ORIGINAL PAPER

SOME NEW HERMITE- HADAMARD TYPE INEQUALITIES FOR STRONGLY HARMONIC $h-convex$ FUNCTIONS

RASHIDA HUSSAIN[1], ASGHAR ALI[1]*, AYESHA NASEEM[1]

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Abstract. *Present study introduces a new class of strongly harmonic $h-convex$ functions and its relationship with harmonically $h-convex$ functions. Some new Hermite-Hadamard type inequalities for class of strongly harmonic $h-convex$ are also established which generalizes previous results.*

Keywords: *Convex function, Hermite- Hadamard type inequalities, Harmonic $h-convex$ functions.*

1. INTRODUCTION

The theory of convexity has been subject so far-reaching research during the past few years due to its efficacy in various branches of pure and applied mathematics. The concept of convexity has been unlimited and universal in several directions. A considerable class of convex functions is that of strongly convex functions introduced by Polyak [1]. Motivated by the work of Polyak [1], Noor et al. [2] considered the strongly harmonically h-convex functions.

In this paper we present a new class of harmonic h-convex function which is called strongly harmonic $h-convex$ functions. We derive some new Hermite-Hadamard type inequalities for strongly harmonic $h-convex$. The established results generalizes Hermite-Hadamard type inequalities obtained in [2] for strongly harmonically h- convex function which was introduced by Noor. Strongly convex functions play an important role in optimization theory, mathematical economics, variational inequalities and other branches of pure and applied mathematics.

2. MATHEMATICAL FORMULATION

2.1. HARMONICALLY CONVEX SET

A set $\hat{I} \subset \bar{R}\{0\}$ is said to be harmonically convex set, if

$$\frac{\upsilon\nu}{t\upsilon+(1-t)\nu} \in \hat{I}, \quad \forall \upsilon,\nu \in \hat{I}, \qquad t \in [0,1]$$

[1]Mirpur University of Science and Technology (MUST), 10250 Mirpur, Pakistan. E-mail: drali@must.edu.pk.

It is well known that harmonic mean has played an important role in different fields of mathematics. Using the concept of weighted harmonic mean, one usually defines the harmonic convex function introduced by Iscan in [3].

2.2. HARMONICALLY CONVEX FUNCTION

A function $\dot{g}: \hat{I} \subset \bar{R} \setminus \{0\} \to \bar{R}$ is said to be harmonically convex function if

$$\dot{g}\left(\frac{\upsilon \nu}{t\upsilon+(1-t)\nu}\right) \leq (1-t)\big(\dot{g}(\upsilon)\big)+t\big(\dot{g}(\nu)\big), \forall \upsilon, \nu \in \hat{I},\ \ t \in [0,1].$$

2.3. HARMONIC $h-Convex$ FUNCTION

Let $h: [0,1] \subset K \to \bar{R}$ be non-negative function. A function $\dot{g}: \hat{I} \subset \bar{R} \setminus \{0\} \to \bar{R}$ is said to be harmonically $h-convex$ function if

$$\dot{g}\left(\frac{\upsilon \nu}{t\upsilon+(1-t)\nu}\right) \leq h(1-t)\dot{g}(\upsilon)+h(t)\dot{g}(\nu), \forall \upsilon, \nu \in \hat{I}, t \in [0,1].$$

2.4. STRONGLY HARMONIC CONVEX FUNCTION

A function$\dot{g}: \hat{I} = [m,n] \subset \bar{R} \setminus \{0\} \to \bar{R}$ is said to be strongly harmonic convex with modulus $c > 0$, if

$$\dot{g}\left(\frac{\upsilon \nu}{t\upsilon+(1-t)\nu}\right) \leq (t)\dot{g}(\upsilon)+\ (1\ -t)\dot{g}(\upsilon)-ct(1\ -t)\left\|\frac{\upsilon-\nu}{\upsilon \nu}\right\|^2$$

$\forall \upsilon, \nu \in \hat{I}$ and d$t \in [0,1]$.

