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PELL POLYNOMIAL APPROACH FOR DIRICHLET PROBLEM RELATED TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. *Dirichlet problem is one of the major problems of the theory of partial differential equations and occurs in several physical applications. In this study, a new numerical technique based on the Pell polynomials and collocation points is offered to obtain the approximate solution of Dirichlet problem for linear partial differential equations with variable coefficients. The method transforms the equation along with Dirichlet boundary conditions into a matrix equation with the unknown Pell coefficients by means of collocation points and operational matrices. The solution of this matrix equation yields the Pell coefficients of the solution function. Thereby, the approximate solution is obtained in the truncated Pell series form. Also, some examples together with error analysis technique based on residual functions are expensed to demonstrate the validity and applicability of the present method; the comparisons are fulfilled with existing results.*

Keywords: *Pell polynomials and series, Matrix and collocation methods, Dirichlet Problem, Partial differential equations, Residual error analysis.*

1. INTRODUCTION

Boundary value problems for partial differential equations are mathematical models of many real processes studied in natural and engineering sciences. In a boundary value problem, boundary conditions are prescribed at the end points of the domain. Boundary conditions can be classified under three canonical types: Dirichlet (or the boundary conditions of the first kind), Neumann (or the boundary conditions of the second kind) and Robin (or the boundary conditions of the third kind). In Dirichlet boundary condition, the value of the dependent variable is prescribed on the boundary. In Neumann boundary condition, the value of the gradient of the dependent variable normal to the boundary is prescribed on the boundary. In Robin boundary condition, a linear combination of the value of the dependent variable and its normal gradient is specified at the boundary [1-3]. Many numerical methods for solving boundary value problems are developed such as finite difference method [3-5], shooting method [3], Sinc-Galerkin method [6], Haar wavelet method [7], homotopy perturbation method [8], modified artificial neural network [9], multi-level adaptive technique [10], Monte Carlo methods [11-12], block linear multistep methods [13], Taylor matrix collocation methods [14], Chebyshev matrix collocation methods [15-17], Euler matrix method [18], etc.

In this study, we develop a numerical technique based on Pell polynomials to solve the second order linear partial differential equation

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$$A(x,t)\frac{\partial^2 u}{\partial x^2}+B(x,t)\frac{\partial^2 u}{\partial x\partial t}+C(x,t)\frac{\partial^2 u}{\partial t^2}+D(x,t)\frac{\partial u}{\partial x}+E(x,t)\frac{\partial u}{\partial t}+F(x,t)u=G(x,t) \quad (1)$$

under the initial conditions and Dirichlet boundary conditions for $a \le x \le b$ and $c \le t \le d$

$$u(x,c)=f_1(x), \quad u(x,d)=f_2(x), \quad u(a,t)=m_1(t), \quad u(b,t)=m_2(t). \quad (2)$$

Here $A(x,t), B(x,t), C(x,t), D(x,t), E(x,t), F(x,t)$, $G(x,t)$, $f_1(x)$, $f_2(x)$, $m_1(t)$ and $m_2(t)$ are continuous functions. We assume the approximate solution of the problem $(1)-(2)$ in the truncated Pell series form

$$u(x,t)\cong u_N(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}a_{m,n}P_{m+1,n+1}(x,t);\ \ P_{m+1,n+1}(x,t)=P_{m+1}(x)P_{n+1}(t) \quad (3)$$

where $u_N(x,t)$ is approximate solution of Eq.(1); $a_{m,n}$, $(m,n=0,1,\ldots,N)$ are the unknown Pell polynomials coefficients; N is chosen as any positive integer such that $N \ge 2$.

SOME PROPERTIES OF PELL POLYNOMIALS

Fibonacci and Pell numbers are specific values of Pell polynomials $P_n(x)$ which were studied extensively in 1985 by A.F. Horadam. The polynomials $A_n(x)$ were investigated by Horadam, which are defined by recurrence relations

$$A_n(x)=pxA_{n-1}(x)+qA_{n-2}(x)\ ,\ n\ge 2\ ,\ A_0(x)=0\ ,\ A_1(x)=1. \quad (4)$$

By using the relations (4) and taking $p=2$ and $q=1$, we obtain the recurrence relations of Pell Polynomials [22-24]:

$$P_n(x)=2xP_{n-1}(x)+P_{n-2}(x)\ ,\ n\ge 2\ ,\ P_0(x)=0\ ,\ P_1(x)=1. \quad (5)$$

From standard methods, the explicit representations of Pell polynomials is defined

$$P_n(x)=\sum_{k=0}^{\left[\frac{(n-1)}{2}\right]}\binom{n-k-1}{k}2^{n-2k-1}x^{n-2k-1}. \quad (6)$$

The first ten Pell polynomials are given in Table 1.

Table 1. Pell Polynomials

n	$P_n(x)$	n	$P_n(x)$
1	1	6	$6x+32x^3+32x^5$
2	$2x$	7	$1+24x^2+80x^4+64x^6$
3	$1+4x^2$	8	$8x+80x^3+192x^5+128x^7$
4	$4x+8x^3$	9	$1+40x^2+240x^4+448x^6+256x^8$
5	$1+12x^2+16x^4$	10	$10x+160x^3+672x^5+1024x^7+512x^9$

2. MATERIALS AND METHODS

2.1. FUNDAMENTAL MATRIX RELATIONS

Let us consider Eq. (1) and find the matrix forms of the equation. First, we are able to write Pell polynomials (5) or (6) in the matrix form:

$$u(x,t) \cong u_N(x,t) = \mathbf{P}(x)\overline{\mathbf{P}}(t)\mathbf{A} \tag{7}$$

where

$$\overline{\mathbf{P}}(t) = \begin{bmatrix} \mathbf{P}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{P}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}(t) \end{bmatrix}; \quad \mathbf{P}(x) = \begin{bmatrix} P_1(x) & P_2(x) & \cdots & P_{N+1}(x) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_N \end{bmatrix}^T; \quad \mathbf{A}_i = \begin{bmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,N} \end{bmatrix}^T, \quad (i = 0,1,2,\ldots,N),$$

$$\mathbf{A} = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N} & a_{1,0} & a_{1,1} & \cdots & a_{1,N} & \cdots & a_{N,0} & a_{N,1} & \cdots & a_{N,N} \end{bmatrix}^T.$$

