ORIGINAL PAPER

ON GENERALIZED LUCAS AND PELL-LUCAS SEQUENCES

DURSUN TASCI[1], ZISAN KUSAKSIZ TAS[2]

Manuscript received: 22.05.2019; Accepted paper: 26.08.2019;
Published online: 30.09.2019.

Abstract. *In this paper, we define the generaziled Lucas sequences and the Pell-Lucas sequences. Further we give Binet-like formulas, generating function, sums formulas and some important identities which involving the generalized Lucas and Pell-Lucas Numbers.*

Keywords: *Lucas numbers, Pell-Lucas Numbers, generalized Lucas Numbers and generalized Pell-Lucas Numbers.*

1. INTRODUCTION

Fibonacci sequence, Lucas sequence, Pell sequence and Pell-Lucas sequence are most prominent examples of recursive sequences.These sequences arise naturally in many unexpected places and used in equally surprising places like computer algorithms [1-3], some areas of algebra [3, 4], graph theory [5, 6], quasi crystals [1, 7] and many areas of mathematics.

There are various types of generalization of these sequences. For example Tasci [8] defined the generalized order-k Lucas numbers, Er [9] defined the generalized order-k Fibonacci numbers, and Kilic [10] defined the generalized order-k Pell numbers.

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by the recurrence relation,

$$F_{n+1} = F_n + F_{n-1} \ , n \geq 1 \tag{1.1}$$

with initial conditions $F_0 = 0 \, , F_1 = 1$

Well known $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of quadratic equation of the Fibonacci numbers.

The Lucas sequence $\{L_n\}_{n\geq 0}$ is defined by the recurrence relation,

$$L_{n+1} = L_n + L_{n-1} \ , n \geq 1 \tag{1.2}$$

with initial conditions $L_0 = 2 \, , L_1 = 1$.

In [11], Melham gave Binet forms of Fibonacci and Lucas numbers defined as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where α and β are the roots of $x^2 - 2x - 1 = 0$.

The Pell sequence $\{P_n\}_{n\geq 0}$ is defined by

[1] Gazi University, Faculty of Science, Department of Mathematics, Ankara, Turkey. E-mail: dtasci@gazi.edu.tr.
[2] Gazi University, Enstitue of Science, Ankara, Turkey. E-mail: zisan.k.tas@gmail.com.

$$P_{n+1} = 2P_n + P_{n-1} \ , n \geq 1 \tag{1.3}$$

with initial condition $P_0 = 0 \, , P_1 = 1$.

We know that the roots of quadratic equation of Pell sequences are $\gamma = 1 + \sqrt{2}$ and $\theta = 1 - \sqrt{2}$.

Additionaly In [12], Horadam showed that some properties involving Pell numbers.

The Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ is defined by

$$Q_{n+1} = 2Q_n + Q_{n-1} \ , n \geq 1 \tag{1.4}$$

where $Q_0 = 2 \, , Q_1 = 2$.

In [11], Melham gave Binet formulas for the Pell and Pell-Lucas numbers:

$$P_n = \frac{\gamma^n - \theta^n}{\gamma - \theta} \text{ and } Q_n = \gamma^n + \theta^n$$

respectively.

Also, Modified Pell sequence $\{q_n\}_{n\geq 0}$ can be identified by the following recurrence relation

$$q_{n+1} = 2q_n + q_{n-1} \ , n \geq 1 \tag{1.5}$$

where $q_0 = 1 \, , q_1 = 1$

In [13, 14] Horadam the generalized Fibonacci sequence $\{U_n\}$ defined by with initial values $U_0 = 0$, $U_1 = 1$ and the recurrence relations

$$U_{n+1} = p \ U_n + q \ U_{n-1} \ ,$$

and the generalized Lucas sequence $\{V_n\}$ are defined by

$$V_{n+1} = pV_n + qV_{n-1} \ , n \geq 1$$

$V_0 = 2$ and $V_1 = p$, where p and q are nonzero real numbers.

