ORIGINAL PAPER

SOME EXACT SOLUTIONS OF FIFTH ORDER EQUATION OF BURGERS HIERARCHY

UNAL IC[1]

Manuscript received: 19.09.2019; Accepted paper: 19.11.2019;
Published online: 30.12.2019.

Abstract. *In this paper, we implemented an improved tanh function Method for some exact solutions of fifth order equation of Burgers hierarchy. This method is presented by Chen and Zhang.*

Keywords: *Fifth order equation of Burgers hierarchy, improved tanh function method, exact solutions.*

1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics [1, 2]. Many analytical methods have been found in literature [3-8]. Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Using these methods, partial differential equations are transformed into ordinary differential equations. These nonlinear partial differential equations are solved with the help of ordinary differential equations. These methods are given in [9-16]. We used the improved tanh function method to find the exact solutions of fifth order equation of Burgers hierarchy in this study. This method is presented by Chen and Zhang [9].

2. ANALYSIS OF METHOD

Let's introduce the method briefly. Consider a general partial differential equation of two variables,

$$\varphi(v, v_t, v_x, v_{xx}, \dots) = 0, \tag{1}$$

and transform equation (1) with $v(x,t) = v(\emptyset)$, $\emptyset = k(x - \beta t)$ here k, β are constants. With this conversion, we obtain a nonlinear ordinary differential equation for $v(\emptyset)$,

$$\varphi'(v', v'', v''', \dots) = 0. \tag{2}$$

We can express the solution of equation (2) as below,

$$v(\phi) = \sum_{i=0}^{n} a_i F^i(\phi), \tag{3}$$

[1] Firat University, Faculty of Education, 23119 Elazig, Turkey. E-mail: unalic@firat.edu.tr

here is n a positive integer and is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation.

If we write these solutions in equation (2), we obtain a system of algebraic equations for $F(\emptyset), F^2(\emptyset), \dots, F^i(\emptyset)$ after, if the coefficients of $F(\emptyset), F^2(\emptyset), \dots, F^i(\emptyset)$ are equal to zero, we can find the $k, \beta, a_0, a_1, \dots, a_n$ constants.

The basic step of the method is to make full use of the Riccati equation satisfying the tanh function and to use $F(\emptyset)$, solutions. The Riccati equation required in this method is given below

$$F'(\phi) = A + BF(\phi) + CF^2(\phi). \tag{4}$$

Here, $F'(\emptyset) = \frac{dF(\emptyset)}{d\emptyset}$ and A, B and C are constants. The authors expressed the solutions [9].

Example 1. We consider the fifth order equation of Burgers hierarchy,

$$v_t + v_{xxxxx} + 10v_{xx}^2 + 15v_x v_{xxx} + 5vv_{xxxx} + 15v_x^3 + 50vv_x v_{xx} + 10v^2 v_{xxx} + 30v^2 v_x^2 + 10v^3 v_{xx} + 5v^4 v_x = 0. \tag{5}$$

Let us consider the traveling wave solutions $v(x,t) = v(z)$, $z = k(x - \beta t)$ then Eq. (5) becomes

$$-\beta v' + k^4 v^{(5)} + 10k^3 (v'')^2 + 15k^3 v' v''' + 5k^3 v v^{(4)} + 15k^2 (v')^3 + 50k^2 v v' v'' + 10k^2 v^2 v''' + 30kv^2 (v')^2 + 10kv^3 v'' + 5v^4 v' = 0, \tag{6}$$

when balancing $vv'v''$, with $v^{(5)}$ then $n = 1$ gives. The solution is as follows:

$$v = a_0 + a_1 F(\emptyset). \tag{7}$$

(7) is substituted in equation (6), a system of algebraic equations for k, β, a_0, a_1 are obtained. The obtained systems of algebraic equations are as follows:

$$\begin{aligned} &B^4 k^4 a_1 + 22A^2 B^2 C k^4 a_1 + 16A^3 C^2 k^4 a_1 - A\beta a_1 + 5AB^3 k^3 a_0 a_1 + 40A^2 BC k^3 a_0 a_1 \\ &\quad + 10AB^2 k^2 a_0^2 a_1 + 20A^2 C k^2 a_0^2 a_1 + 10ABk a_0^3 a_1 + 5A a_0^4 a_1 + 25A^2 B^2 k^3 a_1^2 \\ &\quad + 30A^3 C k^3 a_1^2 + 50A^2 B k^2 a_1^2 a_0 + 30A^2 k a_0^2 a_1^2 + 15A^3 k^2 a_1^3 = 0, \end{aligned}$$