Also, we can use the relations

$$\mathbf{P}(x) = \mathbf{X}(x)\mathbf{S} \quad \text{and} \quad \overline{\mathbf{P}}(\mathrm{t}) = \overline{\mathbf{X}}(t)\overline{\mathbf{S}}; \tag{8}$$

$$\overline{\mathbf{X}}(t) = \begin{bmatrix} \mathbf{X}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(t) \end{bmatrix}; \quad \mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}$$

$$\mathbf{S}^{\mathbf{T}}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \\ 0 & 2^1\binom{1}{0} & 0 & \cdots & 0 & \\ 2^0\binom{1}{1} & 0 & 2^2\binom{2}{0} & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 2^0\binom{N-\left(\frac{N-1}{2}\right)-1}{\frac{N-1}{2}} & 0 & 2^2\binom{N-\left(\frac{N-3}{2}\right)-1}{\frac{N-3}{2}} & \cdots & 2^{N-1}\binom{N-\left(\frac{N-N}{2}\right)-1}{\left(\frac{N-N}{2}\right)} & \Big\}\text{if } N \text{ is odd} \\ 0 & 2^1\binom{N-\left(\frac{N-2}{2}\right)-1}{\frac{N-2}{2}} & 0 & \cdots & 2^{N-1}\binom{N-\left(\frac{N-N}{2}\right)-1}{\left(\frac{N-N}{2}\right)} & \Big\}\text{if } N \text{ is even} \end{bmatrix}$$

$$\mathbf{S}^0=\mathbf{I}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},\quad \overline{\mathbf{S}}=\begin{bmatrix} \mathbf{S} & 0 & \cdots & 0 \\ 0 & \mathbf{S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{S} \end{bmatrix}.$$

On the other hand, the relation between the matrices $\mathbf{X}(x)$ and its derivatives $\mathbf{X}'(x)$, $\mathbf{X}''(x)$,…, $\mathbf{X}^{(k)}(x)$ are [18-20, 25]

$$\mathbf{X}'(x)=\mathbf{X}(x)\mathbf{B}\ ,\ \mathbf{X}''(x)=\mathbf{X}(x)\mathbf{B}^2\ ,\ldots,\ \mathbf{X}^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k\ , \tag{9}$$

$\overline{\mathbf{X}}(t)$ and its derivatives $\overline{\mathbf{X}'}(t)$, $\overline{\mathbf{X}''}(t)$,…, $\overline{\mathbf{X}^{(k)}}(t)$ are [18-20, 25]

$$\overline{\mathbf{X}'}(t)=\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\ ,\ \overline{\mathbf{X}''}(t)=\overline{\mathbf{X}}(t)\overline{\mathbf{B}^2}\ ,\ldots,\ \overline{\mathbf{X}^{(k)}}(t)=\overline{\mathbf{X}}(t)\overline{\mathbf{B}^{(k)}}\ ; \tag{10}$$

$$\mathbf{B}^0=\mathbf{I}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\quad \overline{\mathbf{B}}=\begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}.$$

Then, we organize the following matrix relations of the derivative forms of $u(x,t)$ with (7)-(10):

$$\begin{aligned} u(x,t) &= \mathbf{P}(x)\overline{\mathbf{P}}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} \\ u_x(x,t) &= \mathbf{P}'(x)\overline{\mathbf{P}}(t)\mathbf{A} = \mathbf{X}'(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} = \mathbf{X}(x)\mathbf{B}\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} \\ u_t(x,t) &= \mathbf{P}(x)\overline{\mathbf{P}'}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}'}(t)\overline{\mathbf{S}}\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\overline{\mathbf{S}}\mathbf{A} \\ u_{xt}(x,t) &= \mathbf{P}'(x)\overline{\mathbf{P}'}(t)\mathbf{A} = \mathbf{X}'(x)\mathbf{S}\overline{\mathbf{X}'}(t)\overline{\mathbf{S}}\mathbf{A} = \mathbf{X}(x)\mathbf{B}\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\overline{\mathbf{S}}\mathbf{A} \\ u_{xx}(x,t) &= \mathbf{P}''(x)\overline{\mathbf{P}}(t)\mathbf{A} = \mathbf{X}''(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} = \mathbf{X}(x)\mathbf{B}^2\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} \\ u_{tt}(x,t) &= \mathbf{P}(x)\overline{\mathbf{P}''}(t)\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}''}(t)\overline{\mathbf{S}}\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}^2}\overline{\mathbf{S}}\mathbf{A} \end{aligned} . \quad (11)$$

2.2. PELL MATRIX COLLOCATION METHOD

By substituting the relations (11) into Eq.(1) we have the fundamental matrix form for Eq.(1):

$$\underbrace{\left\{\begin{matrix} A(x,t)\mathbf{X}(x)\mathbf{B}^2\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}} + B(x,t)\mathbf{X}(x)\mathbf{B}\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\overline{\mathbf{S}} + C(x,t)\mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}^2}\overline{\mathbf{S}} + \\ D(x,t)\mathbf{X}(x)\mathbf{B}\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}} + E(x,t)\mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\overline{\mathbf{S}} + F(x,t)\mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}} \end{matrix}\right\}}_{\mathbf{W}(x,t)}\mathbf{A} = G(x,t);$$

$$\mathbf{W}(x,t)\mathbf{A} = G(x,t)$$

and then, using the collocation points defined by

$$x_i = a + \frac{b-a}{N}i \quad , \quad t_j = c + \frac{d-c}{N}j \quad , \quad i,j = 0,1,2,\ldots,N ,$$

or shortly

$$\mathbf{W}\mathbf{A} = \mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}]; \quad (12)$$

$$\mathbf{W} = [\mathbf{W}_0 \quad \mathbf{W}_1 \quad \cdots \quad \mathbf{W}_N]^T \ ; \ \mathbf{W}_\mathbf{i} = [\mathbf{W}(x_i,t_0) \quad \mathbf{W}(x_i,t_1) \quad \cdots \quad \mathbf{W}(x_i,t_N)]^T$$

$$\mathbf{G} = [\mathbf{G}_0 \quad \mathbf{G}_1 \quad \cdots \quad \mathbf{G}_N]^T \ ; \ \mathbf{G}_i = [G(x_i,t_0) \quad G(x_i,t_1) \quad \cdots \quad G(x_i,t_N)]^T \quad , \quad i = 0,1,2,\ldots,N .$$

