ORIGINAL PAPER

THE DISTANCE MATRIX AND THE DISTANCE ENERGY OF THE POWER GRAPHS OF C_n AND D_{2n}

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Abstract. *In this study, the distance matrix and the distance energy of power graphs on cyclic groups and dihedral groups are considered. Furthermore, some bounds for the largerst eigenvalue of the distance matrix and the distance energy are found. Also, some results are obtained by using these bounds.*

Keywords: *Power graph, distance matrix, distance energy, eigenvalue bound, energy bound.*

1. INTRODUCTION

An undirected power graph $P(G)$ of a group G is an undirected graph whose vertex set is G and two distinct vertices x and y are adjacent if and only if $x^m = y$ or $y^m = x$ for some positive integer m. At the first time, the concept of power graphs was introduced by Kelarev and Quinn but they studied only directed power graphs for semigroups in [1]. A directed power graph of a semigroup S is a directed graph with vertex set S and for $x, y \in S$ there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m [1-3]. Then motivated by concept of directed power graphs, Chakrabarty et al. introduced the undirected power graph [4]. In the same paper, it was shown that the undirected power graph $P(G)$ of any finite group G is complete if and only if G is a cyclic group of order 1 or p^m for some prime number p and positive integer m. Indeed, $P(G)$ is always connected. Throughout this paper, we use the brief term power graph to refer to an undirected power graph.

In this paper, we will examine the undirected power graphs of cyclic groups C_n of order n and dihedral groups D_{2n} of order $2n$. For this aim, we now redefine some graph theoretic concepts for our particular power graphs.

The distance between vertices v_i and v_j of a power graph $P(C_n)$, denoted by $d(v_i, v_j)$, is defined to be the length of the shortest path from v_i to v_j. Let V_1 be the set of the identity and generators of C_n, *so* $|V_1| = 1 + \phi(n) = \ell$ (say), where $\phi(n)$ is Euler's ϕ function. Also, let $V_2 = C_n - V_1$. Then the distance matrix $D\big(P(C_n)\big)$ of the power graph $P(C_n)$ is of the form

$$D(P(C_n)) = \begin{pmatrix} J_{\ell\times\ell} - \mathrm{I}_{\ell\times\ell} & J_{\ell\times(\mathrm{n}-\ell)} \\ J_{(\mathrm{n}-\ell)\times\ell} & D(P(V_2))_{(\mathrm{n}-\ell)\times(\mathrm{n}-\ell)} \end{pmatrix}$$

where I is the identity matrix, J the all-ones matrix, and $D(P(V_2)) = (d_{ij})$ is the distance matrix of the power graph induced by the vertex set V_2, i.e.,

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$$d_{ij} = \begin{cases} d(v_i, v_j) & \text{if } v_i \neq v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of the distance matrix of $P(C_n)$ are denoted by

$$\mu_1(P(C_n)), \mu_2(P(C_n)), \dots, \mu_n(P(C_n)).$$

Since $D\big(P(C_n)\big)$ is a real symmetric matrix, its eigenvalues are real and can be ordered as

$$\mu_1(P(C_n)) \geq \mu_2(P(C_n)) \geq \cdots \geq \mu_n(P(C_n)).$$

For each positive integer $n \geq 3$, the dihedral group $D_{2n} =< a, b >$ is a non-commutative group of order $2n$ whose generators a and b satisfy $o(a) = n, o(b) = 2$, and $ba = a^{-1}b = a^{\mathrm{n-1}}b$. Since $o(a) = n$, the cyclic group $C_n =< a >$ is a subgroup of D_{2n} of order n. So $P(C_n)$ is a connected subgraph of $P(D_{2n})$. The power graph $P(D_{2n})$, can be considered as a copy of $P(C_n)$ and n copies of the complete graph K_2 which share the identity. Moreover, the distance matrix $D\big(P(D_{2n})\big)$ of $P(D_{2n})$ is of the form

$$D\big(P(D_{2n})\big) = \begin{pmatrix} D\big(P(C_n)\big) & M_n \\ M_n^T & 2J_{\mathrm{n\times n}} - 2\mathrm{I}_{\mathrm{n\times n}} \end{pmatrix},$$

where

$$M_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ 2 & 2 & \dots & 2 \end{pmatrix}.$$