In this study, we define and study the generalized Pell and the generalized Pell-Lucas sequences. We give generating functions and Binet formulas for these sequences. Moreover, we obtain some important identities involving the generalized Pell and Pell-Lucas numbers.

2. THE GENERALIZED LUCAS NUMBERS

Firstly we give the definition of the generalized Lucas sequence.

Definition 2.1. The generalized Lucas sequence is defined by

$$D_{n+1}(a,b) = D_n(a,b) + D_{n-1}(a,b), \ n \geq 1 \tag{2.1}$$

with initial conditions $D_0(a,b) = a + b, D_1(a,b) = b$ where $D_n(a,b)$ is the n^{th}generalized Lucas numbers.

The first few terms of generalized Lucas numbers are

$$a + b \, , b \, , a + 2b \, , a + 3b \, , 2a + 5b \, , 3a + 8b \, , 5a + 13b$$

and so on. We remark that

If $a = 1, b = 1$,then we obtain Lucas numbers $D_n(1,1) = L_n$ and
If $a = -1, b = 1$ we get Fibonacci numbers $D_n(-1,1) = F_n$.

Theorem 2.2. The Binet-like formula for the generalized Lucas sequence is

$$\boldsymbol{D_n(a,b) = \left(\frac{b-\beta(a+b)}{\alpha-\beta}\right)\alpha^n + \left(\frac{\alpha.(a+b)-b}{\alpha-\beta}\right)\beta^n} \tag{2.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$ and a, b are real numbers.

Proof: From the theory of difference equations, we know the general term of the generalized Lucas numbers can be expressed in the following form

$$\boldsymbol{D_n(a,b) = c_1\alpha^n + c_2\beta^n}$$

where $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ are the coefficients. Using the values n = 0, 1 we obtain the following the linear equation system:

$$D_0(a,b) = a + b = c_1 + c_2$$

$$D_1(a,b) \;=\; b \quad = c_1\alpha + c_2\beta$$

The solution of this equation system is,

$$\boldsymbol{c_1 = \frac{b-\beta(a+b)}{\alpha-\beta}}, \quad \boldsymbol{c_2 = \frac{\alpha.(a+b)-b}{\alpha-\beta}}$$

Thus it is easily seen that the formula (2.2).

Theorem 2.3. The generating function of the generalized Lucas number is

$$\sum_{n=0}^{\infty} D_n(a,b)x^n = \frac{a+b-ax}{1-x-x^2} \tag{2.3}$$

Proof: Let $f(x)$ be the generating function of the generalized Lucas sequences. Then we can write

$$f(x) = \sum_{n=0}^{\infty} D_n(a,b)x^n = D_0(a,b) + D_1(a,b)x^1 + D_2(a,b)x^2 + \cdots + D_n(a,b)x^n + \cdots$$

On the other hand, since

$$\begin{aligned} xf(x) &= \sum_{n=0}^{\infty} D_n(a,b)x^{n+1} \\ &= D_0(a,b)x^1 + D_1(a,b)x^2 + D_2(a,b)x^3 + \cdots + D_{n-1}(a,b)x^n + \cdots \end{aligned}$$

and

$$x^2 f(x) = \sum_{n=0}^{\infty} D_n(a,b)x^{n+2}$$

$$= D_0(a,b)x^2 + D_1(a,b)x^3 + D_2(a,b)x^4 + .. + D_{n-2}(a,b)x^n + \cdots$$

we write

$$(1-x-x^2)f(x) = D_0(a,b) + \left[\left(D_1(a,b) - D_0(a,b)\right)\right]x$$

Now using

$$D_0(a,b) = a+b,\ \ D_1(a,b) = b \text{ and } D_n(a,b) - D_{n-1}(a,b) - D_{n-2}(a,b) = 0,$$

we obtain

$$f(x) = \frac{a+b-ax}{1-x-x^2}.$$

So the proof is complete.

