ORIGINAL PAPER

ON RULED SURFACES WITH STRICTION SCROLL IN DUAL SPACE

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Manuscript received: 06.08.2019; Accepted paper: 18.10.2019;
Published online: 30.12.2019.

Abstract. *In this article, new ruled surfaces* $\Phi(s,\nu_1)$ *and* $\Phi^*(s,a)$ *are obtained for the base lines* $E_1(s)$ *and* $E_3(s)$*, respectively. As striction curve is taken as the base curve, distribution parameters of these ruled surfaces with striction scroll are calculated in dual space. As these ruled surfaces are accepted as closed, the apex angles and segments are denoted. An example is given to illustrate the main results. Finally, obtained consequences are discussed.*

Keywords: *Ruled surface, striction curve, ruled surface, scroll, distribution parameter.*

1. INTRODUCTION

Surface theory is extensively used in differential geometry, civil engineering, computer-aided design, etc. The properties of surface theory are actively studied by a lot of authors in literature, see for details in [1, 2].

In the last few years there has been a growing interst in ruled surfaces. As known, ruled surface is obtained by moving a line along the curve. In literature, several theories have been proposed to explain the concepts of the ruled surfaces and integral invariants of closed ruled surfaces [3-9].

The concept of scroll was first encountered in Wunderlich's book [10]. According to definition of scroll, there exists a central point Z, the corresponding point of that plane that is orthogonal to the tangent plane. Here the tangent line is called as central tangent. The locus k of all central points Z is defined as striction line of Φ. The ruled surface Φ^* including all central tangents f has the same striction line k and defined as the striction scroll of Φ. Similarly, Φ^* is the striction scroll of Φ.

Ruled surfaces obtained by striction scroll in Euclidean space were studied in [11]. By a rotating frame, the ruled surfaces with striction scroll that is taken as the base curve were defined and some calculations of these ruled surfaces were denoted. New consequences of the ruled surfaces generated by striction scroll were obtained by same authors in Minkowski space [11].

The theory of dual numbers was pioneered by W. K. Clifford in 1873. Then E. Study constructed the correspondence between the geometry of lines and unit dual sphere by the theorem. E. Study mapping says that there is one-to-one correspondence between the oriented lines in Euclidean space and the points on the unit dual sphere, DS^2 [12].

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However, there is a little research about ruled surfaces obtained by striction scroll in dual space. The remainder of this paper is divided into three sections. In Section 2 we explain some basic definitions and theorems about ruled surfaces in Euclidean space and dual space. In Section 3 we introduce new ruled surfaces obtained by striction scroll in dual space. The distribution parameters are calculated. As the ruled surfaces are taken as closed, the integral invariants are presented. After these calculations, we give an example about the obtained corollaries. Section 4 summarizes the result of this work and draw conclusions.

2. PRELIMINARIES

This section deals with some elementary definitions and theorems about ruled surfaces in Euclidean space $\mathbb{R}^3$ and dual space, respectively.

Firstly, basic properties and calculations for ruled surface in $\mathbb{R}^3$ will be given:

Definition 1: Let M be a surface in $\mathbb{R}^3$. M is called ruled surface if along every point p of M there exists a straight line that lies on M. So, a ruled surface $\Phi : I\times\mathbb{R} \to \mathbb{R}^3$ has the following form

$$\Phi(s,\nu) = \alpha(s) + \nu e(s), \nu \in \mathbb{R} \tag{1}$$

where α and e are curves in $\mathbb{R}^3$. Here α and e are called base curve and the base line of $\Phi(s,\nu)$, respectively.

Definition 2: Assume that M is a ruled surface in $\mathbb{R}^3$. The foot of the common perpendicular line of the neighbouring two rulings on the main ruling is called the **s**triction point. So, the geometric locus of these points is defined as the striction curve of M. The striction curve can be given as

$$Z(s) = \alpha(s) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} e(s). \tag{2}$$

Definition 3: Assume that M is a ruled surface in $\mathbb{R}^3$. The shortest distance between the two neighbouring rulings is called the distribution parameter (or drall) of M and can be presented as

$$P_\Phi = \frac{\det(\alpha', e, e')}{\|e'\|^2}. \tag{3}$$

Definition 4: The ruled surface can be denoted as $\Phi(s,\nu) = \alpha(s) + \nu e(s), s \in I, \nu \in \mathbb{R}$.