The eigenvalues of the distance matrix of $P(D_{2n})$ are denoted by

$$\mu_1\big(P(D_{2n})\big), \mu_2\big(P(D_{2n})\big), \dots, \mu_{2n}\big(P(D_{2n})\big).$$

Since $D\big(P(D_{2n})\big)$ is a real symmetric matrix, its eigenvalues are real and can be ordered as

$$\mu_1\big(P(D_{2n})\big) \geq \mu_2\big(P(D_{2n})\big) \geq \cdots \geq \mu_{2n}\big(P(D_{2n})\big).$$

Analog to the definition of the graph energy [5], we can naturally define the distance energy $DE(P(G))$ of the power graph $P(G)$ on a group G as the sum of the absolute values of its distance eigenvalues $\mu_1\big(P(G)\big), \mu_2\big(P(G)\big), \dots, \mu_n\big(P(G)\big)$ i.e.,

$$DE\big(P(G)\big) = \sum_{i=1}^{n} \big|\mu_i\big(P(G)\big)\big|.$$

Since calculating such graph invariants is a hard work, the bounding problem for the largest eigenvalue of the distance matrix and the distance energy of a graph have received much interest. Since the fundamental paper of Ruzieh and Powers [6] in 1990, the bounding problem for the largest eigenvalue of the distance matrix of a graph has appeared frequently in many papers [5, 7-10]. Furthermore, the concept of distance energy for graphs introduced by Indual, Gutman and Vijayakumar [5]. Then, many results on lower and upper bounds for distance energy have been obtained in [9, 11-13].

In this paper, motivated by the definition of the adjacency matrix of a power graph in [17], the distance matrix of the power graph of a finite group are defined. Moreover, its eigenvalues and the sum of the absolute values of its eigenvalues are called distance eigenvalues and the distance energy of a power graph, respectively. In the following parts of this study, sharp upper and sharp lower bounds for the largest distance eigenvalue and the distance energy for the cyclic group C_n and the dihedral group D_{2n} are obtained.

2. BOUNDS FOR THE LARGEST LAPLACIAN EIGENVALUES OF DISTANCE MATRICES OF $P(C_n)$ AND $P(D_{2n})$

Theorem 2.1. Let $P(C_n)$ be the power graph of C_n of order n with $n \geq 3$. Then

$$\mu_1\big(P(C_n)\big) \geq n-1 \tag{1}$$

and

$$\mu_1\big(P(C_n)\big) \leq \frac{2n-\ell-3+\sqrt{\Big(\big(2n-(2\ell+1)\big)^2+\ell(\ell+2)\Big)}}{2}, \tag{2}$$

where $\ell = \phi(n)+1$. Moreover equality holds in (1) and (2) if and only if $n = p^m$, for some prime number p and positive integer m.

Proof. Let V_1 be the set of identity and all generators of C_n and $V_2 = C_n - V_1$. For the sake of simplicity, we should label the vertices of V_1 as $e = v_1, v_2, \dots, v_\ell$ and V_2 as $v_{\ell+1}, v_{\ell+2}, \dots, v_n$. Then $|V_1| = \ell$ and $|V_2| = n-\ell$. Now any row sum of each block $J_{\ell\times\ell} - I_{\ell\times\ell}, J_{\ell\times(n-\ell)}$ and $J_{(n-\ell)\times\ell}$ are $\ell-1$, $n-\ell$ and ℓ, respectively.

Let $x = (x_1, x_2, \dots, x_n)^T$ be a Perron eigenvector of $D\big(P(C_n)\big)$ corresponding to the largest eigenvalue $\mu_1\big(P(C_n)\big)$, and let

$$x_i = \min_{v_k \epsilon V_1} x_k \quad \textit{and} \quad x_j = \min_{v_k \epsilon V_2} x_k. \tag{3}$$

