ORIGINAL PAPER

ON THE FUNCTION SEQUENCES AND SERIES IN THE NON-NEWTONIAN CALCULUS

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Abstract. *The purpose of this study is to examine the function sequences and series in the non-Newtonian real numbers.*

*Firstly, the information about the studies that are done until today and the application areas, was briefly given. Non-Newtonian calculus was introduced which is an alternative to the classical calculus, definitions, theorems and properties were given. *-Function sequence, *-function series, *-pointwise convergence and *-uniform convergence were introduced and theorems were proven which are exposed important differences between *-pointwise convergence and *-uniform convergence. In addition, *-convergence tests such as *-Cauchy criterion and *-Weierstrass M-criterion were obtained. The relationship between *-uniform convergence of the *-continuity, *-integral and *-derivative was examined respectively.*

Keywords: **-Function Sequences; *-Function Series; *-Pointwise Convergence; *-Uniform Convergence; *-Continuity.*

1. INTRODUCTION AND PRELIMINARIES

Non-Newtonian calculus was firstly introduced and studied by Michael Grossman and Robert Katz between 1967 and 1970[1]. Various researchers have been developing its dimensions[2-8]. Grossman worked on some properties of derivatives and integrals[9]. Recently, Duyar, Sağır and Oğur obtained some basic topological properties on non-Newtonian real line[10]. Sağır and Duyar got some results on Lebesgue measure in the sense of non-Newtonian Calculus[11]. In this article, we examine *-function sequences and *-function series.

A generator is defined as an injective function with domain $\mathbb{R}$ and the range of generator is a subset of $\mathbb{R}$. $\mathbb{R}(N)_\alpha = \mathbb{R}(N) = \left\{\alpha(x) : x \in \mathbb{R}\right\}$ is called set of non-Newtonian real numbers where α is a generator. Let take any α generator with range $A = \mathbb{R}(N)_\alpha$. Let define $\alpha -$addition, α-subtraction, $\alpha -$multiplication, $\alpha -$division and $\alpha -$order as follows:

α-addition	$x \dot{+} y = \alpha\left\{\alpha^{-1}(x) + \alpha^{-1}(y)\right\}$
α-subtraction	$x \dot{-} y = \alpha\left\{\alpha^{-1}(x) - \alpha^{-1}(y)\right\}$
α-multiplication	$x \dot{\times} y = \alpha\left\{\alpha^{-1}(x) \times \alpha^{-1}(y)\right\}$
α-division	$x \dot{/} y = \alpha\left\{\alpha^{-1}(x) / \alpha^{-1}(y)\right\}$

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α-order $\quad x \dot{<} y \left(x \dot{\leq} y \right) \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y) \ \left(\alpha^{-1}(x) \leq \alpha^{-1}(y) \right)$.

for $x, y \in \mathbb{R}(N)_\alpha$ [1].

$\left(\mathbb{R}(N)_\alpha, \dot{+}, \dot{\times}, \dot{\leq} \right)$ is totally ordered field[5,12].

The numbers $x \dot{>} \dot{0}$ are α-positive numbers and the numbers $x \dot{<} \dot{0}$ are α-negative numbers in $\mathbb{R}(N)_\alpha$. α-integers are obtained by successive α-addition of $\dot{1}$ to $\dot{0}$ and successive α-subtraction of $\dot{1}$ from $\dot{0}$. Hence α-integers are as follows:

$..., \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), ...$

For each integer n, we set $\dot{n} = \alpha(n)$. If $\dot{n}$ is an $\alpha-$positive integer, then it is n times sum of $\dot{1}$ [1, 5, 13].

$\alpha-$absolute value of a number $x \in \mathbb{R}(N)_\alpha$ is defined by

$$|x|_\alpha = \alpha\left(\left| \alpha^{-1}(x) \right| \right) = \begin{cases} x & if \quad x \dot{>} \dot{0} \\ \dot{0} & if \quad x = \dot{0} \\ \dot{0} \dot{-} x & if \quad x \dot{<} \dot{0} \end{cases} .$$

For $x \in \mathbb{R}(N)_\alpha$, $\sqrt[p]{x}^{\alpha} = \alpha\left(\sqrt[p]{\alpha^{-1}(x)} \right)$ and $x^{p_\alpha} = \alpha\left\{ \left[\alpha^{-1}(x) \right]^p \right\}$ [1,5].

Let $\left(\mathbb{R}(N)_\alpha, |.|_\alpha \right)$ be non-Newtonian metric space. The point a is called $\alpha-$accumulation point of set S (or non-Newtonian accumulation point of set S) if $\left(\left(a \dot{-} \varepsilon, a \dot{+} \varepsilon \right) - \{a\} \right) \cap S \neq \varnothing$ for every $\varepsilon \dot{>} \dot{0}$ where $S \subset \mathbb{R}(N)_\alpha$ and $a \in \mathbb{R}(N)_\alpha$. The set of all $\alpha-$accumulation points of set S is denoted by S'^{α}.

Let sequence (x_n) and a point x be given in non-Newtonian metric space $\left(\mathbb{R}(N)_\alpha, |.|_\alpha \right)$. If for every number $\varepsilon \dot{>} \dot{0}$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|x_n \dot{-} x|_\alpha \dot{<} \varepsilon$ for all $n \geq n_0$, then it is said that the sequence (x_n) is non-Newtonian convergent(or $\alpha-$convergent) and this situation is denoted by ${}^{\alpha}\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\alpha} x$ as $n \to \infty$. When every number $\varepsilon \dot{>} \dot{0}$ is given, if there exists a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|x_n \dot{-} x_m|_\alpha \dot{<} \varepsilon$ for all $n, m \geq n_0$, then the sequence (x_n) is called non-Newtonian Cauchy sequence[5].