2.5. STRONGLY HARMONIC $h-Convex$ FUNCTION

Let $\hat{I} \subseteq R$, c is a positive constant and $h:\ (0,1)\ \to (0,\infty)$ is a given function. We say that a function $\dot{g}: \hat{I} = [m,n] \subset \bar{R} \setminus \{0\}\ \to \bar{R}$ is strongly harmonic h−*convex* with modulus c$>$ 0 if

$$\dot{g}\left(\frac{\upsilon \nu}{t\upsilon+(1-t)\nu}\right) \leq\ h(t)\dot{g}(\nu)+\ h(1\ -\ t)\dot{g}(\upsilon)-\ ct(1\ -\ t)\left\|\frac{\upsilon-\nu}{\upsilon \nu}\right\|^2 \forall\, \upsilon, \nu \in \hat{I}$$

and $t \in [0,1]$.

2.6. HARMONIC SYMMETRIC

A function$\dot{g}: \hat{I} = [m, n] \subset \bar{\underset{.}{R}}\{0\} \to \bar{\underset{.}{R}}$ is said to be harmonic symmetric with respect to $\frac{2mn}{m+n}$, if

$$\dot{g}(\upsilon) = \dot{g}\left(\frac{mn\upsilon}{(m+n)\upsilon - mn}\right) \forall \upsilon \epsilon [m, n].$$

2.7. HERMITE–HADAMARD INEQUALITY (HH)

Let $\dot{g}: S = [m, n] \subset \bar{\underset{.}{R}} \to \bar{\underset{.}{R}}$ be a convex function on the interval S of the real numbers and$m, n \in S$ with $m < n$ then $\dot{g}$ satisfying the following inequality and is called Hermite–Hadamard Inequality.

$$\dot{g}\left(\frac{m+n}{2}\right) \leq \frac{1}{n-m}\int_m^n \dot{g}(x)dx \leq \frac{\dot{g}(m)+\dot{g}(n)}{2} \qquad (1)$$

3. MAIN RESULTS

In this section we start with the following result which gives some relationships between strongly harmonic $h - convex$ functions and harmonic $h - convex$ functions in the Euclidean inner product space R.

Lemma 3.1. Let $\hat{I}$ stands for a convex subset of $\bar{\underset{.}{R}}$, $h : (0, 1) \to (0, \infty)$ is a given function and c is a positive constant. Assume that $h : (0, 1) \to (0, \infty)$ satisfy the condition

$$h(t) \geq t,\ t \in (0,1) \qquad (2)$$

If $\dot{g}: \hat{I} = [m, n] \subset \bar{\underset{.}{R}}\{0\} \to \bar{\underset{.}{R}}$ is harmonic $h -$ convex then$F: \hat{I} = [m, n] \subset \bar{\underset{.}{R}}\{0\} \to \bar{\underset{.}{R}}$ defined by $F(\upsilon) = \dot{g}(\upsilon) + c\left\|\frac{1}{\upsilon}\right\|^2$ is strongly harmonic $h -$ convex with modulus c.

Proof: Assume that $\dot{g}$ is harmonic $h -$ convex. Using properties of the inner product, we have

$$F\left(\frac{\upsilon\nu}{t\upsilon + (1-t)\nu}\right)$$

$$= \dot{g}\left(\frac{\upsilon\nu}{t\upsilon + (1-t)\nu}\right) + c\left\|\frac{t\upsilon + (1-t)\nu}{\upsilon\nu}\right\|^2$$

$$\leq h(1-t)\dot{g}(\upsilon) + h(t)\dot{g}(\nu) + c\left\|\frac{t\upsilon+(1-t)\nu}{\upsilon\nu}\right\|^2,$$

$$= h(1-t)F(\upsilon) + h(t)F(\nu) - ch(1-t)\left\|\frac{1}{\upsilon}\right\|^2 - ch(t)\left\|\frac{1}{\nu}\right\|^2 + c\left\|\frac{t\upsilon+(1-t)\nu}{\upsilon\nu}\right\|^2$$