From the eigenvalue equation, we have

$$D\big(P(C_n)\big)x = \mu_1\big(P(C_n)\big)x. \tag{4}$$

From (3) and the i −th equation of (4), we get

$$\begin{aligned}\mu_1\big(P(C_n)\big)x_i &= \sum d(v_i, v_k)x_k + \sum d(v_i, v_k)x_k \\ &\geq \sum d(v_i, v_k)x_i + \sum d(v_i, v_k)x_j\end{aligned}$$

$$= (\ell - 1)x_i + (n - \ell)x_j,$$

i.e.,

$$\big(\mu_1\big(P(C_n)\big) - \ell + 1\big)x_i \geq (n - \ell)x_j. \tag{5}$$

Similarly, from (3), the j −th equation of (4) and the fact that

$$\min_{\substack{v_j, v_k \epsilon V_2 \\ v_j, \neq v_k}} \{d\big(v_j, v_k\big)\} x_j \geq 1,$$

we have

$$\mu_1\big(P(C_n)\big)x_j \geq \sum d\big(v_j, v_k\big)x_i + \sum d\big(v_j, v_k\big)x_j$$

$$= \ell x_i + (n - \ell - 1) \min_{\substack{v_j, v_k \epsilon V_2 \\ v_j, \neq v_k}} \{d\big(v_j, v_k\big)\} x_j$$

$$\geq \ell x_i + (n - \ell - 1)x_j,$$

i.e.,

$$\Big(\mu_1\big(P(C_n)\big) - (n - \ell - 1)\Big) x_j \geq \ell x_i. \tag{6}$$

From (5) and (6), we have

$$\big(\mu_1\big(P(C_n)\big) - \ell + 1\big) \Big(\mu_1\big(P(C_n)\big) - (n - \ell - 1)\Big) x_i x_j - \ell(n - \ell)x_i x_j \geq 0,$$

also since x_i and x_j are positive, we obtain

$$\big(\mu_1\big(P(C_n)\big) - \ell + 1\big) \Big(\mu_1\big(P(C_n)\big) - (n - \ell - 1)\Big) - \ell(n - \ell) \geq 0,$$

i.e.,

$$\mu_1\big(P(C_n)\big) \geq n - 1.$$

Let

$$x_r = \max_{v_k \epsilon V_1} x_k \quad and \quad x_s = \max_{v_k \epsilon V_2} x_k.$$

By a similar argument, using the fact that $\max_{\substack{v_s, v_k \epsilon V_2 \\ v_s, \neq v_k}} \{d(v_s, v_k)\} \leq 2$, we can show that

$$\big(\mu_1\big(P(C_n)\big) - \ell + 1\big)x_r \leq (n - \ell)x_s \tag{7}$$

and

$$\left(\mu_1(P(C_n)) - 2(n-\ell-1)\right)x_s \le \ell x_r. \tag{8}$$

Since x_r and x_s are positive and from (7) and (8), we have

$$\left(\mu_1(P(C_n)) - \ell + 1\right)\left(\mu_1(P(C_n)) - (n-\ell-1)\right) - \ell(n-\ell) \le 0.$$

Thus

$$\mu_1(P(C_n)) \le \frac{2n-\ell-3+\sqrt{\left(\left(2n-(2\ell+1)\right)^2 + \ell(\ell+2)\right)}}{2}.$$

Equality holds in the lower and upper bounds for $\mu_1(P(C_n))$ if and only if

$$\min_{\substack{v_j,v_k \epsilon V_2\\ v_j, \neq v_k}}\{d(v_j,v_k)\} = \max_{\substack{v_j,v_k \epsilon V_2\\ v_j, \neq v_k}}\{d(v_j,v_k)\}.$$

Then $P(C_n)$ is a complete graph and thus C_n is a cyclic group of order p^m for some prime number p and positive integer m. Conversely, suppose that$n = p^m$ for any prime number p and positive integer m. Then $P(C_n)$ is complete and thus

$$\min_{\substack{v_j,v_k \epsilon V_2\\ v_j, \neq v_k}}\{d(v_j,v_k)\} = \max_{\substack{v_j,v_k \epsilon V_2\\ v_j, \neq v_k}}\{d(v_j,v_k)\}.$$

Hence, the theorem is proved.