Grossman and Katz described the *-calculus with the help of two arbitrary selected generators. In this study, we studied according to *-calculus. Let take any generators α and β and let * ("star") is shown the ordered pair of arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic). The following notations will be used[1].

	$\alpha-$arithmetic	$\beta-$arithmetic
Realm	$A\left(= \mathbb{R}(N)_\alpha\right)$	$B\left(= \mathbb{R}(N)_\beta\right)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$	$\ddot{/}$
Ordering	$\dot{<}$	$\ddot{<}$

In the *-calculus, $\alpha-$arithmetic is used on arguments and $\beta-$arithmetic is used on values.

If the generators α and β are chosen as one of I and $\exp$, the following special calculuses are obtained.

Calculus	α	β
Classic	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp

The isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic is the unique function ι (iota) that possesses the following three properties:

1. ι is one to one,
2. ι is on A and onto B,
3. For any numbers u and v in A,

$$\iota(u \dot{+} v) = \iota(u) \ddot{+} \iota(v),$$
$$\iota(u \dot{-} v) = \iota(u) \ddot{-} \iota(v),$$
$$\iota(u \dot{\times} v) = \iota(u) \ddot{\times} \iota(v),$$
$$\iota(u \dot{/} v) = \iota(u) \ddot{/} \iota(v),\ v \neq \dot{0}$$
$$u \dot{<} v \Leftrightarrow \iota(u) \ddot{<} \iota(v).$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every number x in A, and that $\iota(\dot{n}) = \ddot{n}$ for every integer n [1,14].

Let $X \subset \mathbb{R}(N)_\alpha$, $a \in X'^{\alpha}$, $b \in \mathbb{R}(N)_\beta$ and let $f : X \to \mathbb{R}(N)_\beta$ be a function. If for every $\varepsilon \ddot{>} \ddot{0}$ there exists a number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $| f(x) \ddot{-} b |_\beta \ddot{<} \varepsilon$ for all $x \in X$ which holds condition $\dot{0} \dot{<} | x \dot{-} a |_\alpha \dot{<} \delta$, then it is said that the *-limit of the function *f* (in the sense of Cauchy) at the point *a* is *b* and this is denoted by

$$* - \lim_{x \to a} f(x) = b.$$

If sequence $(f(x_n))$ $\beta-$converges to the number *b* for all sequences $(x_n) \subset X - \{a\}$ which $\alpha-$converge to point *a*, then it is said that the *-limit of the function *f* (*-sequential limit of the function *f*) at the point *a* is *b* and this is denoted by

$$* - \lim_{x \to a} f(x) = b.$$

Let $X \subset \mathbb{R}(N)_\alpha$, $a \in X$ and a function $f : X \to \mathbb{R}(N)_\beta$ be given. If for every $\varepsilon \ddot{>} \ddot{0}$ there exists a number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $| f(x) \ddot{-} f(a) |_\beta \ddot{<} \varepsilon$ for all $x \in X$ which holds condition $| x \dot{-} a |_\alpha \dot{<} \delta$, then it is said that the function *f* is *-continuous at point $a \in X$. The function *f* is *-continuous at the point $a \in X$ iff this point *a* is an element of domain of the function *f* and $* - \lim_{x \to a} f(x) = f(a)$. If ${}^{\beta}\lim_{n \to \infty} f(x_n) = f(a)$ for all sequences (x_n) which hold conditions ${}^{\alpha}\lim_{n\to\infty} x_n = a$ and $x_n \in X$ for $n = 1,2,3,...$, then the function *f* is called sequentially *-continuous at the point $a \in X$.

If the following *-limit exists, we denote it by $\left[\overset{*}{D} f\right](a)$ and call it the *-derivative of f at a, and say that f is *-differentiable at a:

$$* - \lim_{x\to a} \left\{ \left[f(x) \dot{-} f(a) \right] \ddot{/} \left[\iota(x) \dot{-} \iota(a) \right] \right\}.$$

If it exists, $\left[\overset{*}{D} f\right](a)$ is necessarily in B.

The *-average of a *-continuous function f on $[\dot{r}, \dot{s}]$ is denoted by $\overset{*}{M}{}_r^s f$ and defined to be $\beta -$limit of the $\beta -$convergent sequence whose nth term is $\beta -$average of $f(a_1),..., f(a_n)$, where $a_1,...,a_n$ is the $n -$fold $\alpha -$partition of $[\dot{r}, \dot{s}]$.

The *-integral of a *-continuous function f on $[\dot{r}, \dot{s}]$, denoted by ${}^{*}\int_r^s f(x)\, d^* x$, is the number $\left[\iota(s) \dot{-} \iota(r)\right] \ddot{\times} \overset{*}{M}{}_r^s f$ in B [1].

Theorem 1: (First fundamental theorem of *-calculus) If f is *-continuous on $[\dot{r}, \dot{s}]$ and $g(x) = {}^{*}\int_r^x f(t)\, d^* t$ for every $x \in [\dot{r}, \dot{s}]$, then $\overset{*}{D} g = f$ on $[\dot{r}, \dot{s}]$ [1].