Theorem 2.4. The sum of the first n generalized Lucas numbers is expressed as

$$\sum_{i=0}^{n} D_i(a,b) = D_{n+2}(a,b) - b \tag{2.4}$$

Proof: From the definition of the generalized Lucas sequences

$$D_{n+1}(a,b) = D_n(a,b) + D_{n-1}(a,b)$$

For the values of n ,we get the following equalities:

$$D_2(a,b) = D_1(a,b) + D_0(a,b)$$

$$D_3(a,b) = D_2(a,b) + D_1(a,b)$$

$$\vdots$$

$$D_{n+2}(a,b) = D_{n+1}(a,b) + D_n(a,b)$$

If we now sum these equations term by term,

$$\sum_{i=2}^{n+2} D_i(a,b) = \sum_{i=1}^{n+1} D_i(a,b) + \sum_{i=0}^{n} D_i(a,b)$$

then using initial conditions $D_1(a,b) = b$, we obtain

$$\sum_{i=0}^{n} D_i(a,b) = D_{n+2}(a,b) - b.$$

Theorem 2.5 (Catalan Identity):

$$D_{n+r}(a,b)D_{n-r}(a,b) - D_n(a,b)^2 = 5c_1c_2(-1)^{n-r}{F_r}^2, \tag{2.5}$$

where, $n, r \in Z^{+}$, $n > r$, F_r is the $r-th$ Fibonacci number and

$$c_1 = \frac{b-\beta(a+b)}{\alpha-\beta}, \; c_2 = \frac{\alpha(a+b)-b}{\alpha-\beta}$$

Proof: Using the Binet- like formula for the generalized Lucas numbers formula and taking into account that $\alpha\beta = -1$ and $\alpha^r - \beta^r = \sqrt{5}F_r$ we write

$$D_{n+r}(a,b)D_{n-r}(a,b) - D_{n^2}(a,b)$$

$$= (c_1\alpha^{n+r} + c_2\beta^{n+r})(c_1\alpha^{n-r} + c_2\beta^{n-r}) - (c_1\alpha^n + c_2\beta^n)^2$$

$$= c_1c_2\alpha^{n+r}\beta^{n-r} + c_1c_2\alpha^{n-r}\beta^{n+r} - 2c_1c_2(\alpha\beta)^n$$

$$= c_1c_2(\alpha\beta)^n\left[\frac{\alpha^r}{\beta^r} + \frac{\beta^r}{\alpha^r} - 2\right]$$

$$= c_1c_2(\alpha\beta)^{n-r}(\alpha^r - \beta^r)^2$$

$$= 5c_1c_2(-1)^{n-r}{F_r}^2$$

So the theorem is proved.

Corollary 2.6 (Cassini's Identity)

$$D_{n+1}(a,b)D_{n-1}(a,b) - D_n(a,b)^2 = c_1c_2 5(-1)^{n-1} \tag{2.6}$$

Proof: Note that for r = 1, equality (2.3) gives Cassini's identity. Moreover we remark that $F_r = 1$.

Theorem 2.7 (d'Ocagne's Identity): If $m > n$, then

$$D_m(a,b)D_{n+1}(a,b) - D_n(a,b)D_{m+1}(a,b) = -5c_1c_2(-1)^{n+1}F_{m-n} \tag{2.7}$$

where

$$c_1 = \frac{b-\beta(a+b)}{\alpha-\beta}, \; c_2 = \frac{\alpha.(a+b)-b}{\alpha-\beta}$$

Proof: Using Binet- like formula and $\alpha\beta = -1$

$$D_m(a,b)D_{n+1}(a,b) - D_n(a,b)D_{m+1}(a,b)$$

$$= (c_1\alpha^m + c_2\beta^m)\ (c_1\alpha^{n+1} + c_2\beta^{n+1})\ - (c_1\alpha^n + c_2\beta^n)(c_1\alpha^{m+1} + c_2\beta^{m+1})$$

$$= c_1c_2\alpha^m\beta^{n+1} + c_1c_2\alpha^{n+1}\beta^m - c_1c_2\alpha^n\beta^{m+1} - c_1c_2\alpha^{m+1}\beta^n$$

$$= c_1c_2\alpha^n\beta^m(\alpha-\beta) - c_1c_2\alpha^m\beta^n(\alpha-\beta)$$

$$= 5c_1c_2(-1)^{n+1}F_{m-n}$$

Theorem 2.8. For the integer $n \geq 1$

$$D_n(a,b) = aF_{n-1} + bF_{n+1} \tag{2.8}$$

where F_n denotes n^{th} Fibonacci number.

