ORIGINAL PAPER

PERIOD-DOUBLING BIFURCATION ANALYSIS AND STABILITY OF EPIDEMIC MODEL

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Abstract. *This study is concerned with finding the threshold parameter that determines the status of infected individuals in a discrete-time SIS disease model transmitting the infection to other individuals and determining the number of individuals catching the infection. In this study, we firstly examined the equilibrium points of the model, and we determined the presence of a single positive equilibrium point depending on the number of diseased individuals. Then, based on the threshold parameter, we investigated the local asymptotic stability conditions. Moreover, we provided a topological classification of these equilibria. Finally, we obtained the condition providing the emergence of "period-doubling bifurcation" in the given model. The theoretical results that were obtained were verified with numerical examples by using the Mathematica software.*

Keywords: *Epidemic Model, Stability, Bifurcation, Equilibrium Point*

1. INTRODUCTION

One of the prominent ways of investigating the basic problems that exist in life is mathematical modelling. It is possible to investigate problems that arise in biology, zoology, agriculture, chemistry, engineering, physics, medicine and several other branches of science by using discrete-time and continuous-time mathematical models [1-8]. Especially because of the significance of diseases in human life, investigating these models carries great significance. Epidemic diseases are also a part of such research. In such diseases, complete recovery is out of the question. The spreading rate of the disease, the temporary recovery process of the patient or effects that speed up the temporary recovery process are factors that constitute topics of research. The SIS model is one of such models, and it divides a community into two groups as susceptible individuals and infectious individuals. In the model, $S(t)$ refers to susceptible individuals over a time t, while $I(t)$ refers to infectious individuals over a time t. The N value that refers to the total population is constant. In epidemic diseases such as bacterial throat infections, meningitis, gonorrhea and syphilis, infected individuals may recover, but the infection does not form immunity. The first contributions to the epidemic disease model were made by Hamer (1906) [9], Ross (1911) [10] and Kermack and McKendrick (1927) [2]. An in-depth analysis of an SIS epidemic model formed on gonorrhea was discussed by Hethcote et al. [7].

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2. MATERIALS AND METHODS

2.1. MATERIALS

In this study, we will take on the following model:

$$\begin{aligned} S_{t+1} &= S_t - \frac{\beta}{N} I_t S_t + (b+\gamma) I_t \\ I_{t+1} &= I_t (1-\gamma-b) + \frac{\beta}{N} I_t S_t \end{aligned} \tag{1}$$

where $b, \gamma > 0$, the conditions of $0 < b+\gamma < 1$ and $0 < \beta < 1$ are applicable. Additionally, $I_0, S_0 > 0$ and $I_0 + S_0 = N$. The parameter b that is used in this model corresponds to the probability of death. The parameter β refers to the average number of contacts that facilitate infection through infected individuals. The $\frac{\beta S}{N}$ ratio is the number of contacts of a susceptible individual with an infected individual, while the $\frac{\beta SI}{N}$ ratio is the total number of contacts of the infected class that result in infection. Additionally, individuals are born susceptible, and the disease is not carried from mothers to offspring. The parameter γ represents the probability of recovery, and the $\frac{1}{\gamma}$ ratio refers to the average length of infection that does not result in death. The length of the infection period may be shortened due to death. This is why the $\frac{1}{b+\gamma}$ ratio is the length of infection in the case of including deaths.

As the relationship of $S_t + I_t = N$ is known, if $I_t = N - S_t$ placed in Eq. (1),

$$\begin{aligned} S_{t+1} &= S_t - \frac{\beta}{N} I_t S_t + (b+\gamma)(N - S_t) \\ I_{t+1} &= I_t (1-\gamma-b) + \frac{\beta}{N} I_t S_t \end{aligned} \tag{2}$$

is obtained.

Now, let us consider following Jacobian matrix that is taken at an equilibrium point (x, y) to study the dynamics of the system (1)

$$J(x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3}$$

The characteristic equation is written as

$$\lambda^2 - Tr(J)\lambda + \det J = 0 \tag{4}$$

where $Tr(J) = a_{11} + a_{22}$ and $\det(J) = a_{11}a_{22} - a_{12}a_{21}$.

