ORIGINAL PAPER

SOME INTEGRAL TRANSFORM OF THE GENERALIZED FUNCTION $G_{\rho,\eta,r}[a,z]$

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Abstract. *This paper refers to the study of generalized function $G_{\rho,\eta,r}[a,z]$. Using generalized function $G_{\rho,\eta,r}[a,z]$ defined in [2], we derive various integral transform, including Euler transform, Laplace transform, Whittaker transform, Mellin transform, Hankel transform K-transform, P_δ transform and Fractional Fourier transform. Some results are expressed in terms of generalized Wright function. The transforms found here are likely to find useful in problem of Sciences, engineering and technology.*

Keywords: *Special G-function, Integral Transforms, Fractional Fourier transform, Generalized Wright function, Gauss Hyper-geometric function.*

Subject Classification*: Primary 26A33, 44A20; Secondary: 33C20, 33E50.*

1. INTRODUCTION

Integral transform is widely used for problem solving in various fields of physics and applied mathematics. The present paper deals with the evaluation of various integral transforms of the generalized function $G_{\rho,\eta,r}[a,z]$.

Definition 1.1. Lorenzo and Hartley defined the following special function named as generalized function $G_{\rho,\eta,r}[a,z]$ [1-3] as:

$$G_{\rho,\eta,r}[a,z] = z^{r\rho-\eta-1} \sum_{n=0}^{\infty} \frac{(r)_n \ (az^\rho)^n}{\Gamma(n\rho+\rho r-\eta)n!}, \qquad \mathrm{Re}(\rho r-\eta) > 0 \tag{1}$$

On taking $r = 1$ and $z = z - c$ in equation (1), it reduces to special R-function which is defined by Lorenzo and Hartley [2]

$$R_{\rho,\eta}[a,c,z] = (z-c)^{\rho-\eta-1} \sum_{n=0}^{\infty} \frac{[a(z-c)^\rho]^n}{\Gamma(n\rho+\rho-\eta)}, \quad \rho \geq 0, \rho \geq \eta \tag{2}$$

Details related to the function $R_{\rho,\eta}[a,c,z]$ and $G_{\rho,\eta,r}[a,z]$ can be seen in Lorenzo and Hartley [1, 2], H. Nagar and Menaria [3]

Definition 1.2. The Euler transform of a function *f(z)* is defined as [4]

$$B\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)dz \qquad a,b \in C, R(a) > 0, R(b) > 0 \tag{3}$$

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Definition 1.3. The Laplace transform of a function $f(t)$, is given by the equation [4]

$$F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt \qquad R(s) > 0 \tag{4}$$

Provided that the integral (4) is convergent for $t > 0$ and of exponential order as $t \to \infty$. Also

$$\int_0^\infty e^{-st} t^{p-1} dt = \frac{\Gamma(p)}{s^p}, \qquad R(p) > 1, R(s) > 1. \tag{5}$$

Definition 1.4. The Whittaker Transform is defined as [5]:

$$\int_0^\infty e^{-\frac{t}{2}} t^{\zeta-1} W_{\alpha,\beta}(t)dt = \frac{\Gamma\left(\frac{1}{2}+\beta+\zeta\right)\Gamma\left(\frac{1}{2}-\beta+\zeta\right)}{\Gamma(1-\alpha+\zeta)} \tag{6}$$

where $R(\beta \pm \zeta) > -{}^1\!/_2$ and $W_{\alpha,\beta}(t)$ is the Whittaker confluent Hypergeometric function

$$W_{\beta,\zeta}(z) = \frac{\Gamma(-2\beta)}{\Gamma(1/2-\alpha-\beta)} M_{\alpha,\beta}(z) + \frac{\Gamma(2\beta)}{\Gamma(1/2+\alpha+\beta)} M_{\alpha,-\beta}(z) \tag{7}$$

where $M_{\alpha,\beta}(z)$ is given by

$$M_{\alpha,\beta}(z) = z^{1/2+\beta} e^{-1/2z} \, {}_1F_1\left(\frac{1}{2} + \beta - \alpha; 2\beta + 1; z\right) \tag{8}$$