Proof: (By induction on n) If $n = 1$ then the result is obvious. We assume that it is true for n i.e.,

$$D_n(a,b) = aF_{n-1} + bF_{n+1}$$

We'll denote that is true for $n+1$

$$D_{n+1}(a,b) = aF_n + bF_{n+2}$$

By simple calculation using induction's hypothesis we write

$$\begin{aligned} D_{n+1}(a,b) &= D_n(a,b) + D_{n-1}(a,b) \\ &= aF_{n-1} + bF_{n+1} + aF_{n-2} + bF_n \\ &= a(F_{n-1} + F_{n-2}) + b(F_n + F_{n+1}) \\ &= aF_n + bF_{n+2} \end{aligned}$$

which ends the proof.

3. THE GENERALIZED PELL-LUCAS NUMBERS

Definition 3.1. The generalized Pell-Lucas sequences are defined by

$$E_{n+1}(c,d) = 2E_n(c,d) + E_{n-1}(c,d), \qquad n \geq 1 \tag{3.1}$$

with initial conditions

$$E_0(c,d) = c + d, \ E_1(c,d) = 2d,$$

where $E_n(c,d)$ is the $n^{th}$the generalized Pell-Lucas numbers. The first few terms of the generlized Pell-Lucas numbers are

$$c+d\ , 2d\ , c+5d\ , 2c+12d\ , 5c+29d\ , 12c+70d$$

and so on. We remark that:

If $c = 1, d = 1$, then we obtain Pell-Lucas numbers $E_n(1,1) = Q_n$

If $c = -\frac{1}{2}, d = \frac{1}{2}$ we get Pell numbers $E_n\left(-\frac{1}{2},\frac{1}{2}\right) = P_n$

If $c = \frac{1}{2}, d = \frac{1}{2}$ we get Modified Pell-Lucas numbers $E_n\left(\frac{1}{2},\frac{1}{2}\right) = q_n$

We give Binet-like formula of the generalized Pell-Lucas sequences

Theorem 3.2. (Binet-like formula for the generalized Pell-Lucas sequences)

$$E_n(c,d) = \left(\frac{2d-(c+d)\theta}{\gamma-\theta}\right)\gamma^n + \left(\frac{(c+d)\gamma-2d}{\gamma-\theta}\right)\theta^n \tag{3.2}$$

where $\gamma = 1+\sqrt{2},\ \theta = 1-\sqrt{2}$ are roots of the equation $x^2-2x-1=0$.

Proof: We know that the Binet- like formula of the generalized Pell-Lucas numbers

$$E_n(c,d) = a_1\gamma^n + a_2\theta^n$$

On the other hand using the equality and the values $n = 0{,}1$ we obtain the following the linear system:

$$E_0(c,d) = c+d = a_1 + a_2$$

$$E_1(c,d) = 2d = a_1\gamma + a_2\theta$$

The solution of this equation system is,

$$a_1 = \frac{2d-(c+d)\theta}{\gamma-\theta},\ \ a_2 = \frac{(c+d)\gamma-2d}{\gamma-\theta}$$

Thus it is easily seen that the formula (3.2).