Thus, the fundamental matrix equation of Eq.(1) is defined by (12). Similarly, by using the relations (7)-(10), we are able to obtain the corresponding matrix forms for the conditions (2) as follows:

$$\begin{aligned} u(x,c) &= \mathbf{P}(x)\overline{\mathbf{P}}(c)\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(c)\overline{\mathbf{S}}\mathbf{A}, \quad u(x,d) = \mathbf{P}(x)\overline{\mathbf{P}}(d)\mathbf{A} = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(d)\overline{\mathbf{S}}\mathbf{A}, \\ u(a,t) &= \mathbf{P}(a)\overline{\mathbf{P}}(t)\mathbf{A} = \mathbf{X}(a)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A}, \quad u(b,t) = \mathbf{P}(b)\overline{\mathbf{P}}(t)\mathbf{A} = \mathbf{X}(b)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A}. \end{aligned} \quad (13)$$

By substituting the relation (13) into the conditions (2), we have the matrix forms of (2):

$$\underbrace{\mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(c)\overline{\mathbf{S}}}_{\mathbf{U}_1(x,c)}\mathbf{A}=f_1(x) \quad\Rightarrow\quad \mathbf{U}_1(x,c)\mathbf{A}=f_1(x),$$

$$\underbrace{\mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(d)\overline{\mathbf{S}}}_{\mathbf{U}_2(x,d)}\mathbf{A}=f_2(x) \quad\Rightarrow\quad \mathbf{U}_2(x,d)\mathbf{A}=f_2(x),$$

$$\underbrace{\mathbf{X}(a)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}}_{\mathbf{U}_3(a,t)}\mathbf{A}=m_1(t) \quad\Rightarrow\quad \mathbf{U}_3(a,t)\mathbf{A}=m_1(t),$$

$$\underbrace{\mathbf{X}(b)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}}_{\mathbf{U}_4(b,t)}\mathbf{A}=m_2(t) \quad\Rightarrow\quad \mathbf{U}_4(b,t)\mathbf{A}=m_2(t)$$

and by putting the collocation points $x_i=a+\frac{b-a}{N}i$, $t_j=c+\frac{d-c}{N}j$, $i,j=1,2,\ldots,N-1$,

$$\begin{aligned}&\mathbf{U}_1\mathbf{A}=\mathbf{F}_1 \ \left(\text{or } [\mathbf{U}_1;\mathbf{F}_1]\right), \quad \mathbf{U}_2\mathbf{A}=\mathbf{F}_2 \ \left(\text{or } [\mathbf{U}_2;\mathbf{F}_2]\right),\\ &\mathbf{U}_3\mathbf{A}=\mathbf{M}_1 \ \left(\text{or } [\mathbf{U}_3;\mathbf{M}_1]\right), \ \mathbf{U}_4\mathbf{A}=\mathbf{M}_2 \ \left(\text{or } [\mathbf{U}_4;\mathbf{M}_2]\right)\end{aligned} \tag{14}$$

where

$$\mathbf{U}_1=\left[\mathbf{U}_1(x_1,c) \quad \mathbf{U}_1(x_2,c) \quad \mathbf{U}_1(x_3,c) \quad \cdots \quad \mathbf{U}_1(x_N,c)\right]^T$$

$$\mathbf{U}_2=\left[\mathbf{U}_2(x_1,d) \quad \mathbf{U}_2(x_2,d) \quad \mathbf{U}_2(x_3,d) \quad \cdots \quad \mathbf{U}_2(x_N,d)\right]^T$$

$$\mathbf{U}_3=\left[\mathbf{U}_3(a,t_1) \quad \mathbf{U}_3(a,t_2) \quad \mathbf{U}_2(a,t_3) \quad \cdots \quad \mathbf{U}_2(a,t_N)\right]^T$$

$$\mathbf{U}_4=\left[\mathbf{U}_4(b,t_1) \quad \mathbf{U}_4(b,t_2) \quad \mathbf{U}_4(b,t_3) \quad \cdots \quad \mathbf{U}_4(b,t_N)\right]^T$$

$$\mathbf{F}_1=\left[f_1(x_0) \quad f_1(x_1) \quad f_1(x_2) \quad \cdots \quad f_1(x_N)\right]^T, \ \mathbf{F}_2=\left[f_2(x_0) \quad f_2(x_1) \quad f_2(x_2) \quad \cdots \quad f_2(x_N)\right]^T$$

$$\mathbf{M}_1=\left[m_1(t_0) \quad m_1(t_1) \quad m_1(t_2) \quad \cdots \quad m_1(t_N)\right]^T, \ \mathbf{M}_2=\left[m_2(t_0) \quad m_2(t_1) \quad m_2(t_2) \quad \cdots \quad m_2(t_N)\right]^T.$$

To obtain the solution of Eq.(1) under the conditions (2), the following augmented matrix is constructed by replacing the row matrices (12) by the 4(N-1) rows of the matrix (14); so we have the new augmented matrix

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]=\begin{bmatrix}\mathbf{W};\mathbf{G}\\ \mathbf{U}_1;\mathbf{F}_1\\ \mathbf{U}_2;\mathbf{F}_2\\ \mathbf{U}_3;\mathbf{M}_1\\ \mathbf{U}_4;\mathbf{M}_2\end{bmatrix}. \quad (15)$$

Then we solve the Eq. (15) $\mathbf{A}=\left(\tilde{\mathbf{W}}\right)^{-1}\tilde{\mathbf{G}}$ if $rank\left(\tilde{\mathbf{W}}\right)=rank\left(\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right)=(N+1)^2$ and $\mathbf{A}$ is uniquely determined. In this way, the unknown Pell polynomial coefficients are obtained. Thus, the approximate solution $u(x,t)$ is found in the form (3).