Theorem 2.2. Let $P(D_{2n})$ and $P(C_n)$ be the power graphs of D_{2n} and C_n respectively and $n \ge 3$. Then

$$\mu_1(P(C_n)) \le \mu_1(P(D_{2n})) \le \mu_1(P(C_n)) + 2\sqrt{(n-1)}\left(\sqrt{(n-1)} + \sqrt{n}\right) + \sqrt{n}.$$

Proof. It is clear that $D(P(C_n))$ is a principal submatrix of $D(P(D_{2n}))$, and from Cauchy's Interlace Theorem in [5-6], we obtain

$$\mu_1(P(C_n)) \le \mu_1(P(D_{2n})).$$

From the definition of the distance matrix of the power graph $P(D_{2n})$, we have

$$D(P(D_{2n})) = \begin{pmatrix} D(P(C_n)) & M_n \\ M_n^T & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}$$

$$= \begin{pmatrix} D(P(C_n)) & 0_n \\ 0_n & 0_n \end{pmatrix} + \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix} + \begin{pmatrix} 0_n & M_n \\ M_n^T & 0_n \end{pmatrix}$$

$$= \begin{pmatrix} D(P(C_n)) & 0_n \\ 0_n & 0_n \end{pmatrix} + \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J - \mathrm{I})_{\mathrm{n \times n}} \end{pmatrix} + \begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} + \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix},$$

where

$$K_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \quad and \quad L_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \ldots & 2 \end{pmatrix}.$$

Using the relation between the principal minors of a matrix and the coefficients of its characteristic polynomial one can obtain that the characteristic polynomial of

$$\begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}$$

is $p(x) = x^{\mathrm{n}} - 4n(n-1)x^{\mathrm{n}-2}$, and hence,

$$\mu_1 \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix} = 2\sqrt{(n(n-1))}. \tag{9}$$

Also, one can show that

$$\mu_1 \begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} = \sqrt{n}. \tag{10}$$

see p. 64 in [13], we have

$$\mu_1 \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J - \mathrm{I})_{\mathrm{n \times n}} \end{pmatrix} = \mu_1(2(J - I))$$

$$= 2(\mu_1(J) - \mu_1(I)) = 2(n-1). \tag{11}$$

Since

$$\begin{pmatrix} D(P(C_n)) & 0_n \\ 0_n & 0_n \end{pmatrix}, \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J - \mathrm{I})_{\mathrm{n \times n}} \end{pmatrix}, \begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} \text{ and } \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}$$

are symmetric matrices, we obtain

$$\mu_1(P(D_{2n})) = \mu_1 \left[\begin{pmatrix} D(P(C_n)) & 0_n \\ 0_n & 0_n \end{pmatrix} + \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J - \mathrm{I})_{\mathrm{n \times n}} \end{pmatrix} + \begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} + \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix} \right]$$

$$\leq \mu_1 \begin{pmatrix} D(P(C_n)) & 0_n \\ 0_n & 0_n \end{pmatrix} + \mu_1 \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J - \mathrm{I})_{\mathrm{n \times n}} \end{pmatrix} + \mu_1 \begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} + \mu_1 \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}$$

By (9), (10) and (11), we have

$$\mu_1\big(P(D_{2n})\big) = \mu_1\big(P(C_n)\big) + 2(n-1) + \sqrt{n} + 2\sqrt{\big(n(n-1)\big)}$$

$$= \mu_1\big(P(C_n)\big) + 2\sqrt{(n-1)}\left(\sqrt{(n-1)} + \sqrt{n}\right) + \sqrt{n}.$$

The proof is complete.

3. SOME UPPER AND LOWER BOUNDS FOR THE DISTANCE ENERGY OF $P(C_n)$ AND $P(D_{2n})$

Let C_n be a cyclic group of order n with $n \geq 3$ and $P(C_n)$ be its power graph. We denote by $D\big(P(C_n)\big)$ the distance matrix of $P(C_n)$ and by $\mu_1\big(P(C_n)\big), \mu_2\big(P(C_n)\big), \dots, \mu_n\big(P(C_n)\big)$ its eigenvalues in decreasing order. Moreover, naturally define the distance energy $DE\big(P(C_n)\big)$ of the power graph $P(C_n)$ as the sum of the absolute values of its distance eigenvalues, i.e.,

$$DE\big(P(C_n)\big) = \sum_{i=1}^{n} \big|\mu_i\big(P(C_n)\big)\big|.$$

Lemma 3.1. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$. Then

$$\sum_{i=1}^{n} \mu_i\big(P(C_n)\big) = 0$$

and

$$\sum_{i=1}^{n} \mu_i^2\big(P(C_n)\big) = \ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i < j \leq n} d\big(v_i, v_j\big)^2.$$

