ORIGINAL PAPER

ON DARBOUX HELICES IN THE COMPLEX SPACE C^3

YASIN UNLUTURK[1], TALAT KORPINAR[2]

Manuscript received: 09.08.2019; Accepted paper: 15.10.2019;
Published online: 30.12.2019.

Abstract. *In this study, using Darboux vector of an isotropic curve given by Şemin in [1], we give a characterization of isotropic Darboux helices in the complex space C^3. We show that an isotropic curve to be a Darboux helix has to have a constant pseudo curvature. We obtain the position vector of a nonzero fixed direction U. Then we give some characterizations related to the main theorem.*

Keywords: *Isotropic curve, isotropic Darboux helix, isotropic slant helix.*

1. INTRODUCTION

E. Cartan introduced the imaginary curves in the complex space. He defined the moving frame of an imaginary curve and its special equations in C^3 [2]. Şemin had mentioned about complex elements and complex curves in the real space R^3 [1]. The Cartan equations of isotropic curve were studied in the four dimensional Complex space C^4 by Altinişik [3]. Moreover, Pekmen characterized minimal curves by means of E. Cartan equations in C^3 [4]. Characterizations of helices were characterized in the complex space C^3 by [5]. Y lmaz examined the isotropic curves with constant pseudo-curvature which is called the slant isotropic helix in complex space C^4 [6]. Yılmaz and Turgut gave some properties of isotropic helices in C^3 [5]. Recently, the representation formula for an isotropic curve with pseudo arc length parameter and the structure function of such curves were defined by Qian and they characterized the isotropic Bertrand curve and $k-$type isotropic helices by using the representation and the Frenet formulas [7]. Several authors introduced different types of helices and investigated their properties. For instance, Barros et al. studied general helices in 3-dimensional Lorentzian space [4]. Izumiya and Takeuchi defined slant helices by the property that principal normal makes a constant angle with a fixed direction [8]. Kula and Yaylı studied spherical images of tangent and binormal indicatrices of slant helices and also showed that spherical images are spherical helices [9]. Ali and Lopez gave some characterizations of slant helices in Minkowski 3-space E_1^3 [10]. The Darboux vector of isotropic curves was introduced by Şemin [1]. As a special version of helices, Darboux helices are formed by obtaining the relation between the Darboux vector of the curve and a non-zero fixed direction. Darboux helices were studied in both Euclidean and Minkowski 3-spaces by [11-13]. In this study, using Darboux vector of an isotropic curve given by Şemin in [1], we give a characterization of isotropic Darboux helices in the complex space C^3. We show that an isotropic curve to be a Darboux helix has to be a constant pseudo curvature. We obtain the position vector of non zero fixed direction U. Then we give some characterizations related to the main theorem.

[1] Kirklareli University, Department of Mathematics, 39100 Kırklareli, Turkey.
E-mail: yasinunluturk@klu.edu.tr.
[2] Muş Alparsln University, Department of Mathematics, 49250 Muş, Turkey. E-mail: talatkorpinar@gmail.com.

2. MATERIALS AND METHODS

The three dimensional complex space C^3 is given with the standart flat metric as follows:

$$\langle , \rangle = dx_2^2 + 2dx_1 dx_3,$$

where (x_1, x_2, x_3) is a complex coordinate system of C^3.

Let r_p be a complex analytic function of a complex variable t. Then the vector function

$$r(t) = \sum_{p=1}^{3} r_p(t) k_p \,, \tag{2.1}$$

is called an imaginary curve, where $t = t_1 + it_2, r : C \to C^3$ and k_p are standard basis unit vectors of E^3 [1,14].

In this space, a vector which has a minimal direction is called an isotrop vector or minimal vector, that is, a vector u is a minimal vector if and only if $u^2 = 0$ [1]. The curves, of which the square of the distance between the two points is equal to zero, are called minimal or isotropic curves [5]. Let s denote pseudo-arclength, a curve is a minimal (isotropic) curve if and only if $ds^2 = 0$, where s denotes the pseudo arc-length. Thus it is obvious that an isotropic curve satisfies vectorial differential equation

$$[r'(t)]^2 = 0, \tag{2.2}$$

where $\dfrac{dx}{dt} = r'(t) \neq 0.$

For each point r of the isotropic curve, E. Cartan frame is defined (for well-known complex number $i^2 = -1$) as follow [1,14]:

$$\begin{aligned} e_1 &= x', \\ e_2 &= ix'', \\ e_3 &= -\frac{\beta}{2} x' + x''', \end{aligned} \tag{2.3}$$

where $\beta = (r''')^2.$ The moving E. Cartan frame along the isotropic curve x in C^3 is given by (2.3) which is denoted by $\{e_1, e_2, e_3\}.$ The inner products of these frame vectors are given by

$$e_j . e_k = \begin{cases} 0 \text{ if } j+k = 1,2,3; \text{Mod}(4), \\ 1 \text{ if } j+k = 4. \end{cases} \tag{2.4}$$