3. ERROR ANALYSIS

We can easily check the accuracy of the method. Since the finite seri solution (3) is an approximate solution of Eq.(1), when the approximate solution $u_N(x,t)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is,for $x=x_r$, $a\le x_r\le b$ and $t=t_s$, $c\le t_s\le d$ [14]:

$$R_N(x_r,t_s)=\left|\begin{array}{l}A(x_r,t_s)(u_N)_{xx}(x_r,t_s)+B(x_r,t_s)(u_N)_{xt}(x_r,t_s)+C(x_r,t_s)(u_N)_{tt}(x_r,t_s)+\\ D(x_r,t_s)(u_N)_x(x_r,t_s)+E(x_r,t_s)(u_N)_t(x_r,t_s)+F(x_r,t_s)(u_N)(x_r,t_s)-G(x_r,t_s)\end{array}\right|\cong 0$$

where $R_N(x_r,t_s)\le 10^{-k_{rs}}=10^{-k}$ (k is positive integer) . If $\max 10^{-k_{rs}}=10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_N(x_r,t_s)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand we use absolute error and relative error for measuring errors. If $\tilde{u}(x,t)$ is an approximation to $u(x,t)$, then the absolute error and relative error can be defined as [25-28]

$$\text{absolute error}=\left|u(x,t)-\tilde{u}(x,t)\right|,\quad \text{relative error}=\left|\frac{u(x,t)-\tilde{u}(x,t)}{u(x,t)}\right|.$$

3.1. ESTIMATION THE UPPER BOUND OF THE ERROR BASED ON RESIDUAL FUNCTION

Let L be the linear operator and $L[u(x,t)]=g(x,t)$, $a\le x\le b$, $c\le t\le d$. By means of residual function defined by $R_N(x,t)$ and the mean value of the function $\left|R_N(x,t)\right|$, the accuracy of the solution can be controlled and the error can be estimated. Thus, we can estimate the upper bound of the mean error, that is, $\overline{R_N}$ as follows [26-27]:

$$R_N(x,t)=L[u_N(x,t)]-g(x,t)$$

$$\left|\iint_D R_N(x,t)dA\right| \le \iint_D \left|R_N(x,t)\right|dA.$$

The mean value theorem for double integrals says that if R_N is a continuous function on a plane region D, then there is exists a point $(x_0,t_0)\in D$ such that

$$\iint_D R_N(x,t)dA = R_N(x_0,t_0)A(D),$$

$$\left|\iint_D R_N(x,t)dA\right| = \left|R_N(x_0,t_0)\right|\left|A(D)\right|$$

$$\left|R_N(x_0,t_0)\right|\left|A(D)\right| \le \iint_D \left|R_N(x,t)\right|dA$$

$$\Rightarrow \left|R_N(x_0,t_0)\right| \le \frac{\iint_D \left|R_N(x,t)\right|dA}{A(D)} = \overline{R_N}\ .$$

where $A(D)$ denotes the area of D.

4. NUMERICAL EXAMPLES

The method of this study is useful in finding the solutions in terms of Pell polynomials. We illustrate it by the following examples.

Example 1. Let us consider Laplace equation given by [2]

$$u_{xx}(x,t)+u_{tt}(x,t)=0$$

subject to the Dirichlet conditions

$$u(x,0)=0,\quad u(x,1)=x,\quad u(0,t)=0,\quad u(1,t)=t\ ,\quad 0\le x\le 1,\ 0\le t\le 1.$$

The exact solution is $u(x,t)=xt$. Approximate solution of the problem is obtained by using Pell matrix collocation method for $N=3$. For this aim, approximate solution is investigated in form of the truncated Pell series form (3). Thereafter, the collocation points are determined for $0\le x\le 1$ and $0\le t\le 1$ as follows

$$\left\{x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_3=1\right\} \text{ and } \left\{t_0=0, t_1=\frac{1}{3}, t_2=\frac{2}{3}, t_3=1\right\}.$$

Fundamental matrix equation for Pell matrix collocation method of Laplace equation as

$$\left\{\mathbf{X}(x)\mathbf{B}^2\mathbf{S}\bar{\mathbf{X}}(t)\bar{\mathbf{S}}+\mathbf{X}(x)\mathbf{S}\bar{\mathbf{X}}(t)\overline{\mathbf{B}^2}\bar{\mathbf{S}}\right\}\mathbf{A}=g(x,t);$$

$$\mathbf{W}(x,t)=\mathbf{X}(x)\mathbf{B}^2\mathbf{S}\bar{\mathbf{X}}(t)\bar{\mathbf{S}}+\mathbf{X}(x)\mathbf{S}\bar{\mathbf{X}}(t)\overline{\mathbf{B}^2}\bar{\mathbf{S}}$$

where

$$\mathbf{S}=\begin{bmatrix}1&0&1&0\\0&2&0&4\\0&0&4&0\\0&0&0&8\end{bmatrix},\quad \mathbf{B}^2=\begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \mathbf{X}(x)=\begin{bmatrix}1&x&x^2&x^3\end{bmatrix}$$

$$\bar{\mathbf{X}}(x)=diag\left[\mathbf{X}(x),\mathbf{X}(x),\mathbf{X}(x),\mathbf{X}(x)\right],\quad \bar{\mathbf{B}}^2=diag\left[\mathbf{B}^2,\mathbf{B}^2,\mathbf{B}^2,\mathbf{B}^2\right],$$

$$\bar{\mathbf{S}}=diag\left[\mathbf{S},\mathbf{S},\mathbf{S},\mathbf{S}\right]\ \text{ and }\ g(x,t)=0.$$

By putting the collocation points for $i,j=0,1,2,3$ into the fundamental matrix equation, then we have the augmented matrix as follows

$$[\mathbf{W};\mathbf{G}]=\left[\begin{array}{cccccccccccccccccc}
0&0&8&0&0&0&0&0&8&0&16&0&0&0&0&0&;&0\\
0&0&8&16&0&0&0&0&8&16/3&176/9&784/27&0&0&0&0&;&0\\
0&0&8&32&0&0&0&0&8&32/3&272/9&1952/27&0&0&0&0&;&0\\
0&0&8&48&0&0&0&0&8&16&48&144&0&0&0&0&;&0\\
0&0&8&0&0&0&16/3&0&8&0&176/9&0&16&0&784/27&0&;&0\\
0&0&8&16&0&0&16/3&32/3&8&16/3&208/9&976/27&16&32/3&976/27&1408/27&;&0\\
0&0&8&32&0&0&16/3&64/3&8&32/3&304/9&2336/27&16&64/3&1552/27&3584/27&;&0\\
0&0&8&48&0&0&16/3&32&8&16&464/9&496/3&16&32&2512/27&2432/9&;&0\\
0&0&8&0&0&0&32/3&0&8&0&272/9&0&32&0&1952/27&0&;&0\\
0&0&8&16&0&0&32/3&64/3&8&16/3&304/9&1552/27&32&64/3&2336/27&3584/27&;&0\\
0&0&8&32&0&0&32/3&128/3&8&32/3&400/9&3488/27&32&128/3&3488/27&8704/27&;&0\\
0&0&8&48&0&0&32/3&64&8&16&560/9&688/3&32&64&5408/27&5632/9&;&0\\
0&0&8&0&0&0&16&0&8&0&48&0&48&0&144&0&;&0\\
0&0&8&16&0&0&16&32&8&16/3&464/9&2512/27&48&32&496/3&2432/9&;&0\\
0&0&8&32&0&0&16&64&8&32/3&560/9&5408/27&48&64&688/3&5632/9&;&0\\
0&0&8&48&0&0&16&96&8&16&80&336&48&96&336&1152&;&0
\end{array}\right].$$

