ORIGINAL PAPER

ON THE MATRIX VERSIONS OF PSEDUO JACOBI POLYNOMIALS

M. HIDAN[1], Z. MOSTEFAOUI[1], M. ABDALLA[1,2]

Manuscript received: 09.05.2019; Accepted paper: 22.08.2019;
Published online: 30.09.2019.

Abstract. *In this paper, Pseudo-Jacobi matrix polynomials are introduced, starting from the hypergeometric matrix representation. Some properties of these matrix polynomials such as generating matrix functions, Rodrigues' formula, matrix recurrence relations and expansions are deduced.*

Keywords: *Pseudo-Jacobi matrix polynomials, Generalized hypergeometric matrix functions, Generating matrix functions.*
MSC 2010: *15A60, 33C05, 33C20.*

1. INTRODUCTION

The classical Jacobi polynomials have been used extensively in mathematical analysis and practical applications (see for instance [1, 2]). In the recent papers, matrix polynomials have significant emergent. Some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials [3]. In [4], these matrix polynomials are orthogonal as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. Recently, Jacobi matrix polynomials have been introduced and studied in [3, 5, 6]. Our main aim in this paper is introduce and study of the Pseudo-Jacobi matrix polynomials. The generating matrix functions, Rodrigues'-type formula, matrix recurrence relations and expansions these new matrix polynomials are obtained. In the due course, we observe that some these results the extended matrix versions for [7].

Throughout this paper, the set of all eigenvalues of matrix A in $\mathbb{C}^{N\times N}$ is denoted by $\sigma(A)$. We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $Re(\lambda)>0$ for all $\lambda\in\sigma(A)$ [6,8, 9].

The reciprocal Gamma function denoted by $\Gamma^{-1}(z)=\dfrac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Furthermore, if

$$A+nI \quad \textit{is invertible for all integer } n\geq 0, \tag{1.1}$$

then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and from [10, 11], one gets

[1] King Khalid University, Faculty of Science, Mathematics Department, 61421 Abha, Saudi Arabia.
E-mail: mhedan@kku.edu.sa; z.mostefaoui26@yahoo.fr; moabdalla@kku.edu.sa.
[2] Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt.

$$\begin{aligned}(A)_n &= A(A+I)...(A+(n-1)I)\\ &= \Gamma(A+nI)\Gamma^{-1}(A); \quad n\geq 1; \quad (A)_0 = I.\end{aligned} \tag{1.2}$$

Definition 1.1 [3, 8, 12] *Let* A , B *and* C *be a matrices in* $\mathbb{C}^{N\times N}$, *such that* $C+nI$ *is invertible for all integer* $n>0$ *the hypergeometric matrix function is given by*

$$F(A,B;C;z) = \sum_{n>0} \frac{(A)_n (B)_n [(C)_n]^{-1}}{n!} z^n. \tag{1.3}$$

Definition 1.2 [3] *Let* A , B, C *and* C' *be a matrices in* $\mathbb{C}^{N\times N}$, *such that* $C+nI$ *and* $C'+nI$ *are invertible for all integer* $n>0$ *the Appell's matrix function* F_4 *is given by*

$$F_4(A,B;C,C';z,w) = \sum_{m,n=0}^{\infty} (A)_{m+n}(B)_{m+n}[(C)_m]^{-1}[(C')_n]^{-1} \frac{z^m w^n}{m!n!}. \tag{1.4}$$

Definition 1.3 [3] *Let* A *and* B *be a matrices in* $\mathbb{C}^{N\times N}$, *such that* $B+nI$ *is invertible for all integer* $n>0$ *the Humbert matrix function* Φ_3 *is given by*

$$\Phi_3(A;B;z,w) = \sum_{m,n=0}^{\infty} (A)_m [(B)_{m+n}]^{-1} \frac{z^m w^n}{m!n!}. \tag{1.5}$$

2. DEFINITION AND SOME GENERATING MATRIX FUNCTIONS

Starting, we introduce Pseudo-Jacobi matrix polynomials $J_n(A,B;z)$ as follows:

$$J_n(A,B;z) = \frac{(A)_{2n}}{n!}[(A)_n]^{-1}.\ {}_2F_1\left(-nI, A+B+(n+1)I; A+nI; \frac{1-z}{2}\right). \tag{2.1}$$

where $z \in \mathbb{C}$ and A, B are matrices in $\mathbb{C}^{N\times N}$ satisfying the following conditions

Re(z)>-1 for all z in $\sigma(A)$, Re(z)>-1 for all z in $\sigma(B)$ and AB=BA.