Definition 1.5. The Mellin transform of $f(t)$ is given as [6]

$$M\{f(t)\}(s) = \int_0^\infty t^{s-1} f(t)dt, \qquad R(s) > 0. \tag{9}$$

Definition 1.6. The Hankel transform of $f(x)$, denoted by $g(p; v)$ is defined as [7]

$$g(p; v) = \int_0^\infty (px)^{\frac{1}{2}} J_v(px) f(x)\, dx; \quad p > 0 \tag{10}$$

The following formula can be used to solve the integral in equation (10) [8]

$$\int_0^\infty x^{\lambda-1} J_v(ax)dx = 2^{\lambda-1} a^{-\lambda} \frac{\Gamma\left(\frac{\lambda+v}{2}\right)}{\Gamma\left(1+\frac{v-\lambda}{2}\right)} \tag{11}$$

Definition 1.7. The K-transform is defined by the integral equation [9]

$$\Re_v[f(x); p] = g[p; v] = \int_0^\infty (px)^{1/2} K_v(px) f(x)dx \tag{12}$$

Here $R(v) > 0$; $K_v(x)$ is second kind Bessel function which is defined in [9]

$$K_v(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,v}(2z)$$

where $W_{0,v}(.)$ is the Whittaker function defined in equation (7).

In evaluating the integrals, the following result given by Mathai et al. [8] will be used

$$\int_0^\infty t^{\rho-1}K_\upsilon(ax)dx = 2^{\rho-2}a^{-\rho}\Gamma\left(\frac{\rho\pm\upsilon}{2}\right); \Re_\upsilon(a) > 0; \Re_\upsilon(\rho\pm\upsilon) > 0. \quad (13)$$

Definition 1.8. The $P_\delta[f(t);s]$ transform of a function *f(t)* ,in which s is a complex variable is defined as [10]:

$$P_\delta[f(t);s] = F_{P_\delta}(s) = \int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}} f(t)dt; \delta > 1 \quad (14)$$

P_δ-Transform of the power function $t^{\beta-1}$ is given by

$$P_\delta\left[t^{\beta-1};s\right] = \left(\frac{\delta-1}{ln[1+(\delta-1)]s}\right)^\beta \Gamma(\beta) \quad (15)$$

Definition 1.9. The fractional Fourier Transform of order $\alpha, 0 < \alpha \le 1$ is defined by [11]

$$\widehat{u_\alpha}(\omega) = \Im_\alpha[u](\omega) = \int_R e^{i\omega^{\left(\frac{1}{\alpha}\right)}t}u(t)dt \quad (16)$$

On taking$\alpha = 1$, equation (16) reduces to the conventional Fourier transform and on$\omega > 0$, it reduces to Fractional Fourier transform given by Luchko et al. [11]

Definition 1.10. Some compositions are expressed as generalized Wright Hypergeometric function [12] ${}_p\psi_q(z)$(for detail, see [15]), for $z \in C, a_i, b_j \in C$ and $\alpha_i, \beta_j \in R/\{0\}$, $(i = 1,2,\dots,p; j = 1,2,\dots q)$, is defined as [12-14]:

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p} \\ (b_i,\beta_i)_{1,q}\end{matrix}|z\right] = \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i+\alpha_i k)z^k}{\prod_{j=1}^q \Gamma(b_j+\beta_j k)k!} \quad (17)$$

Definition 1.11. Generalized Hypergeometric Function is defined by [15]:

$${}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix} z\right] = \sum_{n=0}^\infty \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} \quad (18)$$

This series (1.18) is known as the generalized Gauss series or simply the Generalized Hyper-geometric function [16]. Here p and q are positive integers or zero, the numerator parameters $\alpha_1,\dots,\alpha_p$ and the denominator parameters $\beta_1,\dots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\dots; (j = 1,\dots,q).$$

2. INTEGRAL TRANSFORMS OF $G_{\rho,\eta,r}[a,z]$

Theorem 2.1. Let $\rho,\eta,r,p,q \in C$,be such that

$$B\{\, G_{\rho,\eta,r}[a,z];p,q\} = \frac{\Gamma(q)}{\Gamma(r)}\ {}_2\psi_2\left[\begin{matrix}(r,1),(r\rho+p-\eta-1,\rho)\\(r\rho-\eta,\rho),(r\rho+p+q-\eta-1,\rho)\end{matrix};a\right]$$

where $R(p) > 0, R(q) > 0$.