Moreover, the matrix equations for Dirichlet conditions become

$$U_1(x,0)=\mathbf{X}(x)\mathbf{S}\bar{\mathbf{X}}(0)\bar{\mathbf{S}},\qquad U_2(x,1)=\mathbf{X}(x)\mathbf{S}\bar{\mathbf{X}}(1)\bar{\mathbf{S}}$$

$$U_3(0,t)=\mathbf{X}(0)\mathbf{S}\bar{\mathbf{X}}(t)\bar{\mathbf{S}},\qquad U_4(1,t)=\mathbf{X}(1)\mathbf{S}\bar{\mathbf{X}}(t)\bar{\mathbf{S}}.$$

By putting the collocation points for $i, j = 1, 2$ into these matrix equations, then we have the matrices of the conditions as

$$\left[\mathbf{U}_1;\mathbf{\Phi}_1\right]=\begin{bmatrix} 1 & 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{13}{9} & 0 & \frac{13}{9} & 0 & \frac{44}{27} & 0 & \frac{44}{27} & 0 & ; & 0 \\ 1 & 0 & 1 & 0 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & \frac{25}{9} & 0 & \frac{25}{9} & 0 & \frac{136}{27} & 0 & \frac{136}{27} & 0 & ; & 0 \end{bmatrix}$$

$$\left[\mathbf{U}_2;\mathbf{\Phi}_2\right]=\begin{bmatrix} 1 & 2 & 5 & 12 & \frac{2}{3} & \frac{4}{3} & \frac{10}{3} & 8 & \frac{13}{9} & \frac{26}{9} & \frac{65}{9} & \frac{52}{3} & \frac{44}{27} & \frac{88}{27} & \frac{220}{27} & \frac{176}{9} & ; & \frac{1}{3} \\ 1 & 2 & 5 & 12 & \frac{4}{3} & \frac{8}{3} & \frac{20}{3} & 16 & \frac{25}{9} & \frac{50}{9} & \frac{125}{9} & \frac{100}{3} & \frac{136}{27} & \frac{272}{27} & \frac{680}{27} & \frac{544}{9} & ; & \frac{2}{3} \end{bmatrix}$$

$$\left[\mathbf{U}_3;\mathbf{\Phi}_3\right]=\begin{bmatrix} 1 & \frac{2}{3} & \frac{13}{9} & \frac{44}{27} & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{13}{9} & \frac{44}{27} & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & \frac{4}{3} & \frac{25}{9} & \frac{136}{27} & 0 & 0 & 0 & 0 & 1 & \frac{4}{3} & \frac{25}{9} & \frac{136}{27} & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

$$\left[\mathbf{U}_4;\mathbf{\Phi}_4\right]=\begin{bmatrix} 1 & \frac{2}{3} & \frac{13}{9} & \frac{44}{27} & 2 & \frac{4}{3} & \frac{26}{9} & \frac{88}{27} & 5 & \frac{10}{3} & \frac{65}{9} & \frac{220}{27} & 12 & 8 & \frac{52}{3} & \frac{176}{9} & ; & \frac{1}{3} \\ 1 & \frac{4}{3} & \frac{25}{9} & \frac{136}{27} & 2 & \frac{8}{3} & \frac{50}{9} & \frac{272}{27} & 5 & \frac{20}{3} & \frac{125}{9} & \frac{680}{27} & 12 & 16 & \frac{100}{3} & \frac{544}{9} & ; & \frac{2}{3} \end{bmatrix}.$$

Then, by replacing last eight rows of the augmented matrix $\left[\mathbf{W};\mathbf{G}\right]$ with rows of the matrices of the conditions, so we have the new augmented matrix