Note that $\mathrm{J_n(A,B;\ z)}$ is a matrix polynomial of degree n in the form

$$J_n(A,B,1) = \frac{(A)_{2n}}{n!}[(A)_n]^{-1} \ . \tag{2.2}$$

Applying Euler's transformation (cf. [3, 13]) to (2.1), we have

$$J_n(A,B;z) = \frac{(A)_{2n}}{n!}[(A)_n]^{-1}\left(\frac{z+1}{2}\right)^n.\ {}_2F_1\left(-nI, -(B+I); A+nI; \frac{z-1}{z+1}\right) . \tag{2.3}$$

According to (2.1) and (2.3) Pseudo-Jacobi matrix polynomials $J_n(A,B;z)$ have the following finite series:

$$J_n(A,B;\ z)=\sum_{k=0}^{n}\frac{(A)_{2n}(A+B+I)_{n+k}}{k!\,(n-k)!}\,[(A)_{n+k}]^{-1}\,[(A+B+I)_n]^{-1}\left(\frac{z-1}{2}\right)^k, \tag{2.4}$$

And

$$J_n(A,B;\ z)=\sum_{k=0}^{n}\frac{(A)_{2n}\big(-(B+I)\big)_k}{k!\,(n-k)!}\,[(A)_{n+k}]^{-1}\left(\frac{1-z}{2}\right)^k\left(\frac{1+z}{2}\right)^{n-k}. \tag{2.5}$$

Next, the following generating matrix functions for Pseudo-Jacobi matrix polynomials $J_n(A,B;\ z)$:

$$\sum_{n=0}^{\infty}J_n(A,B;\ z)\,t^n=\{1-2(z+1)t\}^{\frac{-1}{2}}\left\{\frac{1}{2}\left(1+\sqrt{1-2(z+1)t}\right)-t\right\}^{B+I}$$

$$\times\left\{\frac{2}{1+\sqrt{1\text{-}2(z+1)t}}\right\}^{2B+A+I} \tag{2.6}$$

$$\sum_{n=0}^{\infty}\left[\left(\frac{A+I}{2}\right)_n\right]^{-1}J_n(A,B;z)t^n$$
$$=exp\,\{(z+1)t\}\,\Phi_3\left\{-(B+I);\frac{A+I}{2};\frac{1}{2}(1-z)t;\frac{1}{4}(z+1)^2t^2\right\}, \tag{2.7}$$

$$\sum_{n=0}^{\infty}J_n(A-nI,B-nI;z)t^n$$
$$=2^{A+B}R^{-1}(1+t+R)^{-(B+I)}(1-t+R)^{I-A}, \tag{2.8}$$

where $R=(1-2zt+t^2)^{\frac{1}{2}}$.

Proof of (2.6): Using (2.5), we find that

$$\sum_{n=0}^{\infty}J_n(A,B;\ z)\,t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(A)_{2n}\big(-(B+I)\big)_k}{k!\,(n-k)!}[(A)_{n+k}]^{-1}\left(\frac{1-z}{2}\right)^k\left(\frac{z+1}{2}\right)^{n-k}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{2^{2n}\left(\frac{A}{2}\right)_n\left(\frac{A+1}{2}\right)_n}{k!\,(n-k)!}\big(-(B+I)\big)_k[(A)_{n+k}]^{-1}\ \left(\frac{1-z}{2}\right)^k\left(\frac{z+1}{2}\right)^{n-k}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}A+kI\right)_n\left(\frac{A+I}{2}+kI\right)_n}{n!\,k!}\big(-(B+I)\big)_k[(A+2kI)_n]^{-1}\{2(z+1)t\}^n\left\{\frac{1}{2}(1-z)t\right\}^k$$

$$= \sum_{k=0}^{\infty} \big(-(B+I)\big)_k \left\{\frac{1}{2}(1-z)t\right\}^k \ {}_2F_1\left(\frac{1}{2}A + kI, \frac{A+I}{2} + kI, A + 2kI; 2(z+1)t\right).$$