Proof: On using (1.1) and (1.3), we get

$$B\{\, G_{\rho,\eta,r}[a,z];p,q\} = \int_0^1 z^{p-1}\,(1-z)^{q-1}\, G_{\rho,\eta,r}[a,z]dz$$

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta)\, n!} \int_0^1 z^{r\rho+n\rho+p-\eta-1-1}\,(1-z)^{q-1}dz$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(r+n)\, a^n}{\Gamma(r)\Gamma(r\rho+n\rho-\eta)\, n!}\, B(\, r\rho+n\rho+p-\eta-1,q)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(r+n) a^n}{\Gamma(r)\Gamma(r\rho+n\rho-\eta)\, n!}\, \frac{\Gamma(r\rho+n\rho+p-\eta-1)\Gamma(q)}{\Gamma(r\rho+n\rho+p+q-\eta-1)}$$

According to the definition of (1.17), we arrive at the result (2.1), which completes the proof of the theorem.

Theorem 2.2. Let $\rho, \eta, r, \in C$ *and* $R(s) > 0$, be such that

$$L\{\, G_{\rho,\eta,r}[a,z];s\} = \frac{s^{\eta-r\rho}}{\Gamma(r)} \sum_{n=0}^{\infty} \Gamma(r+n)\left(\frac{a}{s^{\rho}}\right)^n$$

Proof: On using (1.1) and (1.4), we get

$$L\{\, G_{\rho,\eta,r}[a,z];s\} = \int_0^{\infty} e^{-sz}\, G_{\rho,\eta,r}[a,z]\, dz$$

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta) n!} \int_0^{\infty} z^{r\rho+n\rho-\eta-1}\, e^{-sz}\, dz$$

On using (1.5)

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta) n!}\, \frac{\Gamma(r\rho+n\rho-\eta)}{s^{r\rho+n\rho-\eta}}$$

This directly completes the proof.

Theorem 2.3. Let $\rho, \eta, r, \in C; R(\zeta) > 0, R(\beta \pm \zeta) > -1/2$ be such that

$$\int_0^\infty e^{-\frac{t}{2}} t^{\zeta-1} W_{\alpha,\beta}(t)\, G_{\rho,\eta,r}[a,t]dt$$

$$= \frac{1}{\Gamma(r)}\ {}_3\psi_2\left[\begin{matrix}(r,1),\left(\beta + r\rho + \zeta - \eta - \frac{1}{2}, \rho\right), \left(r\rho + \zeta - \beta - \eta - \frac{1}{2}, \rho\right) \\ (r\rho - \eta, \rho), \left(r\rho + \zeta - \alpha - \eta - \frac{1}{2}, \rho\right)\end{matrix} \middle| a\right]$$

Proof: On using (1.1), we get

$$\int_0^\infty e^{-\frac{t}{2}} t^{\zeta-1} W_{\alpha,\beta}(t)\, G_{\rho,\eta,r}[a,t]dt = \int_0^\infty e^{-\frac{t}{2}} t^{\zeta-1} W_{\alpha,\beta}(t) \left(t^{r\rho-\eta-1} \sum_{n=0}^\infty \frac{(r)_n (at^\rho)^n}{\Gamma(n\,\rho + r\rho - \eta) n!}\right) dt$$

$$= \sum_{n=0}^\infty \frac{(r)_n a^n}{\Gamma(n\,\rho + r\rho - \eta) n!} \int_0^\infty e^{-\frac{t}{2}} t^{r\rho+n\rho-\eta+\zeta-1-1} W_{\alpha,\beta}(t) dt$$