If $\Phi(s + 2\pi, \nu) = \Phi(s,\nu)$ for all $s \in I$, the ruled surface is called closed.

Let Φ be the non-developable ruled surface obtained by base line e. Assume that k is the striction curve with parametric equation $Z = Z(s)$. Here $Z = Z(s)$ is the base curve of Φ. Let e_1 be the unit vector of Φ in the direction of e. Here the set $\{e_1, e_2, e_3\}$ is an orthonormal frame at the striction curve Z. The correspondence between these frame vectors and their derivatives according to the arc parameter of the striction curve is

$$\begin{cases} e_1'(s) = \kappa(s)e_2(s), \\ e_2'(s) = -\kappa(s)e_1(s) + \tau(s)e_3(s), \\ e_3'(s) = -\tau(s)e_2(s). \end{cases}$$

The parametric representation of Φ can be written in the following form

$$\Phi\left(s,\nu_1\right) = Z\left(s\right) + \nu_1 e_1(s), \nu_1 \in \mathbb{R} \tag{4}$$

Definition 5: $\kappa = \left\|e_1'\left(s\right)\right\|$, $\tau = \left\langle e_2', e_3\right\rangle$ and $\lambda = \left\|e_2'\right\| = \sqrt{\kappa^2 + \tau^2}$ are called the natural curvature. the natural torsion and the Lancret curvature of Φ, respectively.

Moreower, the direction of the central tangent vector e_3 forms a ruled surface Φ^*.

The parametric representation of Φ^* can be showed as follows:

$$\Phi^*\left(s,a\right) = Z(s) + ae_3(s), a \in \mathbb{R} \tag{5}$$

Definition 6: Φ^* is called the striction scroll of Φ. Similarly, Φ is called the striction scroll of Φ^*.

Definition 7: Assume that z' is the tangent vector for the striction line *k*. The angle between z' and the line *e* is called the striction angle and presented as σ. This angle provides in the following equation:

$$z' = e_1 \cos\sigma + e_3 \sin\sigma. \tag{6}$$

The distribution parameters of Φ and Φ^* are presented as follows:

$$P_\phi = \frac{\sin\sigma}{\kappa} \quad , \quad P_{\phi^*} = \frac{\cos\sigma}{\tau}.$$

Corollary 1: The absolute value of the distribution parameters of Φ and Φ^* are equal. That is, $\left|P_\phi\right| = \left|P_{\phi^*}\right|$.

Secondly, basic properties and definitions about dual vectors are presented.

The set of dual numbers is defined as

$$ID = \{P = p + \varepsilon p^* : (p, p^*) \in IR \times IR, \varepsilon^2 = 0\}. \tag{7}$$

The combination of p and p^* is called dual vectors in IR^3. These vectors are real and dual part of P, respectively. If p and p^* are vectors in IR^3, $P = p + \varepsilon p^*$ is called as dual vector. Assume that $P = p + \varepsilon p^*$ and $Q = q + \varepsilon q^*$ are dual vectors. The addition, inner product and vector product are presented as follows:

The addition is

$$P + Q = (p + q) + \varepsilon(p^* + q^*)$$

and the inner product is

$$P.Q = \langle p,q\rangle + \varepsilon(\langle p^*,q\rangle + \langle p,q^*\rangle).$$

The vector product is also given as

$$P\times Q = p\times q + \varepsilon(p^*\times q + p\times q^*).$$

The norm of $P = p+\varepsilon p^*$ is given as

$$\|P\| = \sqrt{p.p} + \frac{p.p^*}{\sqrt{p.p}}.$$

The norm of $P = p+\varepsilon p^*$ exists for $p \neq 0$. If the norm $\|P\|$ is equal to 1, the dual vector is called unit dual vector.