Theorem 1: Let λ_1 and λ_2 be the roots of the characteristic equation (4). If $|\lambda_{1,2}| < 1$ or $|Tr(J)| < 1 + \det J < 2$, the equilibrium point (x, y) is locally asymptotically stable [4].

Definition 1: The following situaitons are valid for the equilibrium point (S, I) of any system [6]

(i) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, it is a **sink** point and locally asymptotically stable;

(ii) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, it is a **source** point and locally unstable;

(iii) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), it is a **saddle** point;

(iv) If $|\lambda_1| = 1$ or $|\lambda_2| = 1$ it is **non-hyperbolic**.

Theorem 2: Let's take $F(x) = x^2 + Bx + C$ such that $F(1) > 0$ [6]. Also x_1 and x_2 are two roots of $F(x) = 0$. So

(i) If $F(-1) > 0$ and $C < 1, |x_1| < 1$ and $|x_2| < 1$;

(ii) If $F(-1) < 0, |x_1| > 1$ and $|x_2| < 1$ (or $|x_1| < 1$ and $|x_2| > 1$);

(iii) If $F(-1) > 0$ and $C > 1$, $|x_1| > 1$ and $|x_2| > 1$;

(iv) If $F(-1) = 0$ and $B \neq 0, 2$, $x_1 = -1$ and $|x_2| \neq 1$;

(v) If $B^2 - 4C < 0$ and $C = 1$, x_1 and x_2 complex roots, and $|x_1| = |x_2| = 1$

are valid.

2.2. METHODS

This section provides the mathematical analyses required to examine the status of the individuals who are infectious in the population discussed in System (1).

2.2.1. Dynamics of System (1) Depending on Constrained Positive Parameters

In this section, under the conditions of $0 < b + \gamma < 1$ and $0 < \beta < 1$, the stability analyses of System (1) are carried out such that $b, \gamma > 0$. We obtained the following theorem for System (1).

Theorem 3: System (1)

(i) has disease-free (not containing the number of infected individuals) equilibrium point $E_0 = (N, 0)$ for all positive parameter values.

(ii) has a unique diseased (containing the number of infected individuals) equilibrium point $E_1 = \left(\frac{N(\gamma + b)}{\beta}, N\left[\frac{\beta - (\gamma + b)}{\beta} \right] \right)$ when $\beta > \gamma + b$.

Proof: Equilibrium points of System (1) are

$$\bar{S} = \bar{S} - \frac{\beta}{N}\overline{IS} + (b+\gamma)(N-\bar{S})$$
$$\bar{I} = \bar{I}(1-\gamma-b) + \frac{\beta}{N}\overline{IS}.$$

By solving the equations above, we obtain $E_0 = (N,0)$ for all positive parameter values and $E_1 = \left(\bar{S},\bar{I}\right) = \left(\frac{N(\gamma+b)}{\beta}, N\left[\frac{\beta-(\gamma+b)}{\beta}\right]\right)$ when $\beta > \gamma + b$.

Let us give the theorem below which provides information about the local asymptotic stability of the equilibrium points E_0.

Theorem 4: If $R_0 = \frac{\beta}{\gamma+b} < 1$, then equilibrium point $(N,0)$ is locally asymptotically stable.

Proof: The Jacobian matrix of System (1) is found with a simple calculation as

$$J(S,I) = \begin{pmatrix} 1-\frac{\beta}{N}I-(\gamma+b) & \frac{-\beta}{N}S \\ \frac{\beta}{N}I & (1-\gamma-b)+\frac{\beta}{N}S \end{pmatrix}. \tag{5}$$

Moreover, the Jacobian matrix in the neighbourd of the disease-free equilibrium point is

$$J(N,0) = \begin{pmatrix} 1-(\gamma+b) & -\beta \\ 0 & (1-\gamma-b)+\beta \end{pmatrix}. \tag{6}$$

The eigenvalues of this matrix are obtained as $\lambda_1 = 1-(\gamma+b)$ and $\lambda_2 = (1-\gamma-b)+\beta$. From Theorem 1, it is clear that the equilibrium point $(N,0)$ is locally asymptotically stable.