The vector and mixed products of these frame vectors are given by

$$e_j \times e_k = ie_{j+k-2}, e_1.(e_2 \times e_3) = i,$$

for $j, k = 1,2,3.$

The pseudo-arclength

$$s=\int_{t_0}^{t}-[(x'')^2]^{\frac{1}{4}}dt$$

is an invariant with respect to parameter t [1]. Thus the vectors e_1 and e_3 are isotropic vectors, while e_2 is a real vector. E. Cartan derivative formulas can be deduced from (2.3) as follows

$$\begin{aligned} e_1' &= -ie_2, \\ e_2' &= i(\kappa e_1+e_3), \\ e_3' &= -i\kappa e_2, \end{aligned} \tag{2.5}$$

where

$$\kappa=\frac{1}{2}\langle r'''(s),r'''(s)\rangle$$

is called pseudo-curvature of isotropic curve $r=r(s)$ [4]. These equations can be used if the minimal curve is at least of class C^4. Here (') denotes derivative according to pseudo arclength s. In the rest of the paper, we suppose that pseudo-curvature κ is non-vanishing except in the case of an isotropic cubic.

Definition 2.1. An isotropic curve $r=r(s)$ in C^3 is called an isotropic cubic if pseudo-curvature κ of $r(s)$ is congruent to zero [4].

Definition 2.2. An isotropic curve $r=r(s)$ in C^3 is called an isotropic helix if the tangent vector e_1 of $r(s)$ is isotropic vector [9].

Let $r=r(s)$ be an isotropic curve with the pseudo-curvature $\kappa\neq 0$, the pseudo-Darboux vector of the curve is defined as

$$e_q' = w\times e_q,\ \ (q=1,...,3).$$

If we write the pseudo-Darboux vector of the curve as follows,

$$w=\sum_{q=1}^{3} r_q e_q,$$

then we obtain

$$w=\kappa e_1-e_3.$$

The norm of Darboux vector of the curve $r=r(s)$ in C^3 is defined as

$$\|w\|=\sqrt{(\kappa e_1-e_3)^2}=i\sqrt{2\kappa}$$

which is called pseudo-Lancret curvature [9].

Definition 2.3. Let $r(s)$ be an isotropic curve framed by $\{e_1,e_2,e_3\}$ in C^3. If there exists a nonzero constant vector field $U\in C^3$ such that $\langle e_k,U\rangle$ $(k=0,1,2)$ is a (complex) constant, then it is said to be a $k-$type ($k=0,1,2,$ respectively) isotropic helix and U is called the axis of $r(s)$ [15].

3. RESULTS AND DISCUSSION

Definition 3.1. An isotropic curve $r: I \subset C \to C^3$ with the Darboux vector w is called Darboux helix, if there exists a non-zero fixed direction $U \in C^3$ such that $\langle w, U \rangle$ is a (complex) constant.

Theorem 3.2. Let $r = r(s)$ be an isotropic curve in C^3. $r(s)$ is a Darboux helix if and only if the pseudo curvature κ is a constant.

Proof: **($\Rightarrow$):** Assume that $r(s)$ is a Darboux helix. Hence there is a relation between the Darboux vector w and the non-zero fixed direction U such that

$$\langle w, U \rangle = C_0, \tag{3.1}$$

where C_0 is a complex constant from definition of Darboux helix. Considering Cartan frame, the axis U can be decomposed by

$$U = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \tag{3.2}$$

where $\alpha_i = \alpha_i(s),\ (i = 1, 2, 3)$ are analytic functions of pseudo arc length s. Putting (2.3) and (3.2) into (3.1), we obtain

$$\kappa\alpha_3 - \alpha_1 = C_0. \tag{3.3}$$

Differentiating (3.2) with respect to s, we get

$$(\alpha_1{}' + i\kappa\alpha_2)e_1 + (\alpha_2{}' - i\alpha_1 - i\kappa\alpha_3)e_2 + (\alpha_3{}' + i\alpha_2)e_3 = 0, \tag{3.4}$$

which suggests the following system of differential equations:

$$\begin{cases} \alpha_1{}' + i\kappa\alpha_2 = 0, \\ \alpha_2{}' - i\alpha_1 - i\kappa\alpha_3 = 0, \\ \alpha_3{}' + i\alpha_2 = 0. \end{cases} \tag{3.5}$$