Proof. From the definition of the trace of a matrix, we have

$$\sum_{i=1}^{n} \mu_i\big(P(C_n)\big) = tr\big[D\big(P(C_n)\big)\big] = 0.$$

We now consider the matrix $D\big(P(C_n)\big)^2$.

$$\sum_{i=1}^{n} \mu_i^2\big(P(C_n)\big) = tr\left[D\big(P(C_n)\big)^2\right]$$

$$= \ell(2n-\ell-1) + tr[D(P(V_2))^2].$$

The ii −th entry of $D(P(V_2))^2$ is $\sum_{\substack{j=\ell+1 \\ j\neq i}}^{n} d(v_i, v_j)^2$. Thus

$$\sum_{i=1}^{n} \mu_i^2(P(C_n)) = \ell(2n-\ell-1) + tr\left[D(P(V^2))^2\right]$$

$$= \ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d(v_i, v_j)^2.$$

This completes the proof.

Theorem 3.2. Let $DE(P(C_n))$ be the distance energy of the power graph $P(C_n)$. Then

$$DE(P(C_n)) \geq \sqrt{\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d(v_i, v_j)^2\right)}$$

and

$$DE(P(C_n)) \leq \sqrt{\left(n\ell(2n-\ell-1) + 2n\sum_{\ell+1\leq i<j\leq n} d(v_i, v_j)^2\right)}.$$

Proof. By Lemma 3.1 and the Cauchy-Schwarz inequality, we obtain

$$DE(P(C_n))^2 = \left(\sum_{i=1}^{n} |\mu_i(P(C_n))|\right)^2$$

$$\leq n\sum_{i=1}^{n} \mu_i^2(P(C_n))$$

$$= n\ell(2n-\ell-1) + 2n\sum_{\ell+1\leq i<j\leq n} d(v_i, v_j)^2. \quad (12)$$

For the proof of the first inequality, we have

$$DE(P(C_n))^2 = \left(\sum_{i=1}^{n} |\mu_i(P(C_n))|\right)^2$$

$$\geq \sum_{i=1}^{n} \mu_i^2(P(C_n))$$

$$= \ell(2n - \ell - 1) + 2\sum_{\ell+1 \leq i < j \leq n} d(v_i, v_j)^2. \quad (13)$$

Using (12) and (13), we obtain the required result.

Corollary 3.3. Let C_n be a cyclic group of order p^m, for some prime number p and positive integer m. Then

$$\sqrt{n(n-1)} \leq DE(P(C_n))$$

and

$$DE(P(C_n)) \leq n\sqrt{n-1}.$$

Proof. We know that if C_n is a cyclic group of order p^m for some prime number p and positive integer m then $P(C_n)$ is a complete graph, and hence

$$tr[D(P(V_2))^2] = (n-\ell)(n-\ell-1). \quad (14)$$

By Theorem 3.2 and the equality (14), we have

$$\sqrt{n(n-1)} \leq DE(P(C_n))$$

and

$$DE(P(C_n)) \leq n\sqrt{n-1}.$$

Therefore, the proof is complete.

Theorem 3.4. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$. Then

$$DE(P(C_n)) \leq \mu_1 + \sqrt{(n-1)\left(\ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i < j \leq n} d(v_i, v_j)^2 - \mu_1^2\right)}.$$

Proof. By the Cauchy-Schwarz inequality and Lemma 3.1, we obtain

$$\left(DE(P(C_n)) - \mu^1(P(C_n))\right)^2 = \left(\sum_{i=2}^{n} |\mu_i(P(C_n))|\right)^2$$

$$\leq (n-1)\left(\sum_{i=2}^{n} \mu_i^2\big(P(C_n)\big) - \mu_1^2\big(P(C_n)\big)\right)$$

$$= (n-1)\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 - \mu_1^2\big(P(C_n)\big)\right)$$

and thus

$$DE\big(P(C_n)\big) \leq \mu_1 + \sqrt{(n-1)\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 - \mu_1^2\right)}$$

The proof is complete.