The dual angle between the unit dual vectors A and B is

$$\langle \vec{P},\vec{Q}\rangle = \cos\theta = \cos\varphi - \varepsilon\varphi^*\sin\varphi.$$

Here $\theta=\varphi+\varepsilon\varphi^*$ is a dual number for $0\leq\varphi\leq\pi$ and $\varphi^*\in\mathbb{R}$.

The real numbers φ and φ^* are the angle and minimal distance, respectively.

3. SOME CHARACTERIZATIONS OF THE RULED SURFACES OBTAINED BY STRICTION SCROLL

In this section, firstly, accepting the base curves as $E_1(s)$ and $E_3(s)$, new ruled surfaces $\Phi(s,v)$ and $\Phi^*(s,v)$ are obtained for the orthonormal frame $\{E_1,E_2,E_3\}$. Secondly, distribution parameters of these ruled surfaces are calculated. Accepting these ruled surfaces as periodic, the apex angles and apex segments are presented, respectively. Then we give an example to support the obtained results.

Let $\Phi(s,v)$ be nondevelopable ruled surface obtained by the base line $E = E(s)$. Here $E = E(s)$ is the base curve $\alpha(s)$ in dual space. The parametrization of $\Phi(s,v)$ is

$$\Phi(s,v) = \alpha(s) + vE(s). \tag{8}$$

If the base curve is taken the striction curve $Z = Z(s)$ obtained by the base line $E_1 = E_1(s)$, the parametric representation for $\Phi(s,v_1)$ is can be written in the following form:

$$\Phi(s,v_1) = Z(s) + v_1E_1(s). \tag{9}$$

Here $\{E_1, E_2, E_3\}$ is an orthonormal frame at the central point Z in dual space.

Theorem 1: For the striction curve, the correspondence between these frame vectors and their derivatives is given as follows:

$$\begin{cases} \dot{E}_1(s) = \kappa(s)E_2(s), \\ \dot{E}_2(s) = -\kappa(s)E_1(s) + \tau(s)E_3(s), \\ \dot{E}_3(s) = -\tau(s)E_2(s), \end{cases}$$

where s is the arc parameter of this curve. $E_1(s) = e_1(s) + \varepsilon e_1^*(s), E_2(s) = e_2(s) + \varepsilon e_2^*(s)$ and $E_3(s) = e_3(s) + \varepsilon e_3^*(s)$ are dual vectors, respectively. For the first and second curvatures $\kappa(s) = k_1(s) + \varepsilon k_1^*(s)$ and $\tau(s) = k_2(s) + \varepsilon k_2^*(s)$, real and dual parts of these vectors are presented in the following equations:

$$\begin{cases} \dot{e}_1(s) = k_1(s)e_2(s), \\ \dot{e}_2(s) = k_1(s)e_2^*(s) + k_1^*(s)e_2(s), \\ \dot{e}_3(s) = -k_1(s)e_1(s) + k_2(s)e_3(s), \\ \dot{e}_1^*(s) = -k_1(s)e_1^*(s) - k_1^*(s)e_1(s) + k_2(s)e_3^*(s) + k_2^*(s)e_3(s), \\ \dot{e}_2^*(s) = -k_2(s)e_2(s), \\ \dot{e}_3^* = -k_2(s)e_2^*(s) - k_2^*(s)e_2(s). \end{cases}$$

Definition 8: $\kappa(s) = \|\dot{E}_1(s)\|$, $\tau(s) = \langle \dot{E}_2(s), E_3(s)\rangle$ are called the natural curvature and natural torsion of the ruled surface, respectively.

Moreower, the base line $E_3 = E_3(s)$ is accepted for the ruled surface for the same base curve, the parametric formula of this ruled surface is presented in the following form:

$$\Phi^*(s,a) = Z(s) + aE_3(s), a \in \mathbb{R}. \tag{10}$$

Definition 9: $\Phi^*(s,a)$ is called the striction scroll of $\Phi(s,v_1)$. Similarly, $\Phi(s,v_1)$ is called the striction scroll of $\Phi^*(s,a)$ in dual space.

