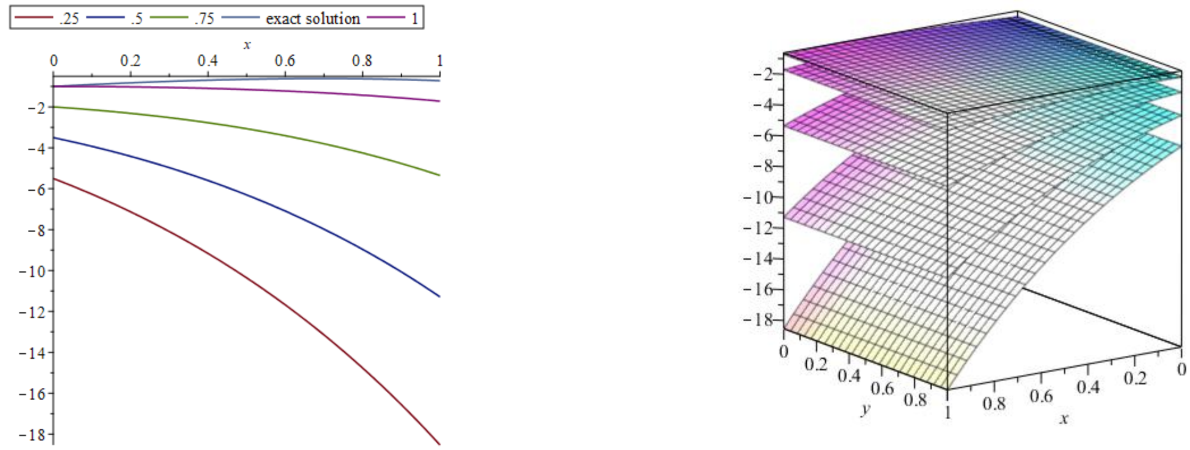
$$v(p) = -1 - 4p - \frac{p^2}{2} - \frac{p^3}{6} + (1-\omega)\left[2 - 3p - p^2 - \frac{p^3}{3} - 5e^{p}\right] - (1-\omega)^2\left[2 + 2p + p^2 + 2e^{p}\right] \tag{26}$$



(i)



(ii)



(iii) (iv)

Figure 4. (i)-(iv) depicts the 2-Dim and 3-Dim multiple graphs of the numerical solution (23) and (26) of the system by the FVIM for the different values of ω and the exact solution. It is obvious from the figure that as we assign different values of ω there comes a slight change in the graph and we found that the numerical solution of fractional IDE is converges to the exact solution for $\omega = 1$. The accuracy can be enhanced by taking higher level of approximate solutions.

Problem 4.5 Suppose the system of linear Volterra FIDEs [4]

$$D^{\omega}u(p) = \frac{1}{2}p^2 + 1 + p + \frac{1}{3}p^3 + \int_0^p [(p-t)u(t) + (p-t+1)v(t)]dt \, , 0 < \omega \le 1 \qquad (27)$$

$$D^{\omega}v(p) = -\frac{3}{2}p^2 - 1 - 3p - \frac{1}{3}p^3 + \int_0^p [(p-t+1)u(t) + (p-t)v(t)]dt \, , \qquad (28)$$

with subject to the condition $u(0) = 1$, $v(0) = 1$. The exact solution is

$$(u, v) = (1 + p + p^2, 1 - p - p^2)$$

The CF is read as

$$u_{k+1}(p) = u_k(p) + {}^{CF}I^{\omega}\left[\mu(s)D^{\omega}u_k(p) - 1 - p - \frac{1}{2}p^2 - \frac{1}{3}p^3 - \int_0^p [(p-t)u_k(t) + (p-t+1)v_k(t)]dt\right]$$

Here $\mu(s)$ is Lagrange multiplier, for 1st order FIDEs, we have

$$\mu(s) = -1 \, .$$

For k = 0

$$u_1(p) = u_0(p) + {}^{CF}I^{\omega}\left[(-1)D^{\omega}u_0(p) - 1 - p - \frac{1}{2}p^2 - \frac{1}{3}p^3 - \int_0^p [(p-t)u_0(t) + (p-t+1)v_0(t)]dt\right]$$

$$u_1(p) = u_0(p) - {}^{CF}I^{\omega}\left[-1 - p - \frac{1}{2}p^2 - \frac{1}{3}p^3 - 2p^2 + p^2 - p\right]$$

$$u_1(p)=1+(1-\omega)+p+2p(1-\omega)+p^2+\frac{1}{3}p^3(1-\omega)$$
$$+\frac{1}{2}p^2(1-\omega)+\frac{1}{6}p^3+\frac{1}{12}p^4 \tag{29}$$

$$u_2(p)=1+2p+\frac{3}{2}p^2+(1-\omega)\left[-p-p^2-\frac{1}{6}p^3\right]$$
$$-(1-\omega)^2\left[1+2p+\frac{1}{2}p^2\right] \tag{30}$$

and the general solution is

$$u(p)=1+2p+\frac{3}{2}p^2+(1-\omega)\left[-p-p^2-\frac{1}{6}p^3\right]$$
$$-(1-\omega)^2\left[1+2p+\frac{1}{2}p^2\right] \tag{31}$$