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]=\begin{bmatrix} 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 8 & 16 & 0 & 0 & 0 & 0 & 8 & \frac{16}{3} & \frac{176}{9} & \frac{784}{27} & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 8 & 32 & 0 & 0 & 0 & 0 & 8 & \frac{32}{3} & \frac{272}{9} & \frac{1952}{27} & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 8 & 48 & 0 & 0 & 0 & 0 & 8 & 16 & 48 & 144 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & \frac{16}{3} & 0 & 8 & 0 & \frac{176}{9} & 0 & 16 & 0 & \frac{784}{27} & 0 & ; & 0 \\ 0 & 0 & 8 & 16 & 0 & 0 & \frac{16}{3} & \frac{32}{3} & 8 & \frac{16}{3} & \frac{208}{9} & \frac{976}{27} & 16 & \frac{32}{3} & \frac{976}{27} & \frac{1408}{27} & ; & 0 \\ 0 & 0 & 8 & 32 & 0 & 0 & \frac{16}{3} & \frac{64}{3} & 8 & \frac{32}{3} & \frac{304}{9} & \frac{2336}{27} & 16 & \frac{64}{3} & \frac{1552}{27} & \frac{3584}{27} & ; & 0 \\ 0 & 0 & 8 & 48 & 0 & 0 & \frac{16}{3} & 32 & 8 & 16 & \frac{464}{9} & \frac{496}{3} & 16 & 32 & \frac{2512}{27} & \frac{2432}{9} & ; & 0 \\ 1 & 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{13}{9} & 0 & \frac{13}{9} & 0 & \frac{44}{27} & 0 & \frac{44}{27} & 0 & ; & 0 \\ 1 & 0 & 1 & 0 & \frac{4}{3} & 0 & \frac{4}{3} & 0 & \frac{25}{9} & 0 & \frac{25}{9} & 0 & \frac{136}{27} & 0 & \frac{136}{27} & 0 & ; & 0 \\ 1 & 2 & 5 & 12 & \frac{2}{3} & \frac{4}{3} & \frac{10}{3} & 8 & \frac{13}{9} & \frac{26}{9} & \frac{65}{9} & \frac{52}{3} & \frac{44}{27} & \frac{88}{27} & \frac{220}{27} & \frac{176}{9} & ; & \frac{1}{3} \\ 1 & 2 & 5 & 12 & \frac{4}{3} & \frac{8}{3} & \frac{20}{3} & 16 & \frac{25}{9} & \frac{50}{9} & \frac{125}{9} & \frac{100}{3} & \frac{136}{27} & \frac{272}{27} & \frac{680}{27} & \frac{544}{9} & ; & \frac{2}{3} \\ 1 & \frac{2}{3} & \frac{13}{9} & \frac{44}{27} & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{13}{9} & \frac{44}{27} & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & \frac{4}{3} & \frac{25}{9} & \frac{136}{27} & 0 & 0 & 0 & 0 & 1 & \frac{4}{3} & \frac{25}{9} & \frac{136}{27} & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & \frac{2}{3} & \frac{13}{9} & \frac{44}{27} & 2 & \frac{4}{3} & \frac{26}{9} & \frac{88}{27} & 5 & \frac{10}{3} & \frac{65}{9} & \frac{220}{27} & 12 & 8 & \frac{52}{3} & \frac{176}{9} & ; & \frac{1}{3} \\ 1 & \frac{4}{3} & \frac{25}{9} & \frac{136}{27} & 2 & \frac{8}{3} & \frac{50}{9} & \frac{272}{27} & 5 & \frac{20}{3} & \frac{125}{9} & \frac{680}{27} & 12 & 16 & \frac{100}{3} & \frac{544}{9} & ; & \frac{2}{3} \end{bmatrix}.$$

Thus, from solution of this system, the matrix of the unknown Pell coefficients is obtained as

$$\mathbf{A}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and thereby, the approximate solution of the problem is determined as

$$u_3(x,t) = \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} \quad \Rightarrow \quad u(x,t) = u_3(x,t) = xt$$

which is the exact solution.

Example 2. Consider Poisson equation given by [12]

$$u_{xx} + u_{tt} = (x^2 + t^2)e^{xt}, \; 0 \le x \le 2, 0 \le t \le 1$$

with Dirichlet boundary conditions

$$u(x,0) = 1, \quad u(x,1) = e^x, \quad u(0,t) = 1, \quad u(2,t) = e^{2t}.$$

The exact solution is given by $u(x,t) = e^{xt}$. We look for the approximate solution $u_N(x,t)$ as

$$u_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} a_{m,n} P_{m+1,n+1}(x,t).$$

By applying the Pell matrix collocation method for $N=10$ and $N=20$ described in the Section 2, the fundamental matrix representation of Poisson equation

$$\left\{\mathbf{X}(x)\mathbf{B}^2\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}} + \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{B}^2}\overline{\mathbf{S}}\right\}\mathbf{A} = (x^2 + t^2)e^{xt}$$

and conditions

$$\mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(0)\overline{\mathbf{S}}\mathbf{A} = 1, \quad \mathbf{X}(x)\mathbf{S}\overline{\mathbf{X}}(1)\overline{\mathbf{S}}\mathbf{A} = e^x, \quad \mathbf{X}(0)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} = 1, \quad \mathbf{X}(2)\mathbf{S}\overline{\mathbf{X}}(t)\overline{\mathbf{S}}\mathbf{A} = e^{2t}$$

are computed and then the Pell coefficients are found. Numerical results can be seen in Tables 2-3 and Figs.1-4.

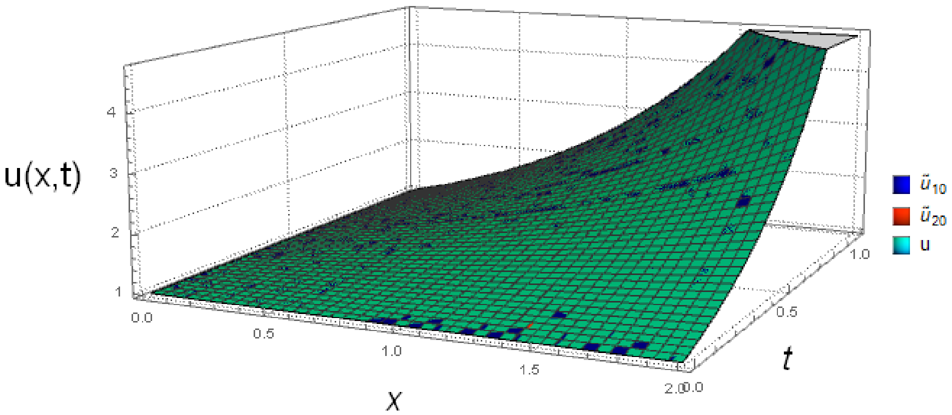

Figure 1. Exact solution and Present Method for $N=10, 20$ in Example 2.

Table 2. Absolute error and upper bound of the error for $N = 10, 20$ in Example 2.

		Absolute Error	
x	**t**	**Present Method for N=10**	**Present Method for N=20**
0.0	0.0	0	0
0.2	0.1	0	0
0.4	0.2	0	0
0.6	0.3	2.22E-16	0
0.8	0.4	9.28E-14	0
1.0	0.5	1.27E-11	0
1.2	0.6	7.18E-10	4.44E-16
1.4	0.7	2.18E-08	4.44E-16
1.6	0.8	4.23E-07	0
1.8	0.9	5.83E-06	0
2.0	1.0	6.13E-05	4.52E-14
$\bar{R}_N$		**7.04E-05**	**5.43E-14**

Table 3. Comparing Monte Carlo Method and Present Method for $N = 10, 20$ in Example 2.