$\leq h(1-t)F(\upsilon)+h(t)F(\nu)-c(1-t)\left\|\frac{1}{\upsilon}\right\|^2-ct\left\|\frac{1}{\nu}\right\|^2+c\left((1-t)^2\left\|\frac{1}{\upsilon}\right\|^2+\frac{2t(1-t)}{\upsilon\nu}+t^2\left\|\frac{1}{\nu}\right\|^2\right)$

$=h(1-t)F(\upsilon)+h(t)F(\nu)-ct(1-t)\left\|\frac{\upsilon-\nu}{\upsilon\nu}\right\|^2$

which shows that F is strongly harmonic h −convex with modulus c.

In a similar way we can prove the next lemma.

Lemma 3.2. Let $\hat{I}$ stands for a convex subset of $\bar{R}$, $h:(0,1)\to(0,\infty)$ is a given function and c is a positive constant. Assume that $h:(0,1)\to(0,\infty)$ satisfy the condition $h(t)\leq t,\ t\in(0,1)$ if $F:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{R}$ is strongly harmonic, $h-$ convex with modulus c. Then there exists a harmonic $h-$ convex function $\dot{g}:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{R}$ such that $F(\upsilon)=\dot{g}(\upsilon)+c\left\|\frac{1}{\upsilon}\right\|^2$ where $\upsilon\in\hat{I}$.

Lemma 3.3. Let $\hat{I}$ stands for a convex subset of $\bar{R}$, $h:(0,1)\to(0,\infty)$ is a given function and c is a positive constant. Assume that $h:(0,1)\to(0,\infty)$ satisfy the condition (2) and $h\left(\frac{1}{2}\right)=\frac{1}{2}$ if $\dot{g}:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{R}$ is harmonic mid h −convex then $F:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{R}$ defined by $F(\upsilon)=\dot{g}(\upsilon)+c\left\|\frac{1}{\upsilon}\right\|^2$ is strongly harmonic mid $h-$ convex with modulus c.

Proof: Assume that ġ is harmonic mid h −convex. Using properties of the inner product, we have

$F\left(\frac{2\upsilon\nu}{\upsilon+\nu}\right)=\dot{g}\left(\frac{2\upsilon\nu}{\upsilon+\nu}\right)+c\left\|\frac{\upsilon+\nu}{2\upsilon\nu}\right\|^2$

$\leq h\left(\frac{1}{2}\right)\dot{g}(\upsilon)+h\left(\frac{1}{2}\right)\dot{g}(\nu)+c\left\|\frac{\upsilon+\nu}{2\upsilon\nu}\right\|^2$

$=h\left(\frac{1}{2}\right)F(\upsilon)+h\left(\frac{1}{2}\right)F(\nu)-ch\left(\frac{1}{2}\right)\left\|\frac{1}{\upsilon}\right\|^2-ch\left(\frac{1}{2}\right)\left\|\frac{1}{\nu}\right\|^2+c\left\|\frac{\upsilon+\nu}{2\upsilon\nu}\right\|^2$

$\leq h\left(\frac{1}{2}\right)F(\upsilon)+h\left(\frac{1}{2}\right)F(\nu)-c\left(\frac{1}{2}\right)\left\|\frac{1}{\upsilon}\right\|^2-c\left(\frac{1}{2}\right)\left\|\frac{1}{\nu}\right\|^2+c\left\|\frac{\upsilon+\nu}{2\upsilon\nu}\right\|^2,$

$\leq h\left(\frac{1}{2}\right)F(\upsilon)+h\left(\frac{1}{2}\right)F(\nu)-c\left(\frac{1}{2}\right)\left\|\frac{1}{\upsilon}\right\|^2-c\left(\frac{1}{2}\right)\left\|\frac{1}{\nu}\right\|^2+\frac{c}{4}\left(\left\|\frac{1}{\upsilon}\right\|^2+\frac{1}{\upsilon\nu}+\left\|\frac{1}{\nu}\right\|^2\right),$

$\leq h\left(\frac{1}{2}\right)F(\upsilon)+h\left(\frac{1}{2}\right)F(\nu)-\frac{c}{4}\left\|\frac{\upsilon-\nu}{\upsilon\nu}\right\|^2,$

$\leq\frac{F(\upsilon)+F(\nu)}{2}-\frac{c}{4}\left\|\frac{\upsilon-\nu}{\upsilon\nu}\right\|^2.$

which shows that F is strongly harmonic mid h −convex with modulus c.