Definition 10: Assume that $\dot{Z} = z_2 + \varepsilon z_2^*$ is the dual tangent vector of the striction curve. Striction angle is defined as an angle between $\dot{Z}$ and the line of $\Phi(s,v)$ and this angle is denoted by Ω.

The real and dual parts of this angle are given as follows:

$$\begin{cases} z_2 = e_1 \cos\gamma + e_3 \sin\gamma, \\ z_2^* = e_1^* cos\gamma + e_3^* sin\gamma, \end{cases}$$

where $\Omega = \gamma + \varepsilon\gamma^*$ is a dual angle.

Proposition 1: Assume that $\Phi(s,v_1)$ is a ruled surface obtained by the base line $E_1 = E_1(s)$. The distribution parameter of $\Phi(s,v_1)$ is

$$P_{\Phi}=\frac{(1+\varepsilon)\sin\gamma}{\kappa}.$$

Proposition 2: Assume that $\Phi^{*}(s,a)$ is a ruled surface obtained by the base line $E_3=E_3(s)$. The distribution parameter of $\Phi^{*}(s,a)$ is

$$P_{\Phi^{*}}=\frac{-(1+\varepsilon)\cos\gamma}{\tau}.$$

as striction curve $Z(s)$ is taken as periodic, the ruled surfaces $\Phi(s,v_1)$ and $\Phi^{*}(s,a)$ generated by $Z(s)$ are closed. Therefore, it is possible to calculate the apex angles and apex segments of these ruled surfaces. Considering this condition, the following consequences are presented:

Corollary 2: The apex angle of $\Phi(s,v_1)$ is given as

$$\lambda_{\Phi}=-\int_{Z}[(1+\varepsilon)(k_1\sin\gamma\cos\gamma-k_2\sin^2\gamma)]ds.$$

Corollary 3: The apex angle of $\Phi^{*}(s,a)$ is given as

$$\lambda_{\Phi^{*}}=-\int_{Z}[(1+\varepsilon)(k_1\cos^2\gamma-k_2\sin\gamma\cos\gamma)]ds.$$

Corollary 4: The apex segment of $\Phi(s,v_1)$ is calculated as

$$L_{\Phi}=\int_{Z}\cos\gamma ds.$$

Corollary 5: The apex segment of $\Phi^{*}(s,a)$ is calculated as

$$L_{\Phi^{*}}=\int_{Z}\varepsilon\sin\gamma ds.$$

Example 1: Let $\alpha(s)=(\cos s,\sin s,0)$ be a smooth curve in dual space. The curve $\alpha(s)$ is also unit speed curve in dual space. The striction curve of $\alpha(s)$ is calculated as follows:

$$Z(s)=(\cos s,\sin s,0)-\frac{\langle(-\sin s,\cos s,0),(-\cos s,-\sin s,0)\rangle}{\langle(-\cos s,-\sin s,0),(-\cos s,-\sin s,0)\rangle}.(-\cos s,-\sin s,0),$$
$$Z(s)=(\cos s,\sin s,0)=\alpha(s).$$

Here $\alpha_s=(-\sin s,\cos s,0)$ and $e_s=(-\cos s,-\sin s,0)$ are first order derivatives $\alpha(s)$ and $e(s)$, respectively. Therefore, the base curve coincides the striction curve.

After some calculations, Frenet frames of $Z(s)$ are given in the following equations:

$$\begin{cases} E_1(s) = (-\sin s, \cos s, 0), \\ E_2(s) = (-\cos s, -\sin s, 0), \\ E_3(s) = (0,0,1). \end{cases}$$

The ruled surface obtained by the line $E_1(s)$ is

$$\Phi(s,v_1) = (\cos s - v_1 \sin s, \sin s + v_1 \cos s, 0). \text{ (Fig. 1)}$$