$$\begin{aligned} &B^5 k^4 a_1 + 52AB^3 C k^4 a_1 + 136A^2 BC^2 k^4 a_1 - B\beta a_1 + 5B^4 k^3 a_0 a_1 + 110AB^2 C k^3 a_0 a_1 \\ &\quad + 80A^2 C^2 k^3 a_0 a_1 + 10B^3 k^2 a_0^2 a_1 + 80ABC k^2 a_0^2 a_1 + 10B^2 k a_0^3 a_1 \\ &\quad + 20AC k a_0^3 a_1 + 5B a_0^4 a_1 + 55AB^3 k^3 a_1^2 + 230A^2 BC k^3 a_1^2 + 120AB^2 k^2 a_1^2 a_0 \\ &\quad + 140A^2 C k^2 a_1^2 a_0 + 90ABk a_0^2 a_1^2 + 20A a_0^3 a_1^2 + 95A^2 B k^2 a_1^3 + 60A^2 k a_0 a_1^3 \\ &\quad = 0, \end{aligned}$$

$$\begin{aligned} &31B^4 C k^4 a_1 + 292C^2 B^2 A k^4 a_1 + 136A^2 C^3 k^4 a_1 - C\beta a_1 + 75B^3 C k^3 a_0 a_1 \\ &\quad + 300C^2 BA k^3 a_0 a_1 - 70CB^2 k^2 a_0^2 a_1 + 80C^2 A k^2 a_0^2 a_1 + 30CBk a_0^3 a_1 \\ &\quad + 5C a_0^4 a_1 + 30B^4 k^3 a_1^2 + 450AB^2 C k^3 a_1^2 + 270A^2 C^2 k^3 a_1^2 + 70B^3 k^2 a_0 a_1^2 \\ &\quad + 460ABC k^2 a_0 a_1^2 + 60B^2 k a_0^2 a_1^2 + 120AC k a_0^2 a_1^2 + 20B a_0^3 a_1^2 \\ &\quad + 155AB^2 k^2 a_1^3 + 165CA^2 k^2 a_1^3 + 150ABk a_0 a_1^3 + 30A a_0^2 a_1^3 + 30A^2 k a_1^4 \\ &\quad = 0, \end{aligned}$$

$$
\begin{aligned}
180B^3C^2k^4a_1 &+ 480ABC^3k^4a_1 + 250B^2C^2k^3a_0a_1 + 200AC^3k^3a_0a_1 + 120BC^2k^2a_0^2a_1 \\
&+ 20C^2ka_0^3a_1 + 255B^3Ck^3a_1^2 + 880ABC^2k^3a_1^2 + 340B^2Ck^2a_0a_1^2 \\
&+ 360AC^2k^2a_0a_1^2 + 150BCka_0^2a_1^2 + 20Ca_0^3a_1^2 + 75B^3k^2a_1^3 + 470ABCk^2a_1^3 \\
&+ 90B^2ka_0a_1^3 + 180ACka_0a_1^3 + 30Ba_0^2a_1^3 + 70ABka_1^4 + 20Aa_0a_1^4 = 0,
\end{aligned}
$$

$$
\begin{aligned}
390B^2C^3k^4a_1 &+ 240AC^4k^4a_1 + 300BC^3k^3a_0a_1 + 60C^3k^2a_0^2a_1 + 665B^2C^2k^3a_1^2 \\
&+ 490BC^2k^2a_0a_1^2 + 90C^2ka_0^2a_1^2 + 315B^2Ck^2a_1^3 + 325AC^2k^2a_1^3 \\
&+ 210BCka_0a_1^3 + 30Ca_0^2a_1^3 + 40B^2ka_1^4 + 80ACka_1^4 + 20Ba_0a_1^4 + 5Aa_0a_1^5 \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
360BC^4k^4a_1 &+ 120C^4k^3a_0a_1 + 690BC^3k^3a_1^2 + 220C^3k^2a_0a_1^2 + 415BC^2k^2a_1^3 \\
&+ 120C^2ka_0a_1^3 + 90BCka_1^4 + 20Ca_0a_1^4 + 5Ba_1^5 = 0,
\end{aligned}
$$

$$120C^5k^4a_1 + 250C^4k^3a_1^2 + 175C^3k^3a_1^3 + 50C^2ka_1^4 + 5Ca_1^5 = 0. \tag{8}$$

If the system is solved, the coefficients are found as,

Case 1:

$$\mathrm{a}_0 = 0, \mathrm{B} = 0 \;\; C \neq 0, \qquad k = -\frac{a_1}{2C}, \qquad \beta = \frac{A^2a_1^4}{C^2},\; a_1 \neq 0.$$

Case 2:

$$Aa_0 \neq 0,\; B = 0, a_1 = \sqrt{\tfrac{2C}{A}}\,a_0,\; k = -\tfrac{a_0^2}{Aa_1},\; \beta = -11a_0^4,\; a_1 \neq 0. \tag{9}$$

with the help of the Mathematica program. After these operations, The solutions of equation (5) for case 1 and case 2 are as follows:

Solutions 1

$$v_1(x,t) = a_1[Coth(a_1x - a_1^5t) \pm Cosech(a_1x - a_1^5t)]$$

$$v_2(x,t) = a_1[Tanh(a_1x - a_1^5t) + iSech(a_1x - a_1^5t)] \tag{10}$$

Solutions 2

$$v_3(x,t) = a_1[Sec(-a_1x + a_1^5t) + Tan(-a_1x + a_1^5t)]$$

$$v_4(x,t) = a_1[Cosec(-a_1x + a_1^5t) - Cot(-a_1x + a_1^5t)]$$

$$v_5(x,t) = a_1[Sec(a_1x - a_1^5t) - Tan(a_1x - a_1^5t)]$$

$$v_6(x,t) = a_1[Cosec(a_1x - a_1^5t) + Cot(a_1x - a_1^5t)] \tag{11}$$

Solutions 3

$$v_7(x,t) = a_1\left[Tanh\left(\frac{a_1}{2}x - \frac{a_1^5}{2}t\right)\right]$$

$$v_8(x,t) = a_1\left[Coth\left(\frac{a_1}{2}x - \frac{a_1^5}{2}t\right)\right] \tag{12}$$

Solutions 4

$$v_9(x,t) = a_1\left[Tan\left(-\frac{a_1}{2}x + \frac{a_1^5}{2}t\right)\right] \tag{13}$$

Solutions 5

$$v_{10}(x,t) = a_1\left[Cot\left(\frac{a_1}{2}x - \frac{a_1^5}{2}t\right)\right] \tag{14}$$

Solutions 6

$$v_{11}(x,t) = a_0 + \sqrt{2}ia_0\left[Coth\left(\sqrt{2}ia_0x + 11\sqrt{2}ia_0^5t\right) \pm Cosech\left(\sqrt{2}ia_0x + 11\sqrt{2}ia_0^5t\right)\right]$$

$$v_{12}(x,t) = a_0 + \sqrt{2}ia_0\left[Tanh\left(\sqrt{2}ia_0x + 11\sqrt{2}ia_0^5t\right) \pm iSech\left(\sqrt{2}ia_0x + 11\sqrt{2}ia_0^5t\right)\right] \tag{15}$$

Solutions 7

$$v_{13}(x,t) = a_0 + \sqrt{2}a_0\left[Sec\left(-\sqrt{2}a_0x - 11\sqrt{2}a_0^5t\right) + Tan\left(-\sqrt{2}a_0x - 11\sqrt{2}a_0^5t\right)\right]$$

$$v_{14}(x,t) = a_0 + \sqrt{2}a_0\left[Cosec\left(-\sqrt{2}a_0x - 11\sqrt{2}a_0^5t\right) - Cot\left(-\sqrt{2}a_0x - 11\sqrt{2}a_0^5t\right)\right]$$

$$v_{15}(x,t) = a_0 + \sqrt{2}a_0\left[Sec\left(\sqrt{2}a_0x + 11\sqrt{2}a_0^5t\right) - Tan\left(\sqrt{2}a_0x + 11\sqrt{2}a_0^5t\right)\right]$$

$$v_{16}(x,t) = a_0 + \sqrt{2}a_0\left[Cosec\left(\sqrt{2}a_0x + 11\sqrt{2}a_0^5t\right) + Cot\left(\sqrt{2}a_0x + 11\sqrt{2}a_0^5t\right)\right] \tag{16}$$

Solutions 8

$$v_{17}(x,t) = a_0 + \sqrt{2}ia_0\left[Tanh\left(\frac{a_0i}{\sqrt{2}}x + \frac{11ia_0^5}{\sqrt{2}}t\right)\right]$$

$$v_{18}(x,t) = a_0 + \sqrt{2}ia_0\left[Coth\left(\frac{a_0i}{\sqrt{2}}x + \frac{11ia_0^5}{\sqrt{2}}t\right)\right] \tag{17}$$

Solutions 9

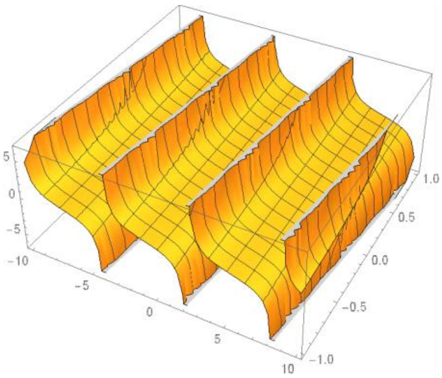
$$v_{19}(x,t) = a_0 + \sqrt{2}a_0\left[Tan\left(-\frac{a_0}{\sqrt{2}}x - \frac{11a_0^5}{\sqrt{2}}t\right)\right] \tag{18}$$