Theorem 2: (Second fundamental theorem of *-calculus) If $\overset{*}{D} h$ is *-continuous on $[\dot{r}, \dot{s}]$, then ${}^{*}\int_r^s \left[\overset{*}{D} h\right](x)\, d^* x = h(s) \dot{-} h(r)$ [1].

2. THE RESULTS AND DISCUSSION

Proposition 1. The definitions *-limit in the sense of Cauchy and *-sequential limit are equivalent.

Proof: Let $* - \lim_{x\to a} f(x) = L$ in the sense of Cauchy. Then, for every $\varepsilon \ddot{>} \ddot{0}$ there exists a number $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $| f(x) \dot{-} L |_{\beta} \ddot{<} \varepsilon$ for all $x \in X$ which holds condition $\dot{0} \dot{<} | x \dot{-} a |_{\alpha} \dot{<} \delta$. Let an arbitrary sequence $(x_n) \subset X - \{a\}$ such that ${}^{\alpha}\lim_{n\to\infty} x_n = a$ be taken. Hence, specially for the number $\delta \dot{>} \dot{0}$ there exist a number $n_0 \in \mathbb{N}$ such that $|x_n \dot{-} a|_{\alpha} \dot{<} \delta$ for all $n > n_0$. Then, $| f(x_n) \dot{-} L |_{\beta} \ddot{<} \varepsilon$ for all $n > n_0$. Namely , it is seen that *-sequential limit is $* - \lim_{x\to a} f(x) = L$.

Conversely, let the *-sequential limit be $*-\lim_{x\to a} f(x)=L$. Assume the contrary. Namely, $*-\lim_{x\to a} f(x)\neq L$ in the sense of Cauchy. In this case, for all number $\delta \dot{>} \dot{0}$ there exist at least a number $\varepsilon \ddot{>} \ddot{0}$ such that $|f(x)\ddot{-}L|_\beta \ddot{\geq} \varepsilon$ for all $x\in X$ which holds condition $\dot{0}\dot{<}|x\dot{-}a|_\alpha \dot{<}\delta$. Then, if $\delta=\frac{\dot{1}}{\dot{n}}\alpha$ is taken for all $n\in\mathbb{N}$, $|f(x_n)\ddot{-}L|_\beta \ddot{\geq}\varepsilon$ for all $x_n\in X$ which holds condition $\dot{0}\dot{<}|x_n\dot{-}a|_\alpha \dot{<}\frac{\dot{1}}{\dot{n}}\alpha$. Thus, a sequence $(x_n)\subset X-\{a\}$ is found which $x_n \xrightarrow{\alpha} a$ but holds ${}^{\beta}\lim_{n\to\infty} f(x_n)\neq L$. This contradicts the hypothesis. Namely, $*-\lim_{x\to a} f(x)=L$ in the sense of Cauchy.

2.1. **-FUNCTION SEQUENCES*

Definition 1. Let S be a nonempty subset of $\mathbb{R}(N)_\alpha$ and let $k\in\mathbb{N}$. The sequence $(f_k)=(f_1,f_2,...,f_k,...)$ is called *-function sequence (or non-Newtonian function sequence) for functions $f_k:S\subseteq\mathbb{R}(N)_\alpha\to\mathbb{R}(N)_\beta$. Here all functions f_k defined on same set. The sequence $(f_k(x_0))$ is *β*-sequence (or non-Newtonian sequence) in $\mathbb{R}(N)_\beta$ for each $x_0\in S$.

Let take *-function sequence (f_k) with $f_k:S\subseteq\mathbb{R}(N)_\alpha\to\mathbb{R}(N)_\beta$ and let take sequence $(f_k(x_0))$ such that *β*-convergent (or non-Newtonian convergent) for $x_0\in S$. Also, let ${}^{\beta}\lim_{k\to\infty} f_k(x_0)=a_{x_0}$. Since *β*-limit of a sequence is unique, the number a_{x_0} is unique. Let define the function *f* as $f(x_0)=a_{x_0}$ at the point x_0. If this process is done for each $x\in S$, then the function *f* is defined as $f:S\to\mathbb{R}(N)_\beta$, $f(x)={}^{\beta}\lim_{k\to\infty} f_k(x)$.

Definition 2. Let *-function sequence (f_k), which $f_k:S\subseteq\mathbb{R}(N)_\alpha\to\mathbb{R}(N)_\beta$ be given. If the sequence $(f_k(x_0))$ is *β*-convergent for $x_0\in S$, then the *-function sequence (f_k) is called *-convergent (or non-Newtonian convergent). The *-function sequence (f_k) is said *-pointwise converges or *-converges to function f, if the sequence $(f_k(x))$ is *β*-convergent for each $x\in S$ and ${}^{\beta}\lim_{k\to\infty} f_k(x)=f(x)$. In this case, the function *f* is called *-limit of the *-function sequence (f_k) and it is shown as follows:

$$*-\lim_{k\to\infty} f_k = f\left(*-pointwise\right) \text{ or } f_k \xrightarrow{*} f\left(*-pointwise\right).$$

Then, the *-function sequence (f_k) *-converges pointwise to the function *f*, if for any given $\varepsilon\ddot{>}\ddot{0}$, there exists a natural number $k_0=k_0(x,\varepsilon)$ such that
$\left|f_k(x)\ddot{-}f(x)\right|_\beta \ddot{<}\varepsilon$ for all $k>k_0$ and for each $x\in S$.