In a similar way we can prove the next lemma.

Lemma 3.4. Let$\hat{I}$stands for a convex subset of$\bar{\mathrm{R}}$, $h:(0,1)\to(0,\infty)$ is a given function and c is a positive constant. Assume that $h:(0,1)\to(0,\infty)$ satisfy the condition $h(t)\le t,\ t\in(0,1)$if$F:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{R}$ is strongly harmonic mid h-convex with modulus c, then there exists a harmonic mid $h-$ convex function$\dot{g}:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{R}$such that $F(v)=\dot{g}(v)+c\left\|\frac{1}{v}\right\|^2$ where$v\in\hat{I}$.

Theorem 3.5. Let $F:\hat{I}=[m,n]\subset\bar{R}\{0\}\to\bar{\mathrm{R}}$ be strongly harmonic$h-$ convex function with modulus $c>0$. If $F\in L[m,n]$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[F\left(\frac{2mn}{m+n}\right)+\frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2\right]\le\frac{mn}{n-m}\int_m^n\frac{F(x)}{x^2}dx$$

$$\le[F(m)+F(n)]\int_0^1 h(t)dt-\frac{c}{6}\left\|\frac{m-n}{mn}\right\|^2 \quad (3)$$

Proof: Fix $m,n\in\hat{I},m<n$, and take $u=\frac{mn}{tm+(1-t)n}$, and $v=\frac{mn}{(1-t)m+tn}$ then the strongly harmonic $h-$convexity of F implies

$$F\left(\frac{2mn}{m+n}\right)=F\left(\frac{uv}{ut+(1-t)v}\right)$$

$$\le h(1-t)F(u)+h(t)F(v)-ct(1-t)\left\|\frac{u-v}{uv}\right\|^2,$$

$$\le h\left(\frac{1}{2}\right)\left(F(u)+F(v)\right)-\frac{c}{4}\left\|\frac{u-v}{uv}\right\|^2,$$

$$\le h\left(\frac{1}{2}\right)\left(F\left(\frac{mn}{tm+(1-t)n}\right)+F\left(\frac{mn}{(1-t)m+tn}\right)\right)-\frac{c}{4}\left\|\frac{(2t-1)m+(1-2t)n}{mn}\right\|^2,$$

Integrating the above inequality over the interval (0,1), we obtain

$$F\left(\frac{2mn}{m+n}\right)\le h\left(\frac{1}{2}\right)\left[\int_0^1 F\left(\frac{mn}{tm+(1-t)n}\right)dt+\int_0^1 F\left(\frac{mn}{(1-t)m+tn}\right)dt\right]-\frac{c}{4}\int_0^1\left\|\frac{(2t-1)m+(1-2t)n}{mn}\right\|^2 dt,$$

$$=h\left(\frac{1}{2}\right)\frac{2mn}{n-m}\int_0^1\frac{F(v)}{v^2}dv-\frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2.$$

This gives the left- hand side of the inequality of (3).

For the proof of the right- hand inequality of (3) we use the second part of the inequality. Integrating over the interval (0, 1), we get

$$\frac{mn}{n-m}\int_m^n\frac{F(v)}{v^2}dv=\int_0^1 F\left(\frac{mn}{tm+(1-t)n}\right)dt$$

$$\leq F(m)\int_0^1 h(1-t)dt + F(n)\int_0^1 h(t)dt - c\left\|\frac{m-n}{mn}\right\|^2\int_0^1 t(1-t)dt$$

$$\leq [F(m)+F(n)]\int_0^1 h(t)dt - \frac{c}{6}\left\|\frac{m-n}{mn}\right\|^2$$

which gives the right –hand side inequality of (3).