Solutions 10

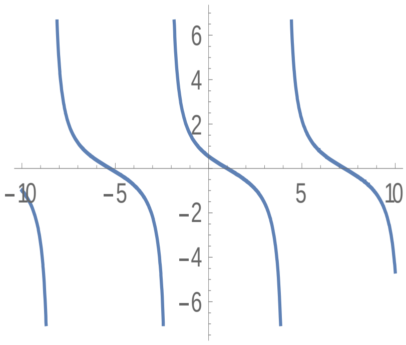
$$v_{20}(x,t) = a_0 + \sqrt{2}a_0 \left[Cot\left(\frac{a_0}{\sqrt{2}}x + \frac{11a_0^5}{\sqrt{2}}t\right)\right] \tag{19}$$

3. EXPLANATIONS AND GRAPHICAL PRESENTMENTS OF SOME OBTAINED SOLUTIONS

The graphs of some solutions of Equation (5) are as follows:

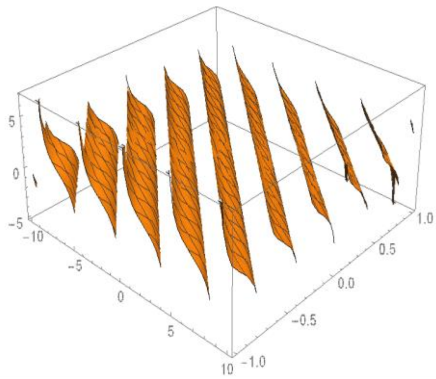


i)

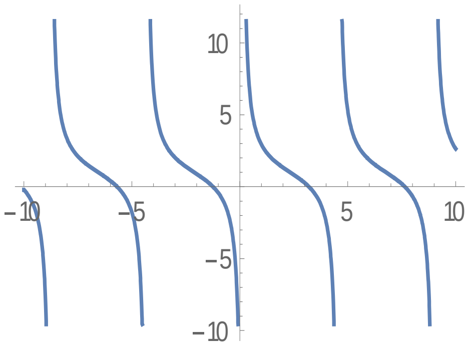


ii)

Figure 1. i) The 3D surfaces of solution v_4 for the value $a_1 = 1$ within the interval $-10 \le x \le 10, -1 \le t \le 1$. ii) The 2D surfaces of solution v_4 for the values $a_1 = 1$ and t=1 within the interval $-10 \le x \le 10$.



i)



ii)

Figure 2. i) The 3D surfaces of solution v_{19} for the value $a_0 = 1$ within the interval $-10 \le x \le 10, -1 \le t \le 1$. ii) The 2D surfaces of solution v_{19} for the values $a_0 = 1$ and t=1 within the interval $-10 \le x \le 10$.

4. CONCLUSION

We used the improved tanh function method to find the exact solutions of fifth order equation of Burgers hierarchy. This method has been successfully applied to solve some nonlinear wave equations and can be used to many other nonlinear equations or coupled ones.

REFERENCES

[1] Debtnath, L., *Nonlinear Partial Differential Equations for Scientist and Engineers*, Birkhauser, Boston, MA, 1997.

[2] Wazwaz, A.M., *Partial Differential Equations: Methods and Applications*, Balkema, Rotterdam, 2002

[3] Shang, Y., *Applied Mathematics and Computation*, **187**, 1286, 2007.

[4] Abourabia, A.M., El Horbaty, M.M., *Chaos Solitons Fractals*, **29**, 354, 2006.

[5] Malfliet, W., *American Journal of Physics*, **60**, 650, 1992.

[6] Cariello, F., Tabor, M., *Physica*, D **39**, 77, 1989.

[7] Fan, E., *Physics Letters A*, **265**, 353, 2000.

[8] Clarkson, P.A., *Journal of Physics A: Mathematical and General*, 22, 2355, 1989.

[9] Chen, H., Zhang, H., *Chaos Soliton Fract*, **19**, 71, 2004.

[10] Fu, Z., Liu, S., Zhao, Q., *Physics Letters A*, **290**, 72, 2001.

[11] Chen, Y., Wang, Q., Li, B., *Verlag der Zeitschrift f ̈ur Naturforschung, Tubingen A*, **59**, 529, 2004.

[12] Chen, Y., Yan, Z., *Chaos Soliton Fract*, **29**, 948, 2006.

[13] Wang, M., Li, X., Zhang, J., *Physics Letters A*, **372**, 417, 2008.

[14] Li, L., Li, E., Wang, M., *Applied Mathematics-A J. Chin. U*, **25**, 454, 2010.

[15] Manafian, J., *Optik*, **127**, 4222, 2016.

[16] He, J.H., Wu, X.H., *Chaos Solitons Fractals*, **30**, 700, 2006.