Example 1. Let the functions $f_k : \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be defined as $f_k(x) = \frac{\iota(x)}{\ddot{k}}\beta$ for all $x \in \mathbb{R}(N)_\alpha$. Then the *- functions sequence (f_k) *-converges pointwise to the function $f = \ddot{0}$

Solution 1. For each $x \in R(N)_\alpha$, we have

$$^{\beta}\lim_{k\to\infty} f_k(x) = {}^{\beta}\lim_{k\to\infty} \frac{\iota(x)}{\ddot{k}}\beta = \iota(x) \ddot{\times} {}^{\beta}\lim_{k\to\infty} \frac{\ddot{1}}{\ddot{k}}\beta = \iota(x) \ddot{\times} \ddot{0} = \ddot{0}.$$

Then, we get $* - \lim_{k\to\infty} f_k = f = \ddot{0}$ $(* - pointwise)$.

Example 2. a) The function $*\sin : \mathbb{R}(N)_\alpha \to [\ \ddot{0} \ddot{-} \ddot{1},\ \ddot{1}\ ]$ defined as $^{*}\sin y = \beta\left[\sin\left(\alpha^{-1}(y)\right)\right]$. Then, the inequality $\left|^{*}\sin y\right|_\beta \ddot{\leq} \ddot{1}$ holds for all $y \in \mathbb{R}(N)_\alpha$.

b) Let $f_k : \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$, $f_k(x) = \frac{\ddot{1}}{\ddot{k}}\beta \ddot{\times} {}^{*}\sin\left(\left(\dot{k} \dot{\times} x\right) \dot{+} \dot{k}\right)$. Then, we have $f_k \xrightarrow{\ *\ } f = \ddot{0}\left(* - pointwise\right)$.

Solution 2. a) We have $\alpha^{-1}(y) \in \mathbb{R}$ for all $y \in \mathbb{R}(N)_\alpha$ and $\left|\sin\left(\alpha^{-1}(y)\right)\right| \leq 1$ for all $\alpha^{-1}(y) \in \mathbb{R}$. Thus

$$\left|^{*}\sin y\right|_\beta = \beta\left(\left|\sin\left(\alpha^{-1}(y)\right)\right|\right) \ddot{\leq} \beta(1) = \ddot{1}.$$

b)Since $\left|^{*}\sin y\right|_\beta \ddot{\leq} \ddot{1}$ for all $y \in \mathbb{R}(N)_\alpha$, we have

$$\begin{aligned} \left|f_k(x) \ddot{-} f(x)\right|_\beta &= \left|\frac{\ddot{1}}{\ddot{k}}\beta \ddot{\times} {}^{*}\sin\left(\left(\dot{k} \dot{\times} x\right) \dot{+} \dot{k}\right) \ddot{-} \ddot{0}\right|_\beta \\ &= \left|\frac{\ddot{1}}{\ddot{k}}\beta \ddot{\times} {}^{*}\sin\left(\left(\dot{k} \dot{\times} x\right) \dot{+} \dot{k}\right)\right|_\beta \\ &\ddot{\leq} \frac{\ddot{1}}{\ddot{k}}\beta. \end{aligned}$$

Thus we get $f_k \xrightarrow{\ *\ } f = \ddot{0}\left(* - pointwise\right)$ for all $k > k_0$, where $\ddot{k}_0 \ddot{>} \frac{\ddot{1}}{\ddot{k}}\beta$, $\frac{\ddot{1}}{\ddot{k}}\beta \ddot{<} \varepsilon$.

Although *-convergence is useful in many cases, there are some special cases, which it is not sufficient. Let (f_k) *-converges pointwise to the function f. In this case, *-limit function f may not be *-continuous even if all of the functions f_k are *-continuous. For example, let the *-function sequence (f_k) be given as follows

$$f_k(x)=\begin{cases} \ddot{0} & , \quad x \dot{\geq} \dfrac{\dot{1}}{k}\alpha \\ \left(\ddot{0} \ddot{-} \ddot{k}\right) \ddot{\times} \iota(x) \ddot{+} \ddot{1} & , \quad \dot{0} \dot{\leq} x \dot{<} \dfrac{\dot{1}}{k}\alpha \end{cases}.$$

Then, $f_k(x)$ is *pointwise converges to the function f, where $f(x)=\begin{cases} \ddot{0} & , \quad x \neq \dot{0} \\ \ddot{1} & , \quad x=\dot{0} \end{cases}$ for all $x \in [\dot{0},\dot{1}]$. The function f is not *-continuous although the function f_k is *-continuous for all number k.

Example 3. Let $f_k : (\dot{0},\dot{1}) \to \mathbb{R}(N)_\beta$ and $f_k(x)=\iota(x)^{k_\beta}$. Then *-pointwise limit of the sequence (f_k) is $\ddot{0} \in \mathbb{R}(N)_\beta$.

Solution 3. Let take arbitrary $\varepsilon \ddot{>} \ddot{0}$ and let take $x_0 \in (\dot{0},\dot{1})$. If the natural number k_0 is chosen as $k_0 \geq \dfrac{\ln \beta^{-1}(\varepsilon)}{\ln \alpha^{-1}(x_0)}$, then

$$\left| f_k(x_0) \ddot{-} f(x_0) \right|_\beta = \left| \iota(x_0)^{k_\beta} \ddot{-} \ddot{0} \right|_\beta = \iota(x_0)^{k_\beta} \ddot{<} \varepsilon$$

for all $k > k_0$. Hence, ${}^{\beta}\lim_{k\to\infty} f_k(x) = {}^{\beta}\lim_{k\to\infty} \iota(x)^{k_\beta} = \ddot{0} = f(x)$ on $\alpha-$interval $(\dot{0},\dot{1})$ since x_0 is arbitrary. Then $f_k \xrightarrow{*} f = \ddot{0}\left(*-pointwise\right)$.