Corollary 3.5. Let C_n be a cyclic group of order p^m, for some prime number *p* and positive integer *m*. Then

$$DE\big(P(C_n)\big) \leq 2(\mathrm{n}-1).$$

Proof. We know that if C_n is a cyclic group of order p^m for some prime number p and positive integer m, then $P(C_n)$ is a complete graph and thus $\mu_1\big(P(C_n)\big) = \mathrm{n}-1$. Using Theorem 3.4, we have

$$DE\big(P(C_n)\big) \leq \mu_1 + \sqrt{(n-1)\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 - \mu_1^2\right)}$$

$$= n-1 + \sqrt{(n-1)\big(\ell(2n-\ell-1) + (n-\ell)(n-\ell-1) - (n-1)^2\,\big)}$$

$$= 2(n-1)$$

so the proof is completed.

Theorem 3.6. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$. Then

$$DE\big(P(C_n)\big) \geq \sqrt{\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 + n(n-1)det\big[D\big(P(C_n)\big)\big]^{(2/n)}\right)}$$

Proof. By Lemma 3.1, we have

$$DE\big(P(C_n)\big)^2 = \left(\sum_{i=1}^{n}\big|\mu_i\big(P(C_n)\big)\big|\right)^2 = \sum_{i=1}^{n}\mu_i^2\big(P(C_n)\big) + 2\sum_{i<j}\big|\mu_i\big(P(C_n)\big)\big|\big|\mu_j\big(P(C_n)\big)\big|$$

$$= \ell(2n-\ell-1) + 2\sum_{\ell+1\le i<j\le n} d\big(v_i,v_j\big)^2 + 2\sum_{i<j}\big|\mu_i\big(P(C_n)\big)\big|\big|\mu_j\big(P(C_n)\big)\big| \qquad (15)$$

Since the geometric mean of nonnegative numbers is smaller than their arithmetic mean. Thus, we have

$$2\sum_{i<j}\big|\mu_i\big(P(C_n)\big)\big|\big|\mu_j\big(P(C_n)\big)\big| = \sum_{i\neq j}\big|\mu_i\big(P(C_n)\big)\big|\big|\mu_j\big(P(C_n)\big)\big|$$

$$\ge n(n-1)\left(\prod_{i\neq j}\big|\mu_i\big(P(C_n)\big)\big|\big|\mu_j\big(P(C_n)\big)\big|\right)^{\frac{1}{n(n-1)}}$$

$$= n(n-1)\left(\prod_{i=1}^{n}\big|\mu_i\big(P(C_n)\big)\big|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= n(n-1)det\big[D\big(P(C_n)\big)\big]^{(2/n)}$$

(16)

By (15) and (16) , we obtain

$$DE\big(P(C_n)\big) \ge \sqrt{\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\le i<j\le n} d\big(v_i,v_j\big)^2 + n(n-1)det\big[D\big(P(C_n)\big)\big]^{(2/n)}\right)}$$

The proof is complete.

Now we consider the power graph $P(D_{2n})$ of the dihedral group D_{2n} with $n \ge 3$. We denote by $D\big(P(D_{2n})\big)$ the distance matrix of $P(D_{2n})$ and by $\mu_1\big(P(D_{2n})\big), \mu_2\big(P(D_{2n})\big), \ldots, \mu_{2n}\big(P(D_{2n})\big)$ its eigenvalues in decreasing order. Moreover, naturally define the distance enrgy $DE\big(P(D_{2n})\big)$ of the power graph $P(D_{2n})$ as the sum of the absolute values of its distance eigenvalues i.e.,

$$DE\big(P(D_{2n})\big) = \sum_{i=1}^{2n}\big|\mu_i\big(P(D_{2n})\big)\big|.$$

Lemma 3.7. Let $P(D_{2n})$ be the power graph of the dihedral group D_{2n} with $n \geq 3$. Then

$$\sum_{i=1}^{2n} \mu_i\big(P(D_{2n})\big) = 0$$

and

$$\sum_{i=1}^{2n} \mu_i^2\big(P(D_{2n})\big) = \ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i,v_j\big)^2 + 2n(6n-5).$$

Proof. From the definition of the trace of a matrix, we have

$$\sum_{i=1}^{2n} \mu_i\big(P(D_{2n})\big) = tr\big[D\big(P(D_{2n})\big)\big] = 0.$$