Theorem 3.3. The generating function of the generalized Pell-Lucas sequence is

$$\sum_{n=0}^{\infty} E_n(c,d)x^n = \frac{c+d-2cx}{1-2x-x^2} \tag{3.3}$$

Proof: Let

$$g(x) = \sum_{n=0}^{\infty} E_n(c,d)x^n$$

be generating function of the generalized Pell-Lucas sequences. Then we have

$$\begin{aligned} 2xg(x) &= \sum_{n=0}^{\infty} 2E_n(c,d)x^{n+1} \\ &= 2E_0(c,d)x^1 + 2E_1(c,d)x^2 + 2E_2(c,d)x^3 + \cdots + 2E_{n-1}(c,d)x^n + \cdots \end{aligned}$$

and

$$\begin{aligned} x^2g(x) &= \sum_{n=0}^{\infty} E_n(c,d)x^{n+2} \\ &= E_0(c,d)x^2 + E_1(c,d)x^3 + E_2(c,d)x^4 + \cdots + E_{n-2}(c,d)x^n + \cdots \end{aligned}$$

we write:

$$(1-2x-x^2)g(x) = E_0(c,d) + \left[\left(E_1(c,d) - 2E_0(c,d)\right)\right]x$$

Now using

$$E_0(c,d) = c + d \, , E_1(c,d) = 2d \text{ and } E_n(c,d) - 2E_{n-1}(c,d) - E_{n-2}(c,d) = 0$$

we obtain

$$g(x) = \frac{c+d-2cx}{1-2x-x^2}.$$

Theorem 3.4. The sum of the first n generalized Pell-Lucas numbers is the following:

$$\sum_{i=0}^{n} E_i(c,d) = \frac{d - c + E_{n+2}(c,d) - E_{n+1}(c,d)}{2} \tag{3.4}$$

Proof: From the definition of the generalized Pell- Lucas sequences

$$E_{n+1}(c,d) = 2E_n(c,d) + E_{n-1}(c,d)$$

For the values of n, we write the following:

$$E_2(c,d) = 2E_1(c,d) + E_0(c,d)$$

$$E_3(c,d) = 2E_2(c,d) + E_1(c,d)$$

$$\vdots$$

$$E_{n+2}(c,d) = 2E_{n+1}(c,d) + E_n(c,d)$$

If we now add these equations term by term and using initial conditions we obtain

$$\sum_{i=0}^{n} E_i(c,d) = \frac{c-d+E_{n+2}(c,d)-E_{n+1}(c,d)}{2}.$$

Theorem 3.5. (Catalan's identity for generalized Pell- Lucas - sequences)

$$E_{n+r}(c,d)E_{n-r}(c,d) - E_n(c,d)^2 = a_1 a_2 (-1)^{n-r} Q_r^2 \tag{3.5}$$

where , $r \in Z^+, n > r$, Q_r is the $r-th$ Pell-Lucas number and

$$a_1 = \frac{2d - (c+d)\theta}{\gamma - \theta}, \ \ a_2 = \frac{(c+d)\gamma - 2d}{\gamma - \theta}.$$

Proof: Considering the Binet- like formula for the generalized Pell-Lucas sequences we have

$$E_{n+r}(c,d)E_{n-r}(c,d) - E_{n^2}(c,d)$$

$$= (a_1\gamma^{n+r} + a_2\theta^{n+r})(a_1\gamma^{n-r} + a_2\theta^{n-r}) - (a_1\gamma^n + a_2\theta^n)^2$$

$$= a_1a_2\gamma^{n+r}\theta^{n-r} + a_1a_2\gamma^{n-r}\theta^{n+r} - 2a_1a_2(\gamma\theta)^n$$

$$= a_1a_2(\gamma\theta)^n\left[\frac{\gamma^r}{\theta^r} + \frac{\theta^r}{\gamma^r} - 2\right]$$

$$= a_1a_2(\gamma\theta)^{n-r}(\gamma^r - \theta^r)^2$$

$$= a_1a_2(-1)^{n-r}(Q_r^2 - 4(-1)^r)$$

as required.