Further simplification yields

$$\sum_{n=0}^{\infty} J_n(A,B;\ z)\, t^n =$$

$$\sum_{k=0}^{\infty} \frac{\big(-(B+I)\big)_k}{k!} \left\{\frac{1}{2}(1-z)t\right\}^k \{1-2(z+1)t\}^{\frac{-1}{2}} \left(\frac{2}{1+\sqrt{1-2(z+1)t}}\right)^{A+(2k-1)I}$$

$$= (1-2(z+1)t)^{\frac{-1}{2}} \left\{\tfrac{1}{2}\left(1+\sqrt{1-2(z+1)t}\right) - t\right\}^{B+I} \left(\tfrac{2}{1+\sqrt{1-2(z+1)t}}\right)^{2B+A+I}.$$

This completes the proof of (2.6).

Proof of (2.7): From L.H.S of (2.7) and using the definition (2.5) of Pseudo- Jacobi matrix polynomials, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{2^{2n}\left(\frac{1}{2}A\right)_n \big(-(B+I)\big)_k [(A)_{n+k}]^{-1}}{k!\,(n-k)!} \left(\frac{1-z}{2}\right)^k \left(\frac{z+1}{2}\right)^{n-k} t^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}A + kI\right)_n (-(B+I))_k [(A+2kI)_n]^{-1}}{k!\,n!} \left[\left(\frac{A+1}{2}\right)_n\right]^{-1} \{2(z+1)t\}^n \left\{\frac{1}{2}(1-z)t\right\}^k$$

$$= \sum_{k=0}^{\infty} \frac{(-(B+I))_k}{k!} \left[\left(\frac{A+I}{2}\right)_k\right]^{-1} \left[\frac{1}{2}(1-z)t\right]^k \ {}_1F_1\left(\frac{1}{2}A + kI; A + 2kI; 2(z+1)t\right).$$

Applying Kummer's second matrix formula [13] and (1.5), we have

$$\sum_{n=0}^{\infty} \left[\left(\frac{A+I}{2}\right)_n\right]^{-1} J_n(A,B;z)t^n = exp\{(z+1)t\} \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-(B+I)_k}{k!\,r!} \left[\left(\frac{A+I}{2}\right)_{k+r}\right]^{-1}$$

$$\times \left\{\tfrac{1}{2}(1-z)t\right\}^k \left[\left\{\tfrac{1}{2}(z+1)t\right\}^2\right]^r$$

$$= exp\,\{(z+1)t\}\Phi_3\left\{-(B+I); \frac{A+I}{2}; \frac{1}{2}(1-z)t; \frac{1}{4}(z+1)^2 t^2\right\}$$

This leads us to the required result.

Proof of (2.8): From (2.5) and (1.4), we see that

$$\sum_{n=0}^{\infty} J_n(A-nI,B-nI;z)t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(A)_n\big(-(B+I)\big)_{n-k}(B+2I)_n(1-z)^k(z+1)^{n-k}}{k!\,(n-k)!\,2^n}\,[(A)_k]^{-1}(-t)^n$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(A)_{n+k}(B+2I)_{n+k}}{k!\,n!}[(A)_k]^{-1}[(B+2I)_n]^{-1}\left\{\frac{1}{2}(z-1)t\right\}^k\left\{\frac{1}{2}(z+1)t\right\}^n$$

In fact, we have

$$\sum_{n=0}^{\infty} J_n(A-nI,B-nI;z)t^n$$

$$=F_4\left(B+2I,A;\ A,B+2I,\frac{1}{2}(z-1)t;\frac{1}{2}(z+1)t\right). \tag{2.9}$$

Now, we have the result

$$F_4\left(P,Q,Q,P,\frac{-u}{(1-u)(1-v)},\frac{-v}{(1-u)(1-v)}\right)=(1-uv)^{-1}(1-u)^P(1-v)^Q. \tag{2.10}$$

where P and Q are matrices are matrices in $\mathbb{C}^{N\times N}$ satisfying the following conditions

Re(z)>-1 for all z in $\sigma(P)$, Re(z)>-1 for all z in $\sigma(Q)$ and PQ=QP.