Definition 3. Let take the *-function sequence (f_k), where $f_k : S \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$. The *-function sequence (f_k) *-uniform converges to the function f on set S, if for any given $\varepsilon \ddot{>} \ddot{0}$, there exists a natural number k_0 depends on number ε but not depend on variable x such that $\left| f_k(x) \ddot{-} f(x) \right|_\beta \ddot{<} \varepsilon$ for all $k > k_0$ and each $x \in S$. We denote *-uniform convergence by $*-\lim_{k\to\infty} f_k = f\left(*-uniform\right)$ or $f_k \xrightarrow{*} f\left(*-uniform\right)$.

Example 4. Let $f_k : \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ and $f_k(x) = \dfrac{{}^{*}\sin x}{\ddot{k}}\beta$. The *-function sequence (f_k) *-uniform converges the function f which $f(x)=\ddot{0}$ on $\mathbb{R}(N)_\alpha$.

Solution 4. For any $\varepsilon \ddot{>} \ddot{0}$ and all $x \in \mathbb{R}(N)_\alpha$, we have

$$\left| f_k(x) \ddot{-} f(x) \right|_\beta = \left| \frac{{}^{*}\sin x}{\ddot{k}}\beta \ddot{-} \ddot{0} \right|_\beta = \frac{\left|{}^{*}\sin x\right|_\beta}{\ddot{k}}\beta \ddot{\leq} \frac{\ddot{1}}{\ddot{k}}\beta$$

and therefore for natural number k_0, which is chosen as $k_0 \geq \frac{1}{\beta^{-1}(\varepsilon)}$, one finds that $\frac{\ddot{1}}{\ddot{k}} \ddot{<} \varepsilon$ where all $k > k_0$. Then we get $f_k \xrightarrow{*} f\,(*-uniform)$ since $k_0 = k_0(\varepsilon)$.

Example 5. If $f_k : [\dot{0}, \dot{+}\dot{\infty}) \to R(N)_\beta$ and $f_k(x) = \frac{\iota(x)}{\ddot{k}}\beta$, the sequence (f_k) is not *-uniform convergent.

Solution 5. It has been shown that this sequence *-pointwise converges to the function $f = \ddot{0}$ (see Example 1). If $f_k \xrightarrow{*} f\,(*-uniform)$ had held, there would existed a natural number k_0 which is corresponds $\varepsilon = \ddot{1}$ such that

$$\left| \frac{\iota(x)}{\ddot{k}}\beta \ddot{-} \ddot{0} \right|_\beta \ddot{<} \ddot{1}$$

for $k > k_0$ and on α-interval $[\dot{0}, \dot{+}\dot{\infty})$. Especially, $\left| \frac{\iota(x)}{\ddot{k}_0 \ddot{+} \ddot{1}}\beta \right|_\beta \ddot{<} \ddot{1}$ is obtained for $k = k_0 + 1$ and all $x \in [\dot{0}, \dot{+}\dot{\infty})$. But

$$\ddot{1} \ddot{>} \left| \frac{\iota(x)}{\ddot{k}_0 \ddot{+} \ddot{1}}\beta \right|_\beta = \left| \frac{\ddot{2} \ddot{\times} (\ddot{k}_0 \ddot{+} \ddot{1})}{\ddot{k}_0 \ddot{+} \ddot{1}}\beta \right|_\beta = \ddot{2}$$

for the point $x = \dot{2} \dot{\times} (\dot{k}_0 \dot{+} \dot{1}) \in [\dot{0}, \dot{+}\dot{\infty})$. This is a contradiction. Then f_k is not *-uniform convergent to the point $\ddot{0}$.

Remark 1. While a *- function sequence is *-uniform convergent on a set, this *-function sequence may not be *-uniform convergent on another set. Every *-uniform convergent sequence is *-pointwise convergent, but every *-pointwise convergent sequence does not have to be *-uniform.

Theorem 3. Let the sequence (f_k) with $f_k : S \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be *-convergent the function f on the set S and let

$$c_k = {}^{\beta}\sup\left\{ \left| f_k(x) \ddot{-} f(x) \right|_\beta : x \in S \right\}.$$

In this case, the sequence is *-uniform convergent to the function f on the set S iff ${}^{\beta}\lim_{k\to\infty} c_k = \ddot{0}$ holds.

Proof: Let the sequence (f_k) be *-uniform convergent on the set S. For arbitrary $\varepsilon \ddot{>} \ddot{0}$, there exist $k_0 \in N$ such that

$$\left| f_k(x) \dot{\dot{-}} f\left(x\right)\right|_\beta \ddot{<} \varepsilon$$

for all $x \in S$ and for $k > k_0(\varepsilon)$. Hence, we have $c_k \ddot{<} \varepsilon$. Since $\varepsilon \ddot{>} \ddot{0}$ is arbitrary, we get ${}^{\beta}\lim_{k\to\infty} c_k = \ddot{0}$.