Remarks:

- In the case $c = 0$ the Hermite-Hadamard- type inequalities coincide with the Hermite-Hadamard- type inequalities for harmonically $h-$ convex functions proved by Noor[4]
- If $h(t) = t, t \in (0,1)$, then the inequalities (3) reduces to strongly harmonic convex function which have been proved by Noor[5]

$$F\left(\frac{2mn}{m+n}\right) + \frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2 \leq \frac{mn}{n-m}\int_m^n \frac{F(v)}{v^2}dv$$

$$\leq \frac{F(m)+F(n)}{2} - \frac{c}{6}\left\|\frac{m-n}{mn}\right\|^2$$

for$c = 0$ we get the classical Hermite-Hadamard inequalities.

- If $h(t) = t^s, t \in (0,1)$, then the inequalities (3) give

$$2^{s-1}\left[F\left(\frac{2mn}{m+n}\right) + \frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2\right] \leq \frac{mn}{n-m}\int_m^n \frac{F(v)}{v^2}dv$$

$$\leq \frac{F(m)+F(n)}{s+1} - \frac{c}{6}\left\|\frac{m-n}{mn}\right\|^2$$

for$c = 0$ it reduces to the Hermite-Hadamard inequalities for harmonically $s-convex$ functions.

- If $h(t) = \frac{1}{t}, t \in (0,1)$, then the inequalities (3) reduces to

$$\frac{1}{4}\left[F\left(\frac{2mn}{m+n}\right) + \frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2\right] \leq \frac{mn}{n-m}\int_m^n \frac{F(v)}{v^2}dv$$

$$\leq (\pm\infty)$$

for$c = 0$ it reduces to the Hermite-Hadamard inequalities for harmonically $Godunova-Levin$ functions.

- If $h(t) = t^{-s}, t \in (0,1)$, then the inequalities (3) reduces to

$$\frac{1}{2^{s+1}}\left[F\left(\frac{2mn}{m+n}\right) + \frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2\right] \leq \frac{mn}{n-m}\int_m^n \frac{F(v)}{v^2}dv$$

$$\leq \frac{F(m)+F(n)}{s-1} - \frac{c}{6}\left\|\frac{m-n}{mn}\right\|^2$$

for$c=0$ it reduces to the Hermite-Hadamard inequalities for harmonically $s-Godunova-Levin$ functions.

- If $h(t)=1, t\in(0,1)$, then the inequalities (3) reduces to

$$\frac{1}{2}\left[F\left(\frac{2mn}{m+n}\right)+\frac{c}{12}\left\|\frac{m-n}{mn}\right\|^2\right] \leq \frac{mn}{n-m}\int_m^n \frac{F(v)}{v^2}\,dv$$

$$\leq F(m)+F(n) - \frac{c}{6}\left\|\frac{m-n}{mn}\right\|^2$$

For $c=0$ it reduces to the Hermite-Hadamard inequalities for harmonically$P-convexfunctions$.

Theorem 3.6. Let $F,\dot{g}: \hat{I}\subset \bar{R}\{0\} \to \bar{R}$be strongly harmonic $h-convex$ functions with modulus $c>0$ if $\mathcal{F},\dot{g}\in L[m,n]$ then

$$\frac{mn}{n-m}\int_m^n \frac{F(x)\dot{g}\left(\frac{mnx}{(m+n)x-mn}\right)}{x^2}\,dx \leq$$

$$[F(m)\dot{g}(m)+F(n)\dot{g}(n)]\int_0^1 h(1-t)h(t)\,dt + F(m)\dot{g}(n)\int_0^1 (h(1-t))^2 dt +$$
$$F(n)\dot{g}(m)\int_0^1 (h(t))^2 dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(n)+\dot{g}(m)]\int_0^1 h(t)t(1-t)dt -$$
$$c\left\|\frac{n-m}{mn}\right\|^2 [F(m)+\dot{g}(n)]\int_0^1 h(1-t)t(1-t)dt + \frac{c^2}{30}\left\|\frac{n-m}{mn}\right\|^4.$$