Remark 1: The $\frac{\beta}{\gamma+b}$ ratio which is denoted by R_0 is the threshold parameter of the *SIS* epidemic model, and this threshold parameter is called the "basic reproduction rate" or the "basic reproduction number". The $R_0 = \frac{\beta}{\gamma+b}$ ratio has a biological interpretation. The R_0 value is the number of secondary infections caused by one infected person in the infection period (Anderson and May, 1991). If $R_0 < 1$ then there is only one equilibrium point as the disease-free equilibrium point, and this equilibrium point is locally asymptotically stable.

Theorem 5: The following topological classifications are applicable for the equilibrium point E_0 of System (1):

(i) If $\beta < \gamma + b$, it is a "**sink**" point,

(ii) If $\beta > \gamma + b$, it is a "**source**" point,
(iii) If $\beta = \gamma + b$, it is a "**non-hyperbolic**" point.

Proof: It is clear from Theorem 4.

Example 1. At a population density of $N = 100$, with the parameters $b = 0.5; \gamma = 0.1;$ $\beta = 0.5$, let's take the following system

$$S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + 0.6(100 - S_t)$$
$$I_{t+1} = 0.4 I_t + \frac{0.5}{100} I_t S_t$$

where the number of susceptible individuals are $S = 70$ and the number of infected individuals are $I = 30$.

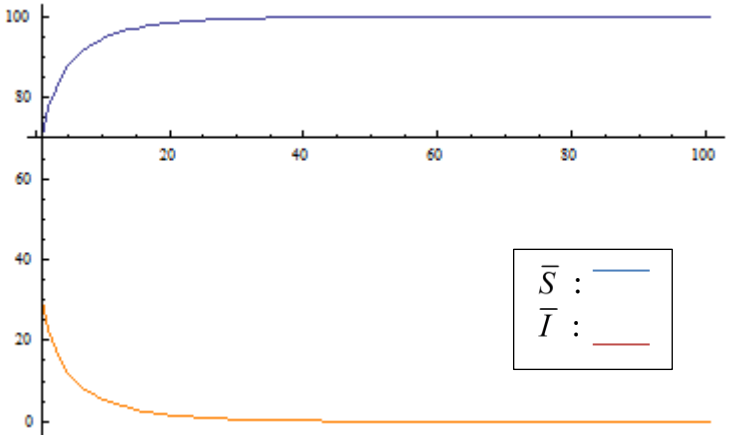


Figure 1. The time series plot of System (1) with the parameter values of $b = 0.5; \gamma = 0.1$ and $\beta = 0.5$

Now, to continue the analysis of the *SIS* epidemic model, let us assume that $R_0 > 1$. When we look at the local asymptotic stability condition for the endemic (diseased) equilibrium point, we obtain the theorem below. With this theorem, we express the local asymptotic stability condition of the equilibrium point E_1 .

Theorem 6: Assume that the trace of the matrix $J(\bar{S}, \bar{I})$ of System (1) is non-negative. If $1 < R_0 < \frac{2}{(b+\gamma)}$ then the endemic equilibrium point is locally asymptotically stable.