Corollary 3.6. (Cassini's Identity)

$$E_{n+1}(c,d)E_{n-1}(c,d) - E_n(c,d)^2 = 8a_1a_2(-1)^{n-1} \tag{3.6}$$

Proof: Substituting $r = 1$ in Catalan's identity the prof is easily seen. Moreover we remark that $Q_r = 2$

Theorem 3.7. (d'Ocagne's Identity) If $m > n$, then

$$E_m(c,d)E_{n+1}(c,d) - E_n(c,d)E_{m+r}(c,d) = -2\sqrt{2}a_1a_2(-1)^{n+1}Q_{m-n} \tag{3.7}$$

where

$$a_1 = \frac{2d - (c+d)\theta}{\gamma - \theta}, \quad a_2 = \frac{(c+d)\gamma - 2d}{\gamma - \theta}$$

Proof: Using Binet- like formula and $\gamma\theta = -1$

$$E_m(c,d)E_{n+1}(c,d) - E_n(c,d)E_{m+1}(c,d)$$

$$= (a_1\gamma^m + a_2\theta^m)(a_1\gamma^{n+1} + a_2\theta^{n+1}) - (a_1\gamma^n + a_2\theta^n)(a_1\gamma^{m+1} + a_2\theta^{m+1})$$

$$= a_1a_2\gamma^m\theta^{n+1} + a_1a_2\gamma^{n+1}\theta^m - a_1a_2\gamma^n\theta^{m+1} - a_1a_2\gamma^{m+1}\theta^n$$

$$= -2\sqrt{2}a_1a_2(\gamma\theta)^n(\gamma^{m-n} - \theta^{m-n})$$

$$= 2\sqrt{2}a_1a_2(-1)^{n+1}\sqrt{Q_{m-n} - 4(-1)^{m-n}}$$

Theorem 3.8. For the integer $n \geq 1$

$$E_n(c,d) = cP_{n-1} + dP_{n+1} \tag{3.8}$$

where P_n denotes n^{th} Pell number.

Proof: (By induction on n) If $n = 1$ then the result is obvious. We assume that it is true for n i.e.,

$$E_n(c,d) = cP_{n-1} + dP_{n+1}$$

We'll denote that is true for $n + 1$,

$$E_{n+1}(c,d) = cP_n + dP_{n+2}$$

By simple calculation using induction's hypothesis we write

$$\begin{aligned} E_{n+1}(c,d) &= 2E_n(c,d) + E_{n-1}(c,d) \\ &= 2(cP_{n-1} + dP_{n+1}) + (cP_{n-2} + dP_n) \\ &= c(2P_{n-1} + P_{n-2}) + d(2P_{n+1} + P_n) \\ &= cP_n + dP_{n+2} \end{aligned}$$

which ends the proof.

4. CONCLUSION

In this study we define two new sequences named generalized Lucas sequence and generalized Pell-Lucas sequence. In the next article, we will study the polynomial of these sequences.

REFERENCES

[1] Stojmenovic, I., *IEEE Trans. Educ.*, **43**, 273, 2000.
[2] Atkins, J., Geist, R., *College Math. J.*, **18**, 328, 1987.
[3] de Souza, J., Curado, E.M.F., Rego-Monteiro, M.A., *J. Phys. A: Math. Gen.*, **39**, 10415, 2006.
[4] Feingold, A.J., *Proc. Amer. Math. Soc.*, **80**, 379, 1980.
[5] Fredman, M.L., Tarjan, R.E., *J. ACM*, **34**, 596, 1987.
[6] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience Publication, New York, 2001.
[7] Chebotarev, P., *Discrete Appl. Math.*, **156**, 813, 2008.
[8] Tasci, D., Kilic, E., *Appl. Math. Comput.*, **155**(3), 637, 2004.
[9] Er, M.C., *Fibonacci Quart.*, **22**(3), 204, 1984.
[10] Kilic, E., Tasci, D., *Taiwanese J. Math.*, **10**(6), 1661, 2006.
[11] Melham, R., *Portugaliae Math.*, **56**(3), 309, 1999.
[12] Horadam, A.F., *Fibonacci Quart.*, **9**(3), 245, 1971.
[13] Horadam, A.F., *Amer. Math. Monthly*, **68**, 455, 1961.
[14] Horadam, A.F., *Amer. Math. Monthly*, **70**, 289, 1963.