		Relative Error		
x	**t**	**Monte Carlo Method [12]**	**Present Method for N=10**	**Present Method for N=20**
2/3	1/3	1.2E-05	1.24E-15	0
2/3	2/3	1.1E-05	2.22E-12	0
4/3	1/3	1.1E-05	2.22E-12	0
4/3	2/3	1.2E-05	3.04E-09	0

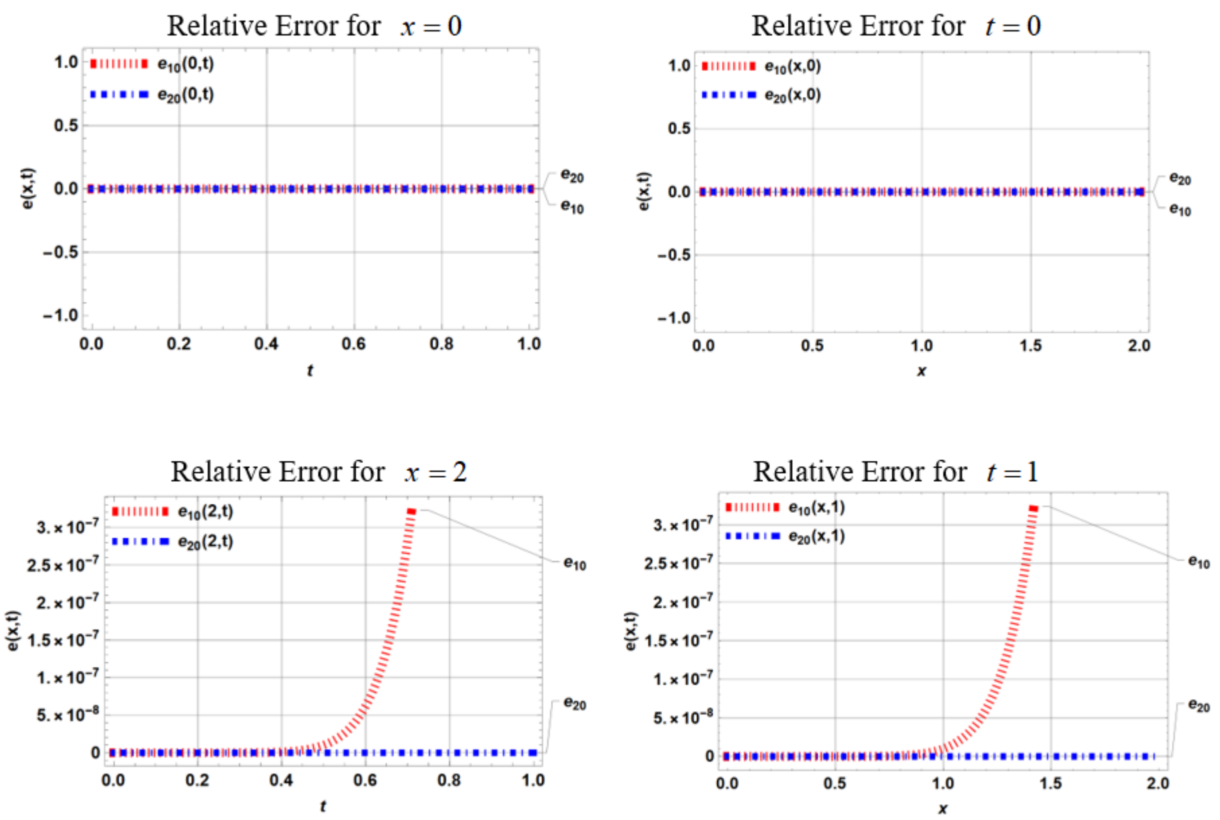

Figure 2. Relative Error for N=10,20 in Example 2.

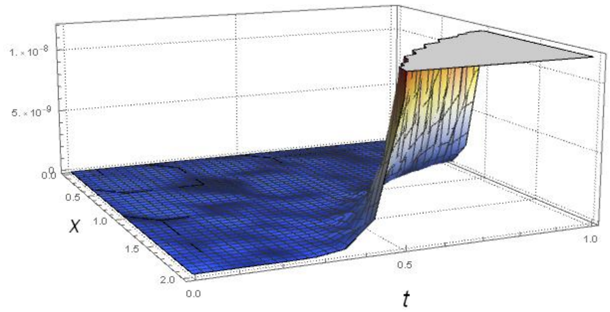

Figure 3. Relative Error for N=10 in Example 2.

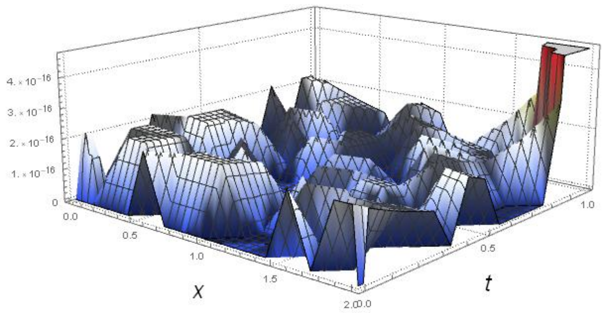

Figure 4. Relative Error for N=20 in Example 2.

Example 3. We consider the two-dimensional convection difusion equation [13]

$$u_{xx}(x,t)+u_{tt}(x,t)+p(x,t)u_x(x,t)+q(x,t)u_t(x,t)=g(x,t)\ ,\ \ 0\le x\le 1,\ 0\le t\le 1.$$

under the Dirichlet conditions

$$u(x,0)=\cos(4x),\ \ u(x,1)=\cos(4x+6),\ u(0,t)=\cos(6t),\ u(1,t)=\cos(6t+4);$$

where $p(x,t)$, $q(x,t)$ and $g(x,t)$ are

$$p(x,t)=10x(x-1)(1-2t),\quad q(x,t)=-10t(t-1)(1-2x)\ ,$$

$$g(x,t)=\text{-}52\cos(4x+6t)+60(-2x+1)t(t-1)\sin(4x+6t)\text{-}40x(x-1)(1-2t)\sin(4x+6t)\,.$$

The exact solution is given by $u(x,t)=\cos(4x+6t)$. Approximate solutions of the problem were calculated by using Pell Matrix Collocation Method (PMCM) for $N=20$, $N=25$ and $N=32$. For numerical result, see Tables 4-5 and Figs. 5-6.