Differentiating (3.3) gives

$$\kappa'\alpha_3 + \kappa\alpha_3{}' = \alpha_1{}'. \tag{3.6}$$

Substituting $(3.5)_1$ and $(3.5)_3$ into (3.6), we find

$$\kappa'\alpha_3 = 0. \tag{3.7}$$

We examine the equation (3.7) according to the following cases:

Case 1. If $\kappa' = 0$, then κ is a constant pseudo curvature. Differentiating the Darboux vector in (2.3) and considering the pseudo curvature as constant, we see that since

$$w' = \kappa' e_1 + \kappa e_1{}' - e_2' = \kappa e_1{}' - e_3{}' = -i\kappa e_2 + i\kappa e_2 = 0, \tag{3.8}$$

the Darboux vector is a fixed vector under this case.

Case 2. If $\alpha_3 = 0$, then using (3.5), $\alpha_1 = \alpha_2 = 0$. Thus the vector U becomes zero vector. So we exclude that case because it contradicts the definition of the fixed axis.
($\Leftarrow$): Conversely, Assume that the pseudo curvature κ is constant. We define the vector field U as

$$\begin{aligned} U = &\{-\sqrt{\frac{\kappa}{2}}C_1 i e^{\sqrt{2\kappa}S} + \sqrt{\frac{\kappa}{2}}C_2 i e^{-\sqrt{2\kappa}S} + C_3\}e_1 \\ &+\{C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S}\}e_2 \\ &+\{-\frac{1}{\sqrt{2\kappa}}C_1 i e^{\sqrt{2\kappa}S} + \frac{1}{\sqrt{2\kappa}}C_2 i e^{-\sqrt{2\kappa}S} + C_4\}e_3, \end{aligned} \tag{3.9}$$

where C_1, C_2, C_3, C_4 are (complex) constants. Differentiating (3.9) and considering the pseudo curvature κ as constant and (2.5), we have $U' = 0$, that is, U is a constant vector field. Because $\langle w, U \rangle = \kappa C_4 - C_3$ is a constant, so we arrive the result that $r(s)$ is a Darboux helix.

Corollary 3.3. Let r be an isotropic Darboux helix in C^3. Then the axis of isotropic Darboux helix is obtained as

$$\begin{aligned} U = &\{-\int (C_1 i\kappa e^{\sqrt{2\kappa}S} + C_2 i\kappa e^{-\sqrt{2\kappa}S})ds\}e_1 \\ &+\{C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S}\}e_2 \\ &+\{-\int (C_1 i e^{\sqrt{2\kappa}S} + C_2 i e^{-\sqrt{2\kappa}S})ds\}e_3, \end{aligned} \tag{3.10}$$

where C_1, C_2, C_3, C_4 are (complex) constants.
Proof: As it is shown in above theorem, the pseudo curvature κ is constant. By taking the pseudo curvature κ constant in the system (3.5), we solve the system as follows:

Differentiating $(3.5)_2$ gives

$$\alpha_2'' = i\alpha_1' + i\kappa\alpha_3'. \tag{3.11}$$

Using the equations $(3.5)_1$ and $(3.5)_2$ in (3.11), we get

$$\alpha_2'' = 2\kappa\alpha_2. \tag{3.12}$$

The solution of the differential equation (3.12) is

$$\alpha_2 = C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S}, \tag{3.13}$$

where C_1, C_2 are constants. From $(3.5)_1$ and $(3.5)_3$, we have the solutions of α_1 and α_3 as follows:

$$\alpha_1 = -\int (C_1 i\kappa e^{\sqrt{2\kappa}S} + C_2 i\kappa e^{-\sqrt{2\kappa}S})ds = -\sqrt{\frac{\kappa}{2}}C_1 i e^{\sqrt{2\kappa}S} + \sqrt{\frac{\kappa}{2}}C_2 i e^{-\sqrt{2\kappa}S} + C_3 \tag{3.14}$$

and

$$\alpha_3 = -\int (C_1 i e^{\sqrt{2\kappa}S} + C_2 i e^{-\sqrt{2\kappa}S})ds = -\frac{1}{\sqrt{2\kappa}}C_1 i e^{\sqrt{2\kappa}S} + \frac{1}{\sqrt{2\kappa}}C_2 i e^{-\sqrt{2\kappa}S} + C_4 \tag{3.15}$$

where C_3, C_4 are (complex) constants. Using (3.13), (3.14), and (3.15) in (3.2), we have the axis U as in (3.10).