Conversely, if ${}^{\beta}\lim_{k\to\infty} c_k = \ddot{0}$, then there exists a number k_0 such that for $k > k_0$ for any $\varepsilon \ddot{>} \ddot{0}$. Since $c_k = {}^{\beta}\left\{\sup\left| f_k(x) \dot{\dot{-}} f\left(x\right)\right|_\beta : x \in S\right\}$, we get

$$\begin{aligned}\left| f_k(x) \dot{\dot{-}} f\left(x\right)\right|_\beta &\ddot{\leq} c_k \\ &\ddot{<} \varepsilon\end{aligned}$$

for all $x \in S$ and for all $k > k_0$. Thus, $f_k \xrightarrow{*} f\left(* - uniform\right)$.

Remark 2. If $\left| f_k(x) \dot{\dot{-}} f(x)\right|_\beta \xrightarrow{\beta} \ddot{0}$ for each $x \in S$, then we have $f_k \xrightarrow{*} f\left(* - pointwise\right)$ and if ${}^{\beta}\sup\left\{\left| f_k(x) \dot{\dot{-}} f(x)\right|_\beta : x \in S\right\} \xrightarrow{\beta} \ddot{0}$, we have $f_k \xrightarrow{*} f\left(* - uniform\right)$.

Example 6. We investigate the *-pointwise limit of the sequence (f_k) where $f_k : [\dot{0}, \dot{1}] \to \mathbb{R}(N)_\beta$, $f_k(x) = \iota(x)^{2_\beta} \dot{\dot{-}} \frac{\iota(x)}{\ddot{k}} \beta$ and show this convergence is *-uniform.

Solution 6. For each $x \in [\dot{0}, \dot{1}]$, since $\frac{\iota(x)}{\ddot{k}} \beta \xrightarrow{*} \ddot{0}\left(* - pointwise\right)$ holds in example 1,

$${}^{\beta}\lim_{k\to\infty} f_k(x) = {}^{\beta}\lim_{k\to\infty}\left(\iota(x)^{2_\beta} \dot{\dot{-}} \frac{\iota(x)}{\ddot{k}} \beta\right) = \iota(x)^{2_\beta} = f(x)$$

hence $f_k \xrightarrow{*} f\left(* - pointwise\right)$ is found. Additionally, by the theorem 3, this convergence is *-uniform since

$$c_k = {}^{\beta}\sup\left\{\left| f_k(x) \dot{\dot{-}} f(x)\right|_\beta : x \in [\dot{0}, \dot{1}]\right\} = {}^{\beta}\sup\left\{\frac{\iota(x)}{\ddot{k}} \beta : x \in [\dot{0}, \dot{1}]\right\} = \frac{\ddot{1}}{\ddot{k}} \beta$$

and ${}^{\beta}\lim_{k\to\infty} c_k = {}^{\beta}\lim_{k\to\infty} \frac{\ddot{1}}{\ddot{k}} = \ddot{0}$.

*2.2. *-FUNCTION SERIES AND CONSEQUENCES OF *-UNIFORM CONVERGENCE*

Definition 4. Let take *-function sequence (f_k) with $f_k : A \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$. The infinite $\beta -$sum

$$_{\beta}\sum_{k=1}^{\infty} f_k = f_1 \ddot{+} f_2 \ddot{+} \ldots \ddot{+} f_k \ddot{+} \ldots$$

is called *-function series (or non-Newtonian function series). The $\beta-$sum $S_k =_{\beta} \sum_{n=1}^{k} f_n$ is called *k*-th partial $\beta-$sum of the series $_{\beta}\sum_{k=1}^{\infty} f_k$ for $k \in N$.

Definition 5. Let the *-function series $_{\beta}\sum_{k=1}^{\infty} f_k$ with $f_k : A \subseteq \mathbb{R}(N)_{\alpha} \to \mathbb{R}(N)_{\beta}$ and the function $f : A \subseteq \mathbb{R}(N)_{\alpha} \to \mathbb{R}(N)_{\beta}$ be given. If the $\beta-$partial sums sequence (S_n), where $S_n =_{\beta} \sum_{k=1}^{n} f_k$ is *-pointwise convergent to the function f, then *-function series $_{\beta}\sum_{k=1}^{\infty} f_k$ *-converges pointwise to the function f on the set A and

$$_{\beta}\sum_{k=1}^{\infty} f_k = f\,(*-pointwise)$$

is written. In this situation, the function f is called $\beta-$sum (or non-Newtonian sum) of *-series $_{\beta}\sum_{k=1}^{\infty} f_k$.

If $S_k \xrightarrow{*} f\,(*-uniform)$, then the *-function series $_{\beta}\sum_{k=1}^{\infty} f_k$ is called *-uniform convergent to the function f on the set A and $_{\beta}\sum_{k=1}^{\infty} f_k = f \quad (*-uniform)$ is written.

The set of numbers x is called *-convergence set (or non-Newtonian convergence set) of the *-function series $_{\beta}\sum_{k=1}^{\infty} f_k$ where the *-function series $_{\beta}\sum_{k=1}^{\infty} f_k(x)$ is *-convergent on.

Example 7. Let the series $_{\beta}\sum_{k=1}^{\infty} f_k$ with $f_k : (\dot{0} \dot{-} \dot{1}, \dot{1}) \to R(N)_{\beta}$, $f_k(x) = \imath(x)^{k_{\beta}}$ be given. We show that this series,

a) is *-pointwise convergent but is not *-uniform convergent on $(\dot{0} \dot{-} \dot{1}, \dot{1})$,

b) is *-uniform convergent on $[\dot{0} \dot{-} a, a]$, where $\dot{0} \dot{<} a \dot{<} \dot{1}$.