Making use of (1.6)

$$= \sum_{n=0}^\infty \frac{(r)_n a^n}{\Gamma(n\,\rho + r\rho - \eta) n!} \frac{\Gamma\left(\frac{1}{2} + \beta + r\rho + n\rho + \zeta - \eta - 1\right) \Gamma\left(\frac{1}{2} - \beta + r\rho + n\rho + \zeta - \eta - 1\right)}{\Gamma\left(r\rho + n\rho + \zeta - \eta - 1 - \alpha + \frac{1}{2}\right)}$$

$$= \sum_{n=0}^\infty \frac{\Gamma(r+n) a^n}{\Gamma(r)\Gamma(n\,\rho + r\rho - \eta) n!} \frac{\Gamma\left(\beta + r\rho + n\rho + \zeta - \eta - \frac{1}{2}\right) \Gamma\left(r\rho + n\rho + \zeta - \beta - \eta - \frac{1}{2}\right)}{\Gamma\left(r\rho + n\rho + \zeta - \eta - \alpha - \frac{1}{2}\right)}$$

It directly completes the proof by using (1.17)

Theorem 2.4. The Mellin Transform of Generalized function $G_{\rho,\eta,r}[a,t]$ is given by:

$$\int_0^\infty t^{s-1} G_{\rho,\eta,r}[a,t]dt = \frac{1}{s + r\rho + n\rho - \eta - 1} \sum_{n=0}^\infty \frac{(r)_n\, a^n}{\Gamma(n\rho + r\rho - \eta)}$$

where $\rho, \eta, r, \in C; R(s) > 0$.

Proof: Using the definition of Mellin Transform (1.9)

$$M\{G_{\rho,\eta,r}[a,t]; s\} = \int_0^\infty t^{s-1} G_{\rho,\eta,r}[a,t]dt$$

and using (1.1)

$$\int_0^\infty t^{s-1}\, G_{\rho,\eta,r}[a,t]dt = \int_0^\infty t^{s-1}\left(t^{r\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(r)_n(at^\rho)^n}{\Gamma(n\rho+r\rho-\eta)n!}\right)dt$$

on changing the order

$$=\sum_{n=0}^{\infty}\frac{(r)_n a^n}{\Gamma(n\rho+r\rho-\eta)n!}\int_0^\infty t^{s+r\rho+n\rho-\eta-1-1}\,dt$$

On setting $t = e^{-x}$, it completes the required result of (2.4).

Theorem 2.5. The Hankel Transform of function $G_{\rho,\eta,r}[a,z]$ is given as

$$g(p\colon v) = \frac{1}{\sqrt{2}}\left(\frac{2}{p}\right)^{r\rho-\eta}\frac{1}{\Gamma(r)}\ {}_2\psi_2\left[\begin{matrix}(r,1),\left(\dfrac{r\rho-\eta-\frac{1}{2}+v}{2},\dfrac{\rho}{2}\right)\\ (r\rho-\eta,\rho),\left(\dfrac{v+\eta-r\rho+\frac{3}{2}}{2},-\dfrac{\rho}{2}\right)\end{matrix};a\left(\frac{2}{p}\right)^{\rho}\right]$$

where $\rho,\eta,r,\in C,p>0$.

Proof: On using (1.10), we get

$$g(p\colon v) = \int_0^\infty (pz)^{\frac{1}{2}} J_v(pz)\ G_{\rho,\eta,r}[a,z]\ dz$$

Making use of (1.1) and changing the order of integration

$$=\sum_{n=0}^{\infty}\frac{(r)_n a^n}{\Gamma(n\rho+r\rho-\eta)n!}\int_0^\infty (pz)^{\frac{1}{2}} J_v(pz)\ z^{r\rho+n\rho-\eta-1}dz$$

$$=\sum_{n=0}^{\infty}\frac{(r)_n a^n}{\Gamma(n\rho+r\rho-\eta)n!}\ p^{\frac{1}{2}}\int_0^\infty z^{r\rho+n\rho-\eta+\frac{1}{2}-1}\ J_v(pz)dz$$