Proof: The Jacobian matrix form taken in the neighbourd of the endemic equilibrium point is obtained as follows:

$$J(\bar{S}, \bar{I}) = \begin{pmatrix} 1-(b+\gamma)R_0 & \frac{-\beta}{R_0} \\ (b+\gamma)(R_0 - 1) & 1 \end{pmatrix}. \quad (7)$$

From this, we get

$$Tr(J)=2-(b+\gamma)R_0 \text{ and } \det(J)=1-(b+\gamma)R_0+\frac{\beta}{R_0}(b+\gamma)(R_0-1). \quad (8)$$

By using theorem 1; the local asymptotic stability condition

$$2-(b+\gamma)R_0<2-(b+\gamma)R_0+\frac{\beta}{R_0}(b+\gamma)(R_0-1)<2 \quad (9)$$

can be written, and if this statement is simplified then we get

$$0<\beta(1-\frac{1}{R_0})<R_0 \ . \quad (10)$$

It is clear that the inequality $\beta(1-\frac{1}{R_0})<1$ holds since $R_0>1$ and $\beta<1$. With the assumption of the trace of the matrix $J(\bar{S},\bar{I})$ is non-negative, $R_0<\frac{2}{b+\gamma}$ is true.

Example 2. At the population density of $N=100$, with the parameter values of $b=0.5$; $\gamma=0.1$; $\beta=0.8$, let's take the following system

$$S_{t+1}=S_t-\frac{\beta}{N}I_tS_t+0.6(100-S_t)$$
$$I_{t+1}=0.4I_t+\frac{0.8}{100}I_tS_t$$

where the number of susceptible individuals are $S=70$ and the number of infected individuals are $I=30$.

The equilibrium point $(75,25)$ of the system is locally asymptotically stable.

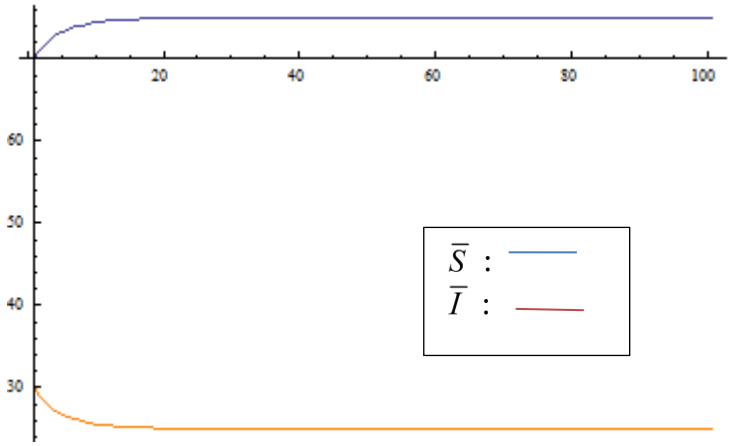


Figure 2. The time series plot of System (1) with the parameter values of $b=0.5$; $\gamma=0.1$ **and** $\beta=0.8$

2.2.2. Dynamics of System (1) Depending on Unconstrained Positive Parameters

Let us take System (1) without any constraint such that $b>0$, $\gamma>0$ and $\beta>0$. Considering Definition 1, a topological classification for the equilibrium point E_1 is given with the following theorem.

Theorem 7: Let's $\beta>\gamma+b$.

(i) If $\beta<\gamma+b+2$ for $0<\gamma+b<2$ then the equilibrium point E_1 of System (1) is a "**sink**" point.

(ii) If $\beta>\gamma+b+2$ for $\gamma+b>2$, then the equilibrium point E_1 of System (1) is a "**source**" point.

(iii) If $0<\gamma+b<2$ and $\gamma+b+2<\beta$ or $\gamma+b>2$ and $\beta<\gamma+b+2$, then the equilibrium point E_1 of System (1) is a "**saddle"** point.

(iv) If $\beta=2+\gamma+b$ for $b+\gamma\neq 2,\beta$ then the equilibrium point E_1 of System (1) is a point of "**period-doubling bifurcation**".