Corollary 3.5. Let r be an isotropic Darboux helix in C^3. Isotropic Darboux helix is not a curve of constant precession.

Proof: Scofield defined that curves of constant precession are curves whose Darboux vectors make a constant angle with a fixed direction and rotate about it with a constant speed [16]. Taking isotropic Darboux helix r with the Darboux vector w into account, the speed of Darboux vector is vanishing, hence the proof is completed [17, 18].

Corollary 3.6. Isotropic cubic is a Darboux helix with the axis

$$U = \{C_1 + C_2\}e_2 + \{-\int (C_1 + C_2) i ds\}e_3.$$

Proof: Isotropic cubic is an isotropic curve with the vanishing pseudo curvature. Thus the proof of corollary is straightforwardly seen by result of Theorem 3.2.

Corollary 3.7. If the pseudo curvature κ vanishes, then the Darboux vector of an isotropic curve is an isotropic vector.

Proof: It is straightforwardly seen by using (2.3).

Theorem 3.8. Let $r = r(s)$ be an isotropic Darboux helix in C^3. Then, $r = r(s)$ is a $0-$type slant helix if and only if the equation

$$C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S} = 0 \tag{3.16}$$

holds. Here C_1, C_2 are constants.

Proof: Let the isotropic Darboux helix $r = r(s)$ be $0-$type slant helix in C^3. Then the relation

$$\langle e_1, U \rangle = \text{constant} \tag{3.17}$$

holds. Differentiating (3.17), we have

$$\langle e_1', U \rangle = \langle -ie_2, U \rangle = -i(C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S}) = 0. \tag{3.18}$$

From (3.18), we obtain the relation in (3.16). Conversely, assume that the relation (3.16) holds. Then, since $r = r(s)$ is a isotropic Darboux helix, the pseudo curvature is constant. Using the axis in (3.9) which a non-zero fixed direction, we have the result

$$\langle e_1, U \rangle = -\frac{1}{\sqrt{2\kappa}} C_1 i e^{\sqrt{2\kappa}S} + \frac{1}{\sqrt{2\kappa}} C_2 i e^{-\sqrt{2\kappa}S} + C_4, \tag{3.19}$$

where C_4 is a constant. Rewriting (3.16) as

$$e^{\sqrt{2\kappa}S} = i\sqrt{\frac{C_2}{C_1}} \tag{3.20}$$

Substituting (3.20) into (3.19) gives

$$\langle e_1, U\rangle = C_4. \tag{3.21}$$

The equation (3.21) means that isotropic Darboux helix $r = r(s)$ is a $0-$type slant helix.

Theorem 3.9. Let $r = r(s)$ be an isotropic Darboux helix in C^3. Then, $r = r(s)$ is a $1-$type slant helix if and only if either

$$\kappa = 0 \quad \text{or} \quad C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S} = 0, \tag{3.22}$$

where C_1, C_2 are constants.

Proof: Let $r = r(s)$ be $1-$type slant helix in C^3. Then the relation

$$\langle e_2, U\rangle = \text{constant} \tag{3.23}$$

holds. Differentiating (3.22), we have

$$\langle e_2', U\rangle = \langle i\kappa e_1 + ie_3, U\rangle = 2\int (C_1 \kappa e^{\sqrt{2\kappa}S} + C_2 \kappa e^{-\sqrt{2\kappa}S}) ds = 0, \tag{3.24}$$

by using the axis U in (3.10). From (3.24), we obtain the relations in (3.21).

Conversely, assume that the relation (3.22) holds. Then, since $r = r(s)$ is a isotropic Darboux helix, the pseudo curvature is constant. Using the axis U in (3.9) which a non-zero fixed direction, we have the result

$$\langle e_2, U\rangle = C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S}. \tag{3.25}$$

Using one of the conditions in (3.22), it turns out that $\langle e_2, U\rangle =$ constant. This means that isotropic Darboux helix $r = r(s)$ is a $1-$type slant helix.