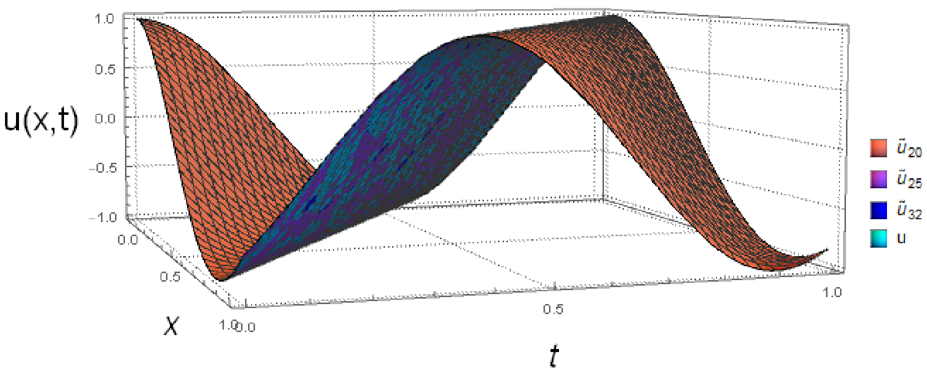

Figure 5. Exact Solution and Present Method for $\mathrm{N}=20,25,32$ **in Example 3.**

Table 4. Comparison of maximum absolute error for N=32 in Example 3.

N	Maximum Absolute Error		
	Present Method	Finite Difference Method [13]	Block Linear Multistep Method [13]
32	1.08E-05	2.65E-03	1.79E-03

Table 5. Absolute error and upper bound of the error for $N = 20, 25, 32$ in Example 3.

x	t	Absolute Error		
		Present Method for N=20	Present Method for N=25	Present Method for N=32
0.1	0.1	8.78E-05	5.60E-07	3.17E-08
0.2	0.2	3.52E-04	2.18E-06	1.29E-07
0.3	0.3	8.04E-04	4.92E-06	2.95E-07
0.4	0.4	1.48E-03	8.93E-06	5.45E-07
0.5	0.5	2.48E-03	1.47E-05	9.13E-07
0.6	0.6	4.06E-03	2.45E-05	1.47E-06
0.7	0.7	6.67E-03	4.58E-05	2.40E-06
0.8	0.8	1.12E-02	1.04E-04	3.98E-06
0.9	0.9	1.63E-02	2.72E-04	6.35E-06
$\bar{R}_N$		**5.16E-01**	**8.00E-03**	**3.04E-04**

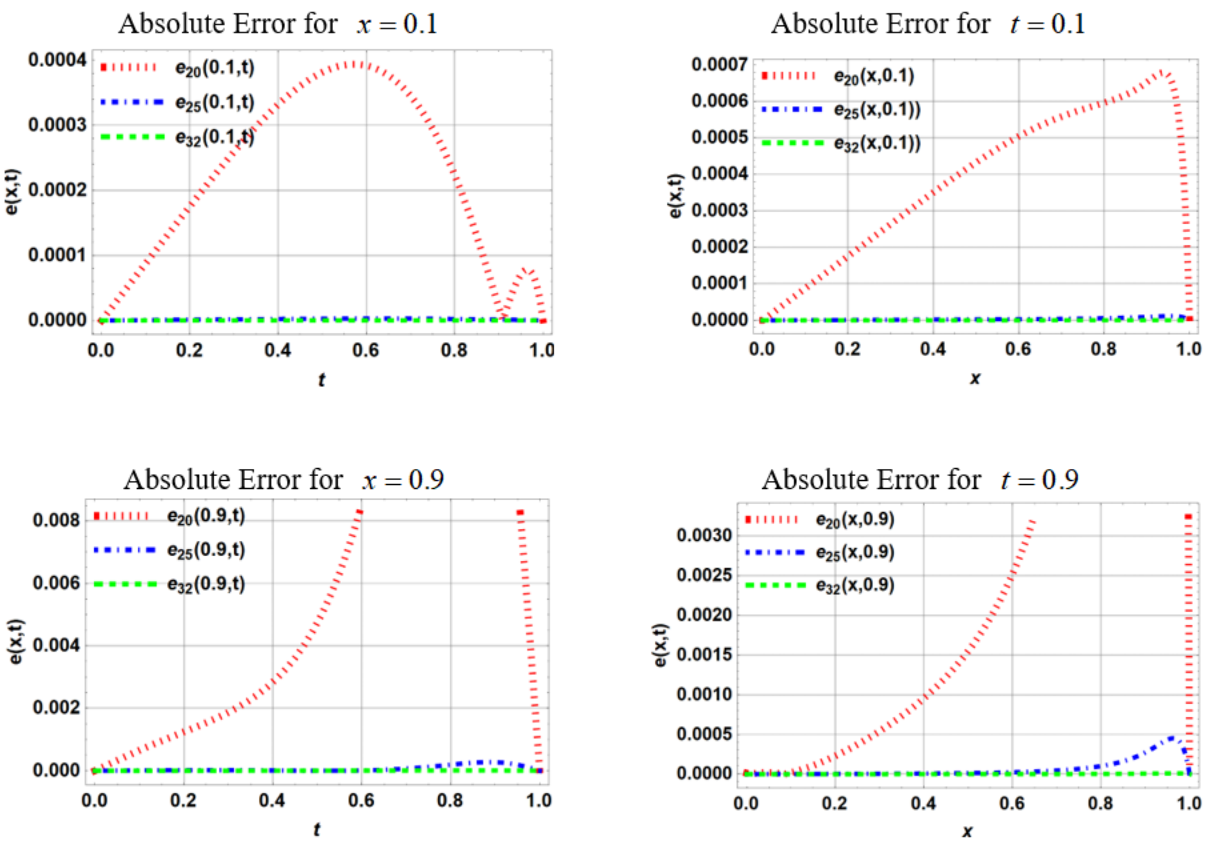

Figure 6. Absolute Error for N=20, 25,32 in Example 3.

5. CONCLUSIONS

A new approach using the Pell polynomials to solve numerically Dirichlet problem for linear partial differential equations with variable coefficients is presented in this study. Linear partial equations with variable coefficients are usually difficult to solve analytically. For this reason, the present method has been proposed for approximate solutions and also polynomial solutions. An error analysis technique based on residual function is also developed for our problems. It is seen that the accuracy improves when N is increased.

Consequently, the present method has been shown to be reliable and effective for solving for linear partial differential equations with variable coefficients. This method can be extended to nonlinear partial differential equations, delay partial differential equations and functional partial integro differential equations but some modifications are required.

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