Now for $v(p)$

$$D^{\omega}v(p)=-1-3p-\frac{3}{2}p^2-\frac{1}{3}p^3+\int_0^p[(p-t+1)u(t)+(p-t)v(t)]dt$$

$$v_{k+1}(p)=v_k(p)+{}^{CF}I^{\omega}\left[\mu(s)D^{\omega}v_k(p)+1+3p+\frac{3}{2}p^2+\frac{1}{3}p^3-\int_0^x[(p-t)v_k(t)+(p-t+1)u_k(t)]dt\right]$$

For k = 0

$$v_1(p)=v_0(p)+{}^{CF}I^{\omega}\left[[-1-3p-\frac{3}{2}p^2-\frac{1}{3}p^3+\int_0^p[(p-t)v_0(t)+(p-t+1)u_0(t)]dt\right]$$

$$v_1(p)=v_0(p)+{}^{CF}I^{\omega}\left[-1-3p-\frac{3}{2}p^2-\frac{1}{3}p^3-2p^2+p^2-p\right]$$

$$v_1(p)=1-(1-\omega)-p-2p(1-\omega)-p^2+\frac{1}{3}p^3(1-\omega)$$
$$-\frac{1}{2}p^2(1-\omega)-\frac{1}{6}p^3+\frac{1}{12}p^4 \tag{32}$$

$$v_2(p)=1-2p-\frac{3}{2}p^2-(1-\omega)\left[-p-p^2-\frac{1}{6}p^3\right]$$
$$+(1-\omega)^2\left[1+2p+\frac{1}{2}p^2\right] \tag{33}$$

and the general solution is

$$v(p) = 1 - 2p - \frac{3}{2}p^2 - (1-\omega)\left[-p - p^2 - \frac{1}{6}p^3\right] + (1-\omega)^2\left[1 + 2p + \frac{1}{2}p^2\right] \quad (34)$$