Solution 7. a)Since k-th partial $\beta-$sum

$$s_k(x) = \ddot{1} \ddot{+} \imath(x) \ddot{+} \imath(x)^{2_{\beta}} \ddot{+} \ldots \ddot{+} \imath(x)^{(k-1)_{\beta}} = \frac{\ddot{1} \ddot{-} \imath(x)^{k_{\beta}}}{\ddot{1} \ddot{-} \imath(x)} \beta$$

and

$$^{\beta}\lim_{k\to\infty} s_k(x) = {}^{\beta}\lim_{k\to\infty} \frac{\ddot{1} \ddot{-} \imath(x)^{k_{\beta}}}{\ddot{1} \ddot{-} \imath(x)} \beta = \frac{\ddot{1}}{\ddot{1} \ddot{-} \imath(x)} \beta$$

the series ${}_\beta\sum_{k=1}^{\infty} f_k(x) = {}_\beta\sum_{k=1}^{\infty} \imath(x)^{k_\beta}$ is *-convergent to the function $f(x) = \frac{\ddot{1}}{\ddot{1} \ddot{-} \imath(x)}\beta$ on $(\dot{0} \dot{-} \dot{1}, \dot{1})$. Therefore ${}_\beta\sum_{k=1}^{\infty} f_k = f \quad (* - pointwise)$. Since the partial $\beta -$sums sequence $(s_k(x))$ is not *-uniform convergent on $(\dot{0} \dot{-} \dot{1}, \dot{1})$, the series ${}_\beta\sum_{k=1}^{\infty} f_k$ is not *-uniform convergent.

b) By (a), we have

$$\left|s_k(x) \ddot{-} f(x)\right|_\beta = \left|\frac{\ddot{1} \ddot{-} \imath(x)^{k_\beta}}{\ddot{1} \ddot{-} \imath(x)}\beta \ddot{-} \frac{\ddot{1}}{\ddot{1} \ddot{-} \imath(x)}\beta\right|_\beta$$

$$= \frac{\left|\imath(x)\right|_\beta^{k_\beta}}{\left|\ddot{1} \ddot{-} \imath(x)\right|_\beta}\beta$$

$$\ddot{\leq} \frac{\imath(a)^{k_\beta}}{\left|\ddot{1} \ddot{-} \imath(a)\right|_\beta}\beta$$

for all $x \in [\dot{0} \dot{-} a, a]$. Since $\dot{0} \dot{<} a \dot{<} \dot{1}$, $\imath(a)^{k_\beta} \xrightarrow{*} \ddot{0}$ holds independently from choosing of $x \in [\dot{0} \dot{-} a, a]$, then the series ${}_\beta\sum_{k=1}^{\infty} \imath(x)^{k_\beta}$ is *-uniform convergent to the function $f(x) = \frac{\ddot{1}}{\ddot{1} \ddot{-} \imath(x)}\beta$ on $\alpha -$interval $[\dot{0} \dot{-} a, a]$, since k depends on only the number ε. Hence ${}_\beta\sum_{k=1}^{\infty} f_k = f \quad (* - uniform)$.

Theorem 4. (*-Cauchy criterion for *-function sequences) Let the *-function sequence (f_k) with $f_k : S \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be given. The sequence (f_k) is *-uniform convergent iff for arbitrary $\varepsilon \ddot{>} \ddot{0}$, there exists a number $k_0 \in N$ such that $\left| f_k(x) \ddot{-} f_p(x)\right|_\beta \ddot{<} \varepsilon$ for $k \geq p > k_0$ and all $x \in S$.

Proof: Let the function sequence $(f_k(x))$ be *-uniform convergent to the function f on the set S and let take arbitrary $\varepsilon \ddot{>} \ddot{0}$. Thus, there exists a natural number k_0 such that

$$\left| f_k(x) \ddot{-} f(x)\right|_\beta \ddot{<} \frac{\varepsilon}{\ddot{2}}\beta$$

for all $x \in S$ and all $k > k_0$. Then,

$$\left| f_k(x) \dot{-} f_p\left(x\right)\right|_\beta = \left| f_k(x) \dot{-} f\left(x\right) \dot{+} f\left(x\right) \dot{-} f_p\left(x\right)\right|_\beta$$
$$\dot{\leq} \left| f_k(x) \dot{-} f\left(x\right)\right|_\beta \dot{+} \left| f_p(x) \dot{-} f\left(x\right)\right|_\beta$$
$$\dot{<} \frac{\varepsilon}{\ddot{2}}\beta \dot{+} \frac{\varepsilon}{\ddot{2}}\beta = \varepsilon$$

for all $x \in S$ and $k \geq p > k_0$.

Conversely, suppose that there exists a positive integer number k_0 for arbitrary $\varepsilon \ddot{>} \ddot{0}$ such that $\left| f_k(x) \dot{-} f_p\left(x\right)\right|_\beta \ddot{<} \varepsilon$ on the set S for $k \geq p > k_0$. This means that the sequence $(f_k(x))$ is a $\beta-$Cauchy (or non-Newtonian Cauchy) sequence for each $x \in S$. Therefore, we get the sequence $(f_k(x))$ is $\beta-$convergent. Let ${}^\beta\lim_{k\to\infty} f_k(x) = f(x)$. The proof is completed if we show this convergence is *-uniform.