Now we consider the matrix

$$D\big(P(D_{2n})\big)^2 = \begin{pmatrix} D\big(P(C_n)\big) & M_n \\ M_n^T & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}^2$$

where I is the identity matrix, J the all-ones matrix and

$$M_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \ldots & 2 \end{pmatrix}$$

Then, it is clear that

$$\sum_{i=1}^{2n} \mu_i^2\big(P(D_{2n})\big) = tr\left[D\big(P(D_{2n})\big)^2\right]$$

$$= tr\left[D\big(P(C_n)\big)^2\right] + tr[M_nM_n^T] + tr[M_n^TM_n] + tr[4(J-I)^2].$$

By Lemma 3.1, we have

$$\sum_{i=1}^{2n} \mu_i^2\big(P(D_{2n})\big) = \ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i,v_j\big)^2$$
$$+4n(n-1) + n + 4n(n-1) + n + 4n(n-1)$$

$$= \ell(2n-\ell-1) + 2\textstyle\sum_{\ell+1\leq i<j\leq n} d\big(v_i,v_j\big)^2 + 2n(6n-5).$$

This completes the proof.

Theorem 3.8. Let $P(D_{2n})$ be the power graph of D_{2n} with $n \geq 3$. Then

$$\sqrt{\ell(2n-\ell-1)+2\sum_{\ell+1\leq i<j\leq n} d(v_i,v_j)^2+2n(6n-5)} \leq DE(P(D_{2n}))$$

and

$$DE(P(D_{2n})) \leq \sqrt{2n\left[\ell(2n-\ell-1)+2\sum_{\ell+1\leq i<j\leq n} d(v_i,v_j)^2+2n(6n-5)\right]}.$$

Proof. By Lemma 3.7 and the Cauchy-Schwarz inequality, we obtain

$$DE(P(D_{2n}))^2 = \left(\sum_{i=1}^{2n}|\mu_i(P(D_{2n}))|\right)^2$$

$$\leq 2n\sum_{i=1}^{2n}\mu_i^2(P(D_{2n}))$$

$$= 2n\ell(2n-\ell-1)+4n\sum_{\ell+1\leq i<j\leq n} d(v_i,v_j)^2+4n^2(6n-5)$$

$$= 2n\left[\ell(2n-\ell-1)+2\sum_{\ell+1\leq i<j\leq n} d(v_i,v_j)^2+2n(6n-5)\right]. \tag{17}$$

For the other side of the inequaities, we have

$$DE(P(D_{2n}))^2 = \left(\sum_{i=1}^{2n}|\mu_i(P(D_{2n}))|\right)^2$$

$$\geq \sum_{i=1}^{2n}\mu_i^2(P(D_{2n}))$$

$$= \ell(2n-\ell-1)+2\sum_{\ell+1\leq i<j\leq n} d(v_i,v_j)^2+2n(6n-5). \tag{18}$$

Using (17) and (18) we get the required result.

Theorem 3.9. Let $P(D_{2n})$ be the power graph of D_{2n} with $n \geq 3$. Then

$$DE\big(P(D_{2n})\big) \leq \mu_1\big(P(D_{2n})\big) + \sqrt{(2n-1)\begin{pmatrix} \ell(2n-\ell-1) + 2\displaystyle\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 \\ +2n(6n-5) - \mu_1^2\big(P(D_{2n})\big) \end{pmatrix}}$$

Proof. By the Cauchy-Schwarz inequality and Lemma 3.7, we obtain

$$\Big(DE\big(P(D_{2n})\big) - \mu_1\big(P(D_{2n})\big)\Big)^2 = \left(\sum_{i=2}^{2n} \big|\mu_i\big(P(D_{2n})\big)\big|\right)^2$$

$$\leq (2n-1)\left(\sum_{i=1}^{2n} \mu_i^2\big(P(D_{2n})\big) - \mu_1^2\big(P(D_{2n})\big)\right)$$

$$= (2n-1)\left(\ell(2n-\ell-1) + 2\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 + 2n(6n-5) - \mu_1^2\big(P(D_{2n})\big)\right)$$

and thus

$$DE\big(P(D_{2n})\big) \leq \mu_1\big(P(D_{2n})\big) + \sqrt{(2n-1)\begin{pmatrix} \ell(2n-\ell-1) + 2\displaystyle\sum_{\ell+1\leq i<j\leq n} d\big(v_i, v_j\big)^2 \\ +2n(6n-5) - \mu_1^2\big(P(D_{2n})\big) \end{pmatrix}}$$

The proof is complete.