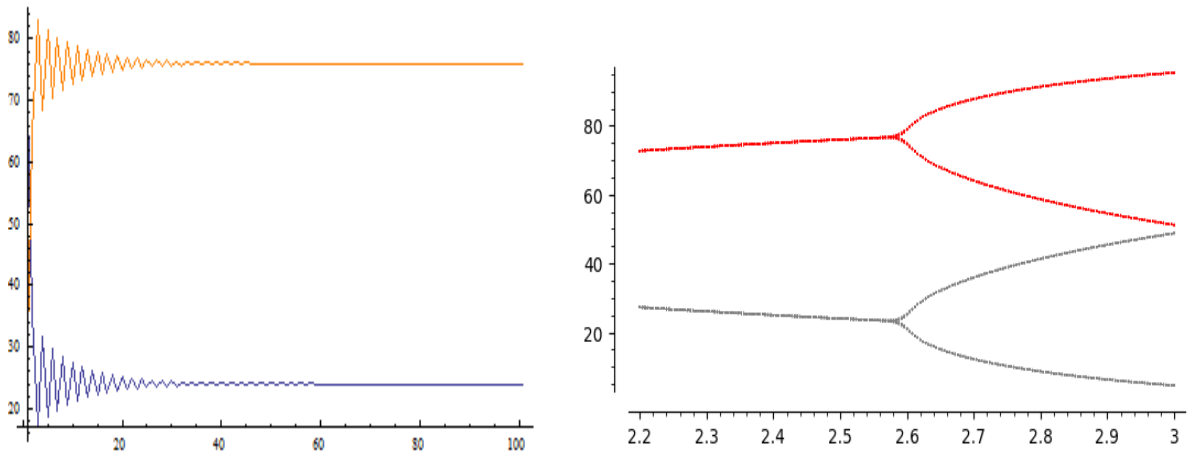
Proof: It is clear from Theorem 2.

Example 3. At the population density of $N=100$, with the parameter values of $b=0.5$; $\gamma=0.1$; $\beta=2.5$, let's take the following system

$$S_{t+1}=S_t-\frac{\beta}{N}I_tS_t+0.6(100-S_t)$$
$$I_{t+1}=0.4I_t+\frac{2.5}{100}I_tS_t$$

where the number of susceptible individuals are $S=70$ and the number of infected individuals are $I=30$.

The equilibrium point $E_1=(24,76)$ of the system is locally asymptotically stable.



(a) **(b)**

Figure 3. (a) The time series plot of System (1) with the parameter values of $b=0.5$; $\gamma=0.1$ and $\beta=2.5$ (b) The bifurcation plot based on $b=0.5$; $\gamma=0.1$ and the values in the interval of $\beta\in(2.2,3)$

3. RESULTS AND DISCUSSION

3.1. RESULTS

From Fig. 1, the equilibrium point $E_0 = (100, 0)$ of the system is locally asymptotically stable. In Example 1, a population of 100 individuals was considered. Looking at the initial status, it is clear that 70 individuals were susceptible, while 30 were carrying the disease. As the rate of contact between infected individuals and susceptible individuals had a small value, it is seen that the 30 patients recovered in time, and 100 individuals continued their lives as susceptible. Considering the selected parameter values in the mathematical sense, it is seen that $\beta < \gamma + b$, and the equilibrium point $(100, 0)$ of System (1) is locally asymptotically stable. Moreover, the condition $R_0 = 0.83 < 1$ shows the presence of an equilibrium point where there are no infected individuals, and this equilibrium point is a unique point.

The Fig. 2 shows that, in a population of 100 where 70 individuals were susceptible and 30 were carrying the disease, the number of infected individuals dropped to 25 in time, and the number of susceptible individuals increased from 70 to 75. Looking at the selected parameter values in the mathematical sense, it is seen that $\beta > \gamma + b$, and the equilibrium point $E_1 = (75, 25)$ of System (1) is locally asymptotically stable. Moreover, $R_0 = 1.33 > 1$, and from Theorem 6, the value $1 < R_0 < 3.33$ shows the presence of an equilibrium point containing the number of infected individuals and that this unique equilibrium point is stable.

As seen in Fig. 3(a), because the rate of contact between infected individuals and susceptible individuals was higher than the rate of recovery, we see that the number of infected individuals increased from 30 to 76 in time, while the number of susceptible individuals decreased from 70 to 24. In this case, the disease was spread. Looking at the selected parameter values in the mathematical sense, it is seen that $\beta < \gamma + b + 2$, and the equilibrium point $E_1 = (24, 76)$ of System (1) is locally asymptotically stable.