Theorem 3.10. Let $r = r(s)$ be an isotropic Darboux helix in $\mathbb{C}^3$. Then, $r = r(s)$ is a $2-$type slant helix if and only if

$$\kappa = 0 \quad \text{or} \quad C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S} = 0, \tag{3.26}$$

Proof: Let the isotropic Darboux helix $r = r(s)$ be $2-$type slant helix in C^3. Then the relation

$$\langle e_3, U\rangle = \text{constant} \tag{3.27}$$

holds. Differentiating (3.27), we have

$$\langle e_3', U\rangle = \langle -i\kappa e_2, U\rangle = -i\kappa(C_1 e^{\sqrt{2\kappa}S} + C_2 e^{-\sqrt{2\kappa}S}) = 0. \tag{3.28}$$

From (3.28), we obtain the relation in (3.25). Conversely, assume that the relation (3.25) holds. Then, since $r = r(s)$ is a isotropic Darboux helix, the pseudo curvature is constant. Using the axis in (3.9) which a non-zero fixed direction, we have the result

$$\langle e_3, U\rangle = -\kappa\int (C_1 i e^{\sqrt{2\kappa}s} + C_2 i e^{-\sqrt{2\kappa}s})ds. \qquad (3.29)$$

Substituting one of the conditions in (3.26) into (3.29) gives

$$\langle e_1, U\rangle = \text{constant}. \qquad (3.30)$$

The equation (3.31) means that isotropic Darboux helix $r = r(s)$ is a $2-$type slant helix. We can give the following result based on the results of Theorem 3.8, 3.9, 3.10:

Corollary 3.11. Let $r = r(s)$ be an isotropic cubic Darboux helix in C^3. Then every isotropic cubic Darboux helices are k-type Darboux helices.

Proof: It is the straightforward result of Theorem 3.8, 3.9, 3.10.

Corollary 3.12. Let $r = r(s)$ be an isotropic Darboux helix in C^3. Then excluding the case to be isotropic cubic curve, the following expressions are equivalent:
(i) $r = r(s)$ is a $0-$type slant helix,
(ii) $r = r(s)$ is a $1-$type slant helix,
(iii) $r = r(s)$ is a $2-$type slant helix,
(v) $C_1 e^{\sqrt{2\kappa}s} + C_2 e^{-\sqrt{2\kappa}s} = 0.$

Proof: It is a direct result of Theorem 3.8, 3.9, 3.10, and Corollary 3.11.

REFERENCES

[1] Şemin, F., *Differential Geometry I (in Turkish)*, Istanbul University, Science Faculty Press, 1983.
[2] Cartan, E., *La Methode du Repere Mobile, la Theorie des Groupes Continus, et les Espaces Generalises*, *Exposes de Geometrie*, Hermann et Cie, Paris, 1935.
[3] Altinişik, E., *PhD Thesis - Complex curves in* R^4, Dokuz Eylül University, Izmir, 1997.
[4] Pekmen, U., *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, **10**, 3, 1999.
[5] Yılmaz, S., Turgut, M., *Int. J. Comput. Math. Sci.*, **2**(2), 107, 2008.
[6] Yılmaz, S., *J. Math. Anal. Appl.*, **374**(2), 673, 2011.
[7] Qian, J., Kim, Y. H., *Bull. Korean Math. Soc.*, **52**(3), 963, 2015.
[8] Barros, M., Ferrandez, A., Lucas, P., Merono, A.M., *Rocky Mountain J. Math.*, **31**(2), 373, 2001.
[9] Kula, L., Yaylı, Y., *Appl. Math. Comput.*, **169**(1), 600, 2005.
[10] Ali, A., Lopez, R., *J. Korean Math. Soc.*, **48**, 159, 2011.
[11] Ilarslan, K, Yildirim, M., *Math Meth Appl Sci.*, 1, 2018.
[12] Şenol, A., Zıplar, E., Yayli, Y., Ghadami, R., *Life Sci. J.*, **9**(4), 5905, 2012.
[13] Zıplar, E., Şenol, A., Yaylı, Y., *Global J Sci Front Res Math Decision Sci.* **12**(13), 73, 2012.
[14] Akbulut, F., *Vector Calculus*, Ege University Press, Izmir, 1981.
[15] Izumiya, S., Takeuchi, N., *Turk. J. Math.* **28**(2), 531, 2004.
[16] Scofield, P.D., *Amer. Math. Monthly*, **102**, 531, 1995.
[17] Barthel, W., *J. Geometry*, **49**(1), 46, 1994.
[18] Öztürk, U., Nešovic, E., *Kuwait J. Sci.* **43**(2), 64, 2016.