The ruled surface obtained by the line $E_3(s)$ is also

$$\Phi^*(s,a) = (\cos s, \sin s, a). \text{ (Fig. 2)}$$

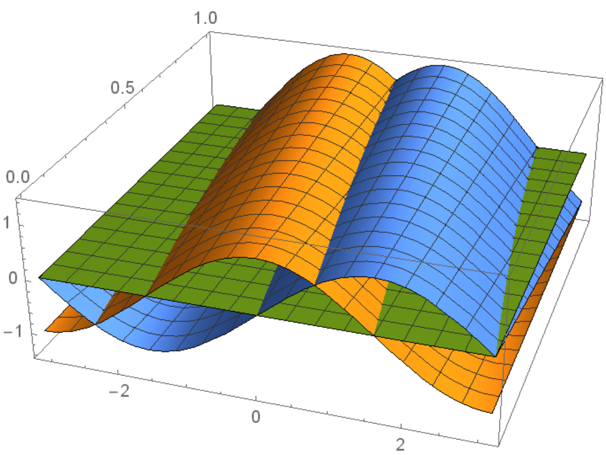

Figure 1. The Ruled Surface Obtained by The Line $E_1(s)$.

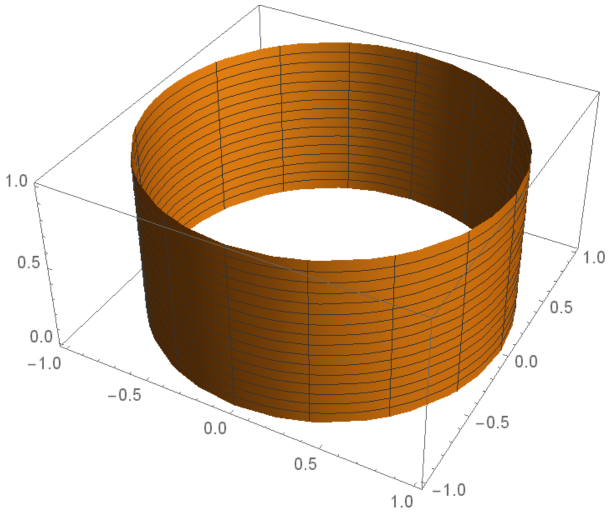

Figure 2. The Ruled Surface Obtained by The Line $E_3(s)$.

$\Phi^*(s,a)$ is called the striction scroll of $\Phi(s,v_1)$. Similarly, $\Phi(s,v_1)$ is called the striction scroll of $\Phi^*(s,a)$ in dual space.

For the dual tangent vector $\dot{Z} = z_2 + \varepsilon z_2^*$, the real and dual parts of dual angle are given as follows:

$$\begin{cases} z_2(s) = (-\sin s \cos\gamma, \cos s \cos\gamma, \sin\gamma), \\ z_2^*(s) = 0. \end{cases}$$

Here $\Omega = \gamma + \varepsilon\gamma^*$ is a dual angle.

The distribution parameter of $\Phi(s,v_1)$ is

$$P_\Phi = (1+\varepsilon)\sin\gamma.$$

The distribution parameter of $\Phi^*(s,a)$ is undefined.

Moreower, the apex angles of $\Phi(s,v_1)$ and $\Phi^*(s,a)$ are calculated in the following equations:

$$\begin{cases} \lambda_{\Phi} = -\int_{Z} (1+\varepsilon)\sin\gamma\cos\gamma ds, \\ \lambda_{\Phi^*} = -\int_{Z} (1+\varepsilon)\cos^2\gamma ds. \end{cases}$$

Also, the apex segments are given as follows:

$$\begin{cases} L_{\Phi} = \int_{Z} \cos\gamma ds, \\ L_{\Phi^*} = \int_{Z} \varepsilon\sin\gamma ds. \end{cases}$$

4. CONCLUSION

This article explains the new ruled surfaces generated by striction scroll in dual space. Using the definitions and propositions in Euclidean space, new calculations and consequences are obtained in dual space. As accepting the closed ruled surface, the apex angles and segments are presented. These results have application areas in geometric modelling, geometry and engineerings which are constructed structures with curvature.

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