Theorem 3.10. Let $P(D_{2n})$ and $P(C_n)$ be the power graphs of D_{2n} and C_n respectively with $n \geq 3$, then

$$DE\big(P(C_n)\big) \leq DE\big(P(D_{2n})\big) \leq DE\big(P(C_n)\big) + 4(n-1) + 2\sqrt{n}\left(1 + 2\sqrt{(n-1)}\right).$$

Proof. Since $D\big(P(C_n)\big)$ is a principal submatrix of $D\big(P(D_{2n})\big)$, and from Cauchy's Interlace Theorem, we have

$$\mu_i\big(P(C_n)\big) \leq \mu_i\big(P(D_{2n})\big),$$

for $1 \leq i \leq n$. Thus

$$DE\big(P(C_n)\big) \leq DE\big(P(D_{2n})\big).$$

From the definition of the distance matrix of the power graph $P(D_{2n})$, we have

$$D\big(P(D_{2n})\big)=\begin{pmatrix} D\big(P(C_n)\big) & M_n \\ M_n^T & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}$$

$$=\begin{pmatrix} D\big(P(C_n)\big) & 0_n \\ 0_n & 0_n \end{pmatrix}+\begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}+\begin{pmatrix} 0_n & M_n \\ M_n^T & 0_n \end{pmatrix}$$

$$=\begin{pmatrix} D\big(P(C_n)\big) & 0_n \\ 0_n & 0_n \end{pmatrix}+\begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}+\begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix}+\begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix},$$

where

$$K_n=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad and \quad L_n=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \dots & 2 \end{pmatrix}.$$

Using the relation between the principal minors of a matrix and the coefficients of its characteristic polynomial one can obtain that the characteristic polynomial of $\begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}$ is

$$p(x)=x^{\mathrm{n}}-4n(n-1)x^{\mathrm{n}-2},$$

and hence,

$$\sum_{i=1}^{2n}\mu_i\ \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}=4\sqrt{\big(n(n-1)\big)}.$$

Also one can show that $2\sqrt{n}$ and $4(n-1)$ are sums of the absolute values of eigenvalues of

$$\begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} \text{ and } \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix},$$

respectively. Since

$$\begin{pmatrix} D\big(P(C_n)\big) & 0_n \\ 0_n & 0_n \end{pmatrix}, \begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}, \begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix} \text{ and } \begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}$$

are symmetric matrices, we obtain

$$DE\big(P(D_{2n})\big)=\sum_{i=1}^{2n}\left|\mu_i\big(P(D_{2n})\big)\right|$$

$$\le \sum_{i=1}^{2n}\left|\mu_i\begin{pmatrix} D\big(P(C_n)\big) & 0_n \\ 0_n & 0_n \end{pmatrix}\right|+\sum_{i=1}^{2n}\left|\mu_i\begin{pmatrix} 0_n & 0_n \\ 0_n & 2(J-\mathrm{I})_{\mathrm{n\times n}} \end{pmatrix}\right|$$

$$+\sum_{i=1}^{2n}\left|\mu_i\begin{pmatrix} 0_n & K_n \\ K_n^T & 0_n \end{pmatrix}\right|+\sum_{i=1}^{2n}\left|\mu_i\begin{pmatrix} 0_n & L_n \\ L_n^T & 0_n \end{pmatrix}\right|$$

$$= DE\big(P(C_n)\big) + 4(n-1) + 2\sqrt{n}\left(1 + 2\sqrt{(n-1)}\right).$$

Hence, the theorem is proved.

Conflict of interests: The authors declare that there is no conlict of interests regarding the publication of this paper.

CONCLUSIONS

In the present paper, the distance matrix and the distance energy of power graphs on cyclic groups and dihedral groups are considered and some bounds for the largerst eigenvalue of the distance matrix and the distance energy are presented at the first time in the literature.

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