In Fig. 3(b), we see how the equilibrium points of the population varied based on the β contact rate values while keeping the $b = 0.5; \gamma = 0.1$ values constant. There was a bifurcation at the $\beta = 2.6$ value. The dynamicity of the population displays a change at this point. This situation provides information on the numbers of carrying the disease as the contact rate of the disease changes.

3.2. DISCUSSION

Keeping the initial conditions $(S, I) = (70, 30)$ constant, when we observe the changes in the case of changing the rates of recovery or mortality, we see that different dynamics are formed.

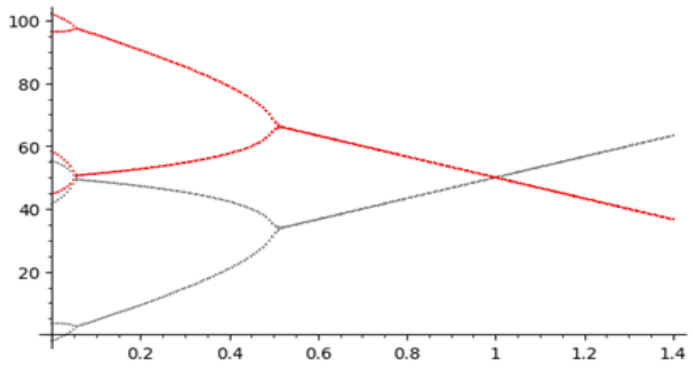
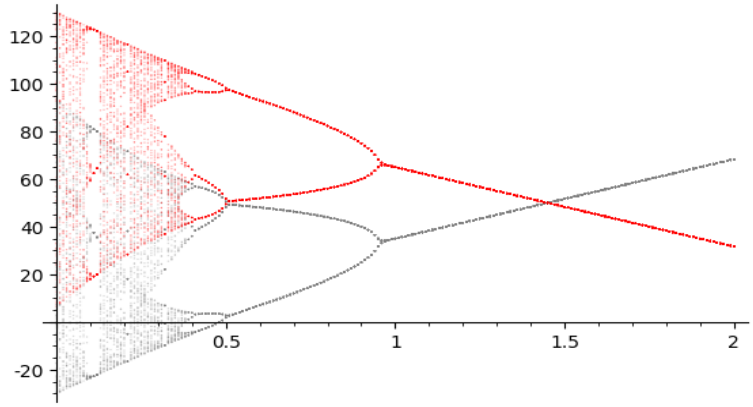
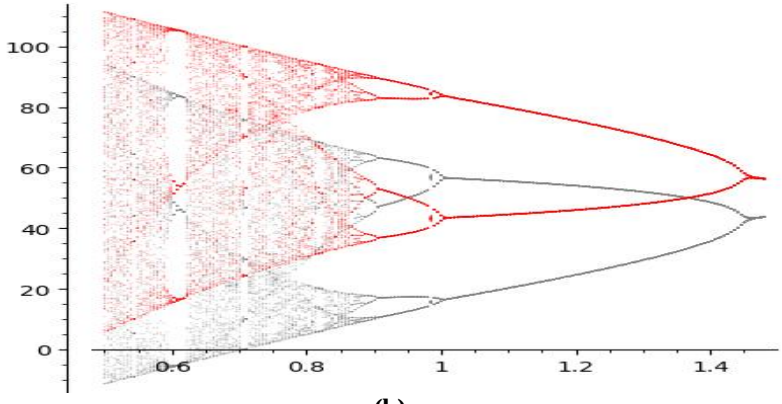


Figure 4. The bifurcation plot based on $b=0.5$; $\beta=3$ and the values in the interval $\gamma\in(0,1.4)$

As seen in Fig. 4, a bifurcation plot was obtained on the varying values of recovery rates where the parameters $\beta=3$, $b=0.5$ were constant. At $\gamma=0.5$, after "period-doubling bifurcation", it is seen that stable equilibrium points emerged against the increasing values of the recovery rate. As the recovery rate was high, it is seen that the infected population decreased in time.