Taking

$$P=2I+B,\ Q=A,\ \frac{-u}{(1-u)(1-v)}=\frac{1}{2}(z-1)t,\ \frac{-v}{(1-u)(1-v)}=\frac{1}{2}(z+1)t\ .$$

Let $R=(1-2zt+t^2)^{\frac{-1}{2}}$ with the following relations

$$u=1-\frac{2}{1+t+R},\qquad v=1-\frac{2}{1-t+R}\quad . \tag{2.11}$$

Using (2.11), we get

$$\frac{-u}{(1-u)(1-v)}=\frac{1}{1-v}\left(1-\frac{1}{1-u}\right)=\frac{1-t+R}{2}\left(1-\frac{1+t+R}{2}\right)$$

$$=\frac{(1-t+R)(1-t-R)}{4}=\frac{(1-t)^2-R^2}{4}=\frac{1}{2}(z-1)t.$$

Similarly

$$\frac{-v}{(1-u)(1-v)}=\frac{(1+t)^2-R^2}{4}=\frac{1}{2}(z+1)t.$$

Also,

$$\frac{1}{1-u}=\frac{1}{2}(1+t+R),\qquad \frac{1}{1-v}=\frac{1}{2}(1-t+R).$$

Therefore, we obtain

$$R=\frac{1}{1-u}+\frac{1}{1-v}-1=\frac{1-uv}{(1-u)(1-v)}\ .$$

Hence,

$$(1-uv)^{-1}(1-u)^{P}(1-v)^{Q}=(1-u)^{P-I}(1-v)^{Q-I}R^{-1}.$$

So that (2.9) and (2.10), imply

$$\sum_{n=0}^{\infty}J_n(A-nI,B-nI;z)t^n=R^{-1}\left(\frac{2}{1+t+R}\right)^{B+I}\left(\frac{2}{1-t+R}\right)^{A-I}.$$

Or

$$\sum_{n=0}^{\infty}J_n(A-nI,B-nI;z)t^n=2^{A+B}R^{-1}(1+t+R)^{-(B+I)}(1-t+R)^{I-A}.$$

where $R=(1-2zt+t^2)^{\frac{1}{2}}$. This given the proof of (2.8).

3. THE RODRIGUES-TYPE FORMULA AND RECURRENCE RELATIONS

Two more basic properties of the Pseudo-Jacobi matrix polynomials $J_n(A,B;z)$ are developed in this section. First, the Rodrigues'-type formula of $J_n(A,B;z)$ is given by

$$J_n(A,B;z)=\frac{(-1)^n(1-z)^{-A+(1-n)I}\,(1+z)^{-B+nI-I}}{2^n\ \ n!}\,\mathbf{D}^n[(1-z)^{A+2n-1}(1+z)^{B+I}],\ \mathbf{D}\equiv\frac{d}{dz}.$$

The proof of this formula is similar to the proof of the Theorem 4.1 of [6].

Second, some matrix recurrence relations for the Pseudo-Jacobi matrix polynomials $J_n(A,B;z)$ are given as follows:

$$(z-1)\begin{bmatrix}(A+B+nI)\,\mathbf{D}J_n(A-nI,B+nI,z)\\+(A+(n-1)I)\,\mathbf{D}J_{n-1}(A-(n-1)I,B+(n-1)I,z)\end{bmatrix}$$

$$=(A+B+nI)\begin{bmatrix}n\,J_n(A-nI,B+nI,z)\\-(A+(n-1)I)\,J_{n-1}(A-nI+I,B+(n-1)I,z)\end{bmatrix},\qquad(3.1)$$

$$(z-1)\mathbf{D}J_n(A-nI,B+nI,z)-n\,J_n(A-nI,B+nI,z)-nJ_n(A-nI,B+nI,z)$$

$$=-(A)_n[(B+A+I)_n]^{-1}\sum_{k=0}^{n-1}(A+B+I)_n[(A)_k]^{-1}$$

$$\times[(A+B+I)J_k(A-Ik,B+Ik,z)+2(z-1)DJ_k(A-Ik,B+Ik,z)] \qquad (3.2)$$

and

$$(z-1)\,\mathbf{D}J_n(A-nI,B+nI,z)-nJ_n(A-nI,B+nI,z)$$
$$=\sum_{k=0}^{n-1}(-1)^{n-k}(A+B+(2k+1)I)(A+B+I)_k[(A)_k]^{-1}\,J_k(A-kI,B+kI,z). \qquad (3.3)$$