On using (1.11), we get

$$=\sum_{n=0}^{\infty}\frac{(r)_n a^n}{\Gamma(n\rho+r\rho-\eta)n!}\ p^{\frac{1}{2}}\, 2^{r\rho+n\rho-\eta+\frac{1}{2}-1}\ p^{\eta-\frac{1}{2}-r\rho-n\rho}\ \frac{\Gamma\left(\dfrac{r\rho+n\rho-\eta+\frac{1}{2}+v}{2}\right)}{\Gamma\left(1+\dfrac{v+\eta-r\rho-n\rho-\frac{1}{2}}{2}\right)}$$

$$= \frac{2^{r\rho-\eta-\frac{1}{2}}\ p^{\eta-r\rho}}{\Gamma(r)} \sum_{n=0}^{\infty} \frac{\Gamma(r+n)\Gamma\left(\frac{r\rho+n\rho-\eta+\frac{1}{2}+v}{2}\right)}{n!\,\Gamma(n\rho+r\rho-\eta)\Gamma\left(\frac{\frac{3}{2}+v+\eta-r\rho-n\rho}{2}\right)} \left(a\left(\frac{2}{p}\right)^{\rho}\right)^{n}$$

Using equation (1.17), this immediately leads to (2.5).

Theorem 2.6. The K-transform of special $G_{\rho,\eta,r}[a,z]$ function for $(p)>0; \rho,\eta,r,\in C$, is given as

$$\int_0^{\infty} t^{\rho'-1} K_v(\omega t)\ G_{\rho,\eta,r}[a,t]\,dt = \frac{1}{2^2}\left(\frac{2}{\omega}\right)^{r\rho+\rho'-\eta-1} \frac{1}{\Gamma(r)}$$
$$\times\ {}_3\psi_1\left[\begin{matrix}(r,1),\left(\frac{r\rho+\rho'+v-\eta-1}{2},\frac{\rho}{2}\right),\left(\frac{r\rho+\rho'-v-\eta-1}{2},\frac{\rho}{2}\right); a\left(\frac{2}{\omega}\right)^{\rho}\\ (r\rho-\eta,\rho)\end{matrix}\right]$$

Proof: On using (1.12) and (1.1) , we get

$$\int_0^{\infty} t^{\rho'-1} K_v(\omega t)\ G_{\rho,\eta,r}[a,t]\,dt = \int_0^{\infty} t^{\rho'-1} K_v(\omega t) \sum_{n=0}^{\infty} \frac{(r)_n (at^{\rho})^n t^{r\rho-\eta-1}}{\Gamma(r\rho+n\rho-\eta)n!}\,dt$$

Changing the order of integration

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta)n!} \int_0^{\infty} t^{r\rho+\rho'+n\rho-\eta-2}\ K_v(\omega t)\,dt$$

Using (1.13) for $R_v(\omega)>0, R_v(\rho'\pm v)>0$

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta)n!}\ 2^{r\rho+\rho'+n\rho-\eta-3}\omega^{1+\eta-r\rho-\rho'-n\rho}$$
$$\times\Gamma\left(\frac{r\rho+\rho'+n\rho-\eta-1+v}{2}\right)\Gamma\left(\frac{r\rho+\rho'+n\rho-\eta-1-v}{2}\right)$$

$$= \frac{2^{r\rho+\rho'-\eta-3}\omega^{1+\eta-r\rho-\rho'}}{\Gamma(r)} \sum_{n=0}^{\infty} \frac{\Gamma(r+n)a^n\omega^{-n\rho}}{\Gamma(r\rho+n\rho-\eta)n!}$$
$$\times\Gamma\left(\frac{r\rho+\rho'-\eta-1+v+n\rho}{2}\right)\Gamma\left(\frac{r\rho+\rho'-\eta-1-v+n\rho}{2}\right)\frac{a^n 2^{n\rho}}{\omega^{n\rho}}$$

The required result in (2.6) directly follows on using (1.17).

Theorem 2.7. For $\delta>1, \rho,\eta,r,\in C$, the following result holds true

$$P_\delta\left[\,G_{\rho,\eta,r}[a,t];s\right]=\frac{1}{\left(\chi(\delta;s)\right)^{r\rho-\eta}}\ {}_1F_0\left[\begin{matrix}r\\ -\end{matrix}\ ;\qquad \frac{a}{\chi(\delta;s)}\right]$$

where $\chi(\delta;s)=\frac{ln[1+(\delta-1)s]}{\delta-1}$.