(a)



(b)

Figure 5. (a) The bifurcation plot based on $b=0.05$; $\beta=3$ and the values in the interval $\gamma\in(0,2)$
(b) The bifurcation plot based on $b=0.05$; $\beta=3.5$ and the values in the interval $\gamma\in(0.5,1.5)$

As seen in Fig. 5(a), a bifurcation plot was obtained on the varying values of recovery rates where the parameters $\beta=3$, $b=0.05$ were constant. For the value of the probability of mortality $b=0.05$, if the rate of the recovery was low, then a chaos state emerged. As seen here, at the value of $\gamma=0.95$, period-doubling bifurcation disappeared, and there was a transition into a stable state as the recovery rate increased. As seen in Fig. 5(b), when the

parameters of $\beta = 3.5$ and $b = 0.05$ are taken, it is seen that there was a chaos corresponding to the low values of γ. As seen here, at the value of $\gamma = 1.45$, period-doubling bifurcation disappeared, and there was a transition into a stable state as the recovery rate increased.

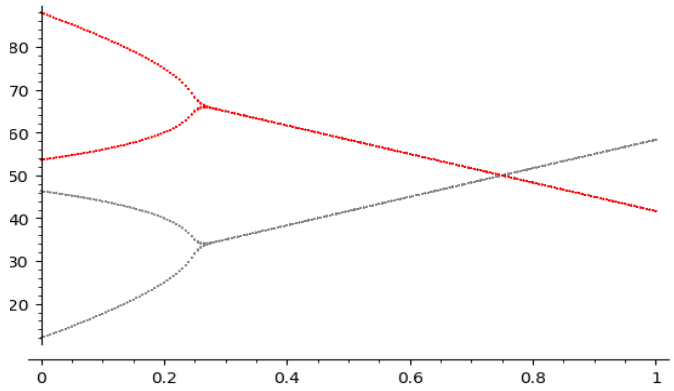


Figure 6. The bifurcation plot based on $\gamma = 0.75$; $\beta = 3$ and the values in the interval $b \in (0,1)$.

As seen in Fig. 6, a bifurcation plot was obtained on the varying values of mortality rates where the parameters $\beta = 3$, $\gamma = 0.75$ were constant. As seen here, at the value of $b = 0.25$, period-doubling bifurcation disappeared, and there was a transition into a stable state with increased recovery rate as the mortality rate increased.

4. CONCLUSION

This study presents the existence of equilibrium points of a discrete-time epidemic system and the spreading analysis of the disease based on these equilibrium points. Constraints on the parameter values and the changes in population in unconstrained conditions were investigated. Using data on the recovery rate, contact rate and mortality rate, the changes in the number of infected individuals were examined via plots.

REFERENCES

[1] Anderson R.M., May, R.M., *Philos Trans. R. Soc. Lond. B*, **314**, 533, 1986.
[2] Kermack, W.O., McKendrick, A.G., *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **115**, 772, 1927.
[3] Edelstein-Keshet, L., *Mathematical Models in Biology*, SIAM, Random House, New York, 2005.
[4] Allen, L.J.S., *An Introduction to Mathematical Biology*, Pearson, New Jersey, 2007.
[5] Din, Q., Gümüş, ÖA, Khalil, H., *Z. Naturforsch. A.*, **72**, 25, 2017.
[6] Liu, X., Xiao, D., *Solution Fract.*, **32**, 80, 2007.
[7] Hethcote, H.W., Yorke, J.A., *Gonorrhea Transmission Dynamics and Control*, Springer-Verlag, Berlin and Heidelberg, 1984.
[8] Ufuktepe, U., Kapcak, S., *RIMS Kyoto Proceeding*, **1909**, 207, 2014.
[9] Hamer, W., *Epidemic disease in England*, **1**, 733, 1906.
[10] Ross, R., *The Prevention of Malaria*, 2nd Ed. John Murray, London, 1911.