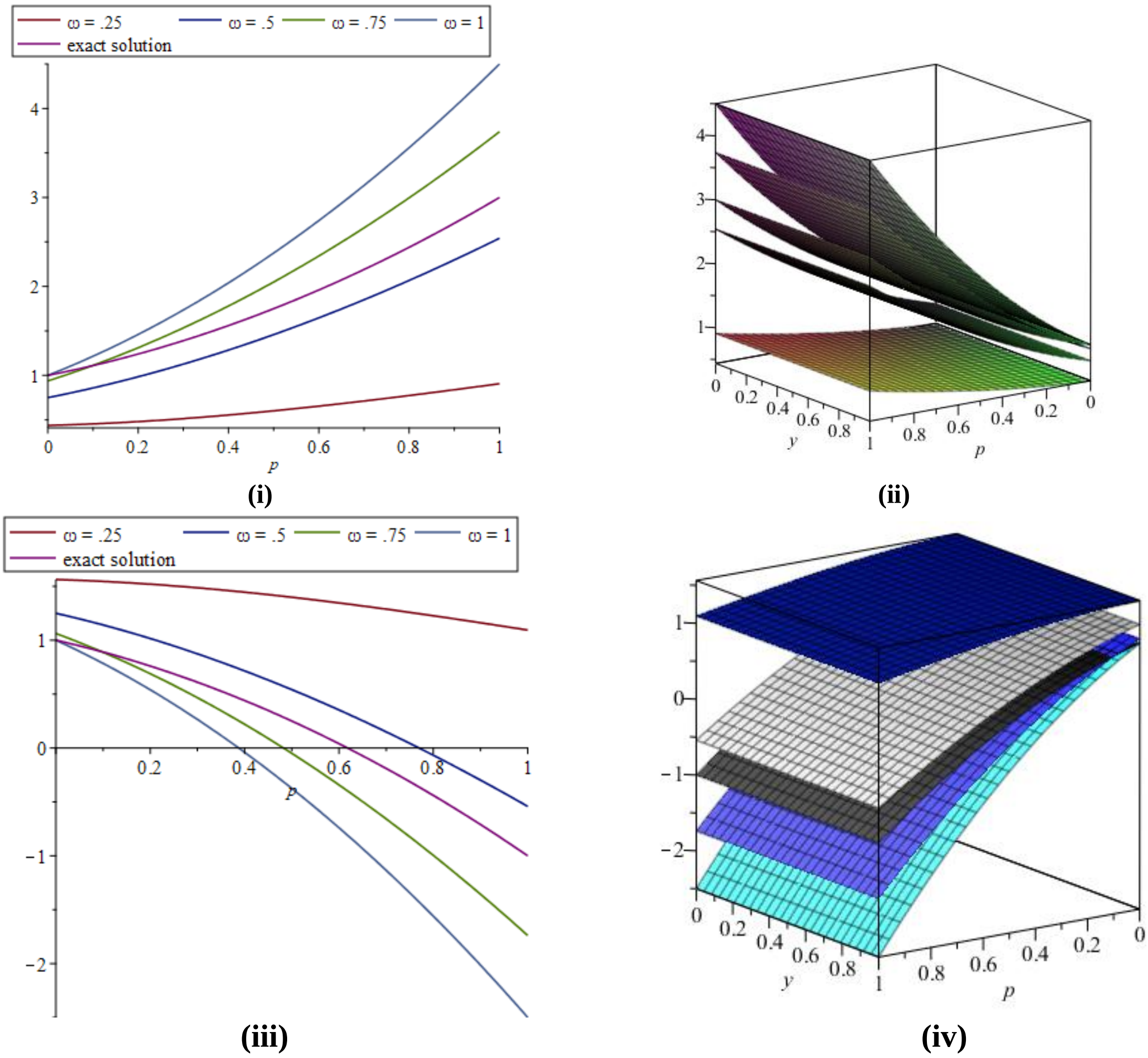


Figure 5. (i)-(iv) demonstrations the 2-Dim and 3-Dim multiple graphs of the numerical solution (31) and (34) of the system by the FVIM for the different values of ω and the exact solution. It is obvious from the figure that as we assign different values of ω there comes a slight change in the graph and we found that the numerical solution of fractional IDE is converges to the exact solution for $\omega = 1$. The accuracy can be enhanced by taking higher level of approximate solutions.

5. CONCLUSION

In this this work, the introduced analytic method (the variational iteration method) has been successfully applied to get invariant solutions of integro-differential equations of fractional order in Caputo Fabrizio sense with appropriate condition. The comparison of the results received by this method fractional VIM with the outcomes of the exact solution. It is worth mentioning that the results obtained are in very good close agreement to the exact solution of these equations using a few iterates of the fractional form of the problem. The methodology can be applied to diverse mathematical equations in Caputo-Fabrizio fractional derivative in scientific fields.

REFERENCES

[1] Rezapour A., Baleanu S.M., Mousalou D., *Advances in Difference Equations*, **2017**, 221, 2017.
[2] Debbouche, A., Nieto, J.J., *Applied Mathematics and Computation*, **245**, 74, 2014.
[3] Khan, A., Abro, K.A., Tassaddiq, A., Khan, I., *Entropy*, **19**, 279, 2017.
[4] Wazwaz, A.M., *Linear and Nonlinear Integral Equations: Methods and Applications*, Springer Publishing Company, 2011.
[5] Diethelm, B.D., Scalas, K., Trujillo, E., *Fractional Calculus Models and Numerical Methods, Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, 2012.
[6] Caputo, M.F., *Progress in Fractional Differentiation and Applications*, **1**(2), 73, 2015.
[7] Mohammed, D.S., *Mathematical Problems in Engineering*, **2014**, 431965, 2014.
[8] Baleanu, D., Mousalou, A., Rezapour, S., *Adavances in Difference Equations*, **2017**, 51, 2017.
[9] Baleanu, D., Agheli, B., Mohamed Al Qurashi, M., *Advances in Mechanical Engineering*, **8**(12), 1, 2016.
[10] Goufo, E.F.D., Pene, M.K., Mwambakana, J.N., *Open Mathematics*, **13**, 839, 2015.
[11] Aminikhan, H., Hosseini, S., *Journal of Applied Mathematics, Statistics and Informatics*, **11**(2), 15, 2015.
[12] Saeedi, H., Samimi, F., *International Journal of Engineering Research and Applications*, **2**(5), 52, 2012.
[13] Wang, H., Fu, H.M., Zhang, H.F., *International Journal of Nonlinear Sciences and Numerical Simulation*, **8**(2), 171, 2007.
[14] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[15] He, J.H., *International Journal of Non-linear Mechanics*, **34**, 699, 1999.
[16] Hou, J., Qin, B., Yang, C., *Journal of Applied Mathematics*, **2012**, 687030, 2012.
[17] Mittal, R.C., *International Journal of Applied Mathematics and Mechanics*, **4**(2), 87, 2008.