Proof: Using equation (1.14), we get

$$P_\delta\left[\,G_{\rho,\eta,r}[a,t];s\right]=\int_0^\infty[1+(\delta-1)s]^{-\frac{t}{\delta-1}}\ G_{\rho,\eta,r}[a,t]dt$$

On changing the order of integration and using (1.1)

$$=\sum_{n=0}^\infty\frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta)n!}\int_0^\infty[1+(\delta-1)s]^{-\frac{t}{\delta-1}}\ t^{r\rho+n\rho-\eta-1}dt$$

Here making use of result (1.15), we obtain

$$=\sum_{n=0}^\infty\frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta)n!}\left(\frac{\delta-1}{ln[1+(\delta-1)s]}\right)^{r\rho+n\rho-\eta}\Gamma(r\rho+n\rho-\eta)$$

$$=\sum_{n=0}^\infty\frac{(r)_n a^n}{n!}\left[\frac{1}{\chi(\delta;s)}\right]^{r\rho+n\rho-\eta}$$

In view of the definition (1.18), we arrive at the required result.

Theorem 2.8. Let $\rho,\eta,r,\in C$; be such that

$$J_\alpha\left[G_{\rho,\eta,r}(a,t)\right]=\sum_{n=0}^\infty\frac{(r)_n a^n}{(i)^{r\rho+n\rho-\eta}(\omega)^{\frac{r\rho+n\rho-\eta}{\alpha}}(-1)^{(r\rho+n\rho-\eta-1)}\,n!}$$

where $\alpha>0$.

Proof: Using (1.1) and (1.16), it gives

$$J_\alpha\left[G_{\rho,\eta,r}(a,t)\right](\omega)=\int_R e^{i\omega^{\left(\frac{1}{\alpha}\right)}t}\,G_{\rho,\eta,r}(a,t)dt$$

$$=\int_R\ e^{i\omega^{\left(\frac{1}{\alpha}\right)}t}\sum_{n=0}^\infty\frac{(r)_n a^n\,t^{r\rho+n\rho-\eta-1}}{\Gamma(r\rho+n\rho-\eta)n!}\ dt$$

$$=\sum_{n=0}^\infty\frac{(r)_n a^n}{\Gamma(r\rho+n\rho-\eta)n!}\int_R\ e^{i\omega^{\left(\frac{1}{\alpha}\right)}t}\ \ t^{r\rho+n\rho-\eta-1}\,dt$$

On setting $i\omega^{\left(\frac{1}{\alpha}\right)} = -u$, then we obtain

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho + n\rho - \eta) n!} \int_{-\infty}^{0} e^{-u} \left(\frac{-u}{i\omega^{\left(\frac{1}{\alpha}\right)}}\right)^{r\rho + n\rho - \eta - 1} \left(\frac{-du}{i\omega^{\left(\frac{1}{\alpha}\right)}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho + n\rho - \eta) n!} \frac{1}{(i)^{r\rho+n\rho-\eta}\ (-1)^{r\rho+n\rho-\eta-1} \omega^{\left(\frac{r\rho+n\rho-\eta}{\alpha}\right)}} \int_{0}^{\infty} e^{-u} u^{r\rho+n\rho-\eta-1} du$$

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{\Gamma(r\rho + n\rho - \eta) n!} \frac{1}{(i)^{r\rho+n\rho-\eta}\ (-1)^{r\rho+n\rho-\eta-1} \omega^{\left(\frac{r\rho+n\rho-\eta}{\alpha}\right)}} \Gamma(r\rho + n\rho - \eta)$$

$$= \sum_{n=0}^{\infty} \frac{(r)_n a^n}{n!} \frac{1}{(i)^{r\rho+n\rho-\eta}\ (-1)^{r\rho+n\rho-\eta-1} \omega^{\left(\frac{r\rho+n\rho-\eta}{\alpha}\right)}}$$

This completes the proof.

3. CONCLUSION

The results obtained in this paper are new and can be further modified in various new and known integral transforms, which are used in various areas of engineering, science, applied mathematics, bio-engineering etc.

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