Proof: Let F,$\dot{g}$ be strongly harmonic $h-$ convex functions with$c>0$. Then

$$\frac{mn}{n-m}\int_m^n \frac{F(x)\dot{g}\left(\frac{mnx}{(m+n)x-mn}\right)}{x^2}\,dx$$

$$= \int_m^n F\left(\frac{mn}{tm+(1-t)n}\right)\dot{g}\left(\frac{mn}{(1-t)m+tn}\right)dt,$$

$$\leq \int_0^1 \left[h(1-t)F(m)+h(t)F(n) - ct(1-t)\left\|\frac{m-n}{mn}\right\|^2\right]\left[h(t)\dot{g}(m)+h(1-t)\dot{g}(n) - ct(1-t)\left\|\frac{m-n}{mn}\right\|^2\right]dt,$$

$$= F(m)\dot{g}(m)\int_0^1 h(1-t)h(t)dt + F(m)\dot{g}(n)\int_0^1 (h(1-t))^2 dt$$
$$+ F(n)\dot{g}(m)\int_0^1 (h(t))^2 dt + F(n)\dot{g}(n)\int_0^1 h(1-t)h(t)\,dt$$

$-c\left\|\frac{m-n}{mn}\right\|^2 F(m)\int_0^1 h(1-t)t(1-t)\,dt - c\left\|\frac{m-n}{mn}\right\|^2 F(n)\int_0^1 h(t)t(1-t)\,dt -$
$c\left\|\frac{m-n}{mn}\right\|^2 \dot{g}(m)\int_0^1 h(t)t(1-t)dt - c\left\|\frac{m-n}{mn}\right\|^2 \dot{g}(n)\int_0^1 h(1-t)t(1-t)dt +$
$c^2\left\|\frac{m-n}{mn}\right\|^4 \int_0^1 t^2(1-t)^2 dt,$

$= [F(m)\dot{g}(m)+F(n)\dot{g}(n)]\int_0^1 h(1-t)h(t)\,dt + F(m)\dot{g}(n)\int_0^1 (h(1-t))^2 dt +$
$F(n)\dot{g}(m)\int_0^1 (h(t))^2 dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(n)+\dot{g}(m)]\int_0^1 h(t)t(1-t) - c\left\|\frac{n-m}{mn}\right\|^2 [F(m)+$
$\dot{g}(n)]\int_0^1 h(1-t)t(1-t)dt + \frac{c^2}{30}\left\|\frac{n-m}{mn}\right\|^4.$

This is the required result.
If $F=\dot{g}$ in Theorem 3.6, then it reduces to the following result.

Corollary 3.7. Let $F:\hat{I}\subset \bar{R}\{0\}\to \bar{R}$ be strongly harmonic $h-convex$ functions with modulus $c>0$ if $F\in L[m,n]$ then

$$\frac{mn}{n-m}\int_m^n \frac{F(x)F\left(\frac{mnx}{(m+n)x-mn}\right)}{x^2}dx \leq$$

$\left[F^2(m)+F^2(n)\right]\int_0^1 h(1-t)h(t)\,dt +$
$F(m)F(n)[\int_0^1 (h(1-t))^2 dt + \int_0^1 (h(t))^2 dt] - c\left\|\frac{n-m}{mn}\right\|^2 [F(n)+F(m)][\int_0^1 h(t)t(1-$
$t)dt + \int_0^1 h(1-t)t(1-t)dt] + \frac{c^2}{30}\left\|\frac{n-m}{mn}\right\|^4.$