Also, we have

$$\boldsymbol{D}^k J_k(A,B,z)=2^{-k}(A+B+(1+n)I)_k\;J_{n-k}(A+2kI,B,z). \qquad (3.4)$$

where $0<k\le n$.

$$(z+1)\mathbf{D}J_n(A,B,z)=n\,J_n(A,B,z)+(B+I)J_{n-1}(A+2I,B-I,z), \qquad (3.5)$$

$$\frac{1}{2}(2I+A+B+2nI)(z-1)J_n(A-I,B,z)$$

$$=(n+1)J_{n+1}(A-I,B+I,z)-(A+2nI)J_n(A,B,z) \qquad (3.6)$$

and

$$(1+z)J_n(A,B+I,z)+(1-z)J_n(A-I,B,z)=2J_n(A,B,z)\,. \qquad (3.7)$$

4. SOME EXPANSIONS OF MATRIX POLYNOMIALS

In this section, we present some expansions for the matrix polynomial $J_n(A,B,z)$, as follows:

$$(1-z)^n=2^n(A)_n\sum_{k=0}^{n}(-nI)_k(A+B+(1+2k)I)\;(A+B+I)_k[(A)_k]^{-1}$$
$$\times\;[(A+B+I)_{n+k+1}]^{-1}J_k(A-kI,B+kI,z). \qquad (4.1)$$

$$J_n(A-nI,B+nI,z)=(A)_n[(A+B+I)_n]^{-1}\sum_{k=0}^{n}\frac{(-1)^{n-k}(B-A)_{n-k}}{(n-k)!}(A+B+I)_{n+k}(I+2A)_k$$

$$\times(2A+(2k+1)I)[(A)_k]^{-1}[(2A+I)_{n+k+1}]^{-1}J_k(A-kI,A+kI,z) \qquad (4.2)$$

$$J_n(A-nI,B+nI,z)=(A)_n[(A+B+I)_n]^{-1}\sum_{k=0}^{n}\frac{(-1)^{n-k}(B)_{n-k}}{(n-k)!}(A+B+I)_{n+k}$$

$$\times(A+1)_k(A+(2k+1)I)[(A)_k]^{-1}[(A+I)_{n+k+1}]^{-1}J_k(A-kI,kI,z). \quad (4.3)$$

REFERENCES

[1] Andrews, G., Askey, R. Roy, R., *Special Functions*, Cambridge University Press, 1999.
[2] Mathai, A.M., *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*, Oxford University Press, Oxford, 1993.
[3] Abdalla, M., *Linear and Multilinear Algebra*, DOI 10.1080/03081087.2018.1497585, 2018.
[4] Jodar, L., Defez, E., Ponsoda, E., *Approximation Theory and its Applications*, **12**(1), 96, 1996.
[5] Cekim B., Altin A. Aktas R., *Filomat*, **27**(4), 713, 2013.
[6] Defez, E., Jodar, L., Law, A., *Computers and Mathematics with Applications*, **48**, 789, 2004.
[7] Khan, M. A., Khan, A. H., Abbas, S. M., *Ain Shams Engineering J*, **4,** 127, 2013.
[8] Bakhet, A., Jiao, Y., He, F.L., *Integral Transforms and Special Functions*, **30**, 138, 2019.
[9] Golub, G., van Loan, C.F., *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 1989.
[10] Cortes, J.C., Jodar, L., Solis, F.J., Ku-Carrillo, R., *Abstract and Applied Analysis*, **2015**, 564287, 2015.
[11] Jodar, L., Cortes, J.C., *Applied Mathematics Letters*, **11**, 89, 1998.
[12] Jodar, L., Cortes, J.C., *Journal of Computational and Applied Mathematics*, **99**, 205, 1998.
[13] Abdalla, M., Bakhet, A., *Iranian Journal of Science and Technology. Transaction A. Science*, **42**, 1465, 2018.