Theorem 3.8. Let $F,\dot{g}:\hat{I}\subset \bar{R}\{0\}\to \bar{R}$be strongly harmonic $h-convex$ functions with modulus $c>0$ if $F,\dot{g}\in L[m,n]$, then

$$\frac{mn}{n-m}\int_m^n \frac{F(x)\dot{g}(x)}{x^2}dx \leq$$

$$F(m)\,\dot{g}(m)\int_0^1 [h(1-t)]^2 dt + [F(m)\,\dot{g}(n)+F(n)\,\dot{g}(m)]\int_0^1 h(1-t)h(t)dt +$$
$$F(n)\,\dot{g}(n)\int_0^1 [h(t)]^2\,dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(m)+\dot{g}(m)]\int_0^1 h(1-t)t(1-t)\,dt$$
$$-\,c\left\|\frac{n-m}{mn}\right\|^2 [F(n)+\dot{g}(n)]\int_0^1 h(t)t(1-t)\,dt + \frac{c^2}{30}\left\|\frac{n-m}{mn}\right\|^4.$$

Proof: Let $F,\dot{g}$ be strongly harmonic $h-convex$ functions with $c>0$. Then

$$\frac{mn}{n-m}\int_m^n \frac{F(x)\dot{g}(x)}{x^2}dx = \int_m^n F\left(\frac{mn}{tm+(1-t)n}\right)\dot{g}\left(\frac{mn}{tm+(1-t)n}\right)dt$$

$$\leq \int_0^1 \left[h(1-t)F(m) + h(t)F(n) - ct(1-t)\left\|\frac{n-m}{mn}\right\|^2\right]\left[h(1-t)\dot{g}(m) + h(t)\dot{g}(n) - ct(1-t)\left\|\frac{n-m}{mn}\right\|^2\right]dt$$

$$= F(m)\dot{g}(m)\int_0^1 [h(1-t)]^2 dt + [F(m)\dot{g}(n) + F(n)\dot{g}(m)]\int_0^1 h(1-t)h(t)dt + F(n)\dot{g}(n)\int_0^1 [h(t)]^2\, dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(m) + \dot{g}(m)]\int_0^1 h(1-t)t(1-t)\,dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(n) + \dot{g}(n)]\int_0^1 h(t)t(1-t)\,dt + c^2\left\|\frac{n-m}{mn}\right\|^4 \int_0^1 t^2(1-t)^2 dt$$

$$= F(m)\dot{g}(m)\int_0^1 [h(1-t)]^2 dt + [F(m)\dot{g}(n) + F(n)\dot{g}(m)]\int_0^1 h(1-t)h(t)dt + F(n)\dot{g}(n)\int_0^1 [h(t)]^2\, dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(m) + \dot{g}(m)]\int_0^1 h(1-t)t(1-t)\,dt - c\left\|\frac{n-m}{mn}\right\|^2 [F(n) + \dot{g}(n)]\int_0^1 h(t)t(1-t)\,dt + \frac{c^2}{30}\left\|\frac{n-m}{mn}\right\|^4$$

which is the required result.

If $F = \dot{g}$ in Theorem 3.8, then it reduces to the following result.

Corollary 3.9. Let $F: \hat{I} \subset \bar{R}\{0\} \to \bar{R}$ be strongly harmonic $h - convex$ functions with modulus $c > 0$ if $F \in L[m,n]$, then

$$\frac{mn}{n-m}\int_m^n \frac{F^2(x)}{x^2}dx \leq F^2(m)\int_0^1 [h(1-t)]^2 dt + 2F(m)F(n)\int_0^1 h(1-t)h(t)dt + F^2(n)\int_0^1 [h(t)]^2\, dt - c\left\|\frac{n-m}{mn}\right\|^2 2F(m)\int_0^1 h(1-t)t(1-t)\,dt - c\left\|\frac{n-m}{mn}\right\|^2 2F(n)\int_0^1 h(t)t(1-t)\,dt + \frac{c^2}{30}\left\|\frac{n-m}{mn}\right\|^4$$

4. CONCLUSION

New Hermite- Hadamard type inequalities for strongly harmonic $h - convex$ are established. This new class unifies several classes of harmonically convex functions which may inspire further research in this field.

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