Let $\varepsilon \ddot{>} \ddot{0}$ be given. By the hypothesis, there exists a natural number k_0 such that $\left| f_k(x) \dot{-} f_p\left(x\right)\right|_\beta \ddot{<} \varepsilon$ for all $x \in S$ and $k \geq p > k_0$. Then, we get

$${}^\beta\lim_{p\to\infty} \left| f_k(x) \dot{-} f_p(x)\right|_\beta = \left| f_k(x) \dot{-} f(x)\right|_\beta \ddot{<} \varepsilon$$

for all $x \in S$ and $k > k_0$. Hence $f_k \xrightarrow{*} f \quad (*-uniform)$.

Corollary 1. (*-Cauchy criterion for *-function series) Let *-series ${}_\beta\sum_{k=1}^{\infty} f_k$ with $f_k : S \subseteq R(N)_\alpha \to R(N)_\beta$ and $\varepsilon \ddot{>} \ddot{0}$ be given. The *-series ${}_\beta\sum_{k=1}^{\infty} f_k$ is *-uniform convergent iff there exists a number $k_0 \in N$ such that

$$\left| s_k(x) \dot{-} s_p\left(x\right)\right|_\beta = \left| {}_\beta\sum_{n=p+1}^{k} f_n(x) \right|_\beta \ddot{<} \varepsilon$$

on the set A for $k \geq p > k_0$.

Corollary 2. Let the *-function series ${}_\beta\sum_{k=1}^{\infty} f_k$ and $\varepsilon \ddot{>} \ddot{0}$ be given. The *-series ${}_\beta\sum_{k=1}^{\infty} f_k$ is *-uniform convergent iff there exists a number $k_0 \in N$ such that

$$\left| R_k(x) \right|_\beta = \left| {}_\beta\sum_{n=k+1}^{\infty} f_n(x) \right|_\beta \ddot{<} \varepsilon$$

for $k > k_0$ and all $x \in A$.

Now an important test known as *-Weierstrass M-criterion will be obtained to determine *-uniform convergence of *-function series.

Theorem 5. (*-Weierstrass M-criterion) If there exist $\beta-$numbers M_k such that $\left|f_k(x)\right|_\beta \ddot{<} M_k$ for all $x \in A$ where $f_k : A \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ and if the series ${}_\beta\sum_{k=1}^{\infty} M_k$ is $\beta-$ convergent, then the series ${}_\beta\sum_{k=1}^{\infty} f_k$ is *-uniform convergent and $\beta-$absolutely convergent.

Proof: By the hypothesis, there exists a number $k_0 \in N$ such that ${}_\beta\sum_{n=p+1}^{k} M_n \ddot{<} \varepsilon$ for $\varepsilon \ddot{>} \ddot{0}$ and $k > p > k_0$. Hence, by the $\beta-$triangle inequality, we have

$$\left|s_k(x) \dot{-} s_p(x)\right|_\beta = \left|{}_\beta\sum_{n=p+1}^{k} f_n(x)\right|_\beta \ddot{\leq}_\beta \sum_{n=p+1}^{k} \left|f_n(x)\right|_\beta \ddot{\leq}_\beta \sum_{n=p+1}^{k} M_n \ddot{<} \varepsilon . \tag{2.1}$$

Then, we get $\left|s_k(x) \dot{-} s_p(x)\right|_\beta \ddot{<} \varepsilon$ for all $x \in A$. Thus, by corollary 2, the series ${}_\beta\sum_{k=1}^{\infty} f_k$ is *-uniform convergent on the set A. Also, by the inequality 2.1, the series ${}_\beta\sum_{k=1}^{\infty} \left|f_k(x)\right|_\beta$ is *-convergent.

Example 8. If the series ${}_\beta\sum_{k=1}^{\infty} a_k$ is $\beta-$absolutely convergent, then the series ${}_\beta\sum_{k=1}^{\infty}\left(a_k \ddot{\times}^* \sin x\right)$ is *-uniform convergent on $R(N)_\alpha$.

Solution 8. The inequality $\left|a_k \ddot{\times}^* \sin x\right|_\beta \ddot{\leq} \left|a_k\right|_\beta$ holds for all $x \in \mathbb{R}(N)_\alpha$. By the hypothesis, ${}_\beta\sum_{k=1}^{\infty} \left|a_k\right|_\beta$ is *-convergent. Then, in view of *-Weierstrass M-criterion, the series ${}_\beta\sum_{k=1}^{\infty}\left(a_k \ddot{\times}^* \sin x\right)$ is *-uniform convergent.

Example 9. Since $\alpha = I$, $\beta = \exp$ in geometric calculus, we have $\mathbb{R}(N)_\alpha = \mathbb{R}$ and $\mathbb{R}(N)_\beta = \mathbb{R}^+$. According to this, the function series ${}_\beta\sum_{n=1}^{\infty} f_n(x) = {}_\beta\sum_{n=1}^{\infty} e^{\frac{3.x^n}{n!}} = \prod_{n=1}^{\infty} e^{\frac{3.x^n}{n!}}$ is uniform convergent with respect to geometric calculus where $f_n : \left[\frac{1}{2}, 2\right] \to \mathbb{R}^+$, $f_n(x) = e^{\frac{3.x^n}{n!}}$.

Solution 9. We have $\left|e^{\frac{3.x^n}{n!}}\right|_\beta = e^{\left|\frac{3.x^n}{n!}\right|} \ddot{\leq} e^{\left|\frac{3.2^n}{n!}\right|}$ for all $x \in \left[\frac{1}{2}, 2\right]$ since $\left|\frac{3.x^n}{n!}\right| \leq \frac{3.2^n}{n!}$. Let $M_n = e^{\frac{3.2^n}{n!}}$. By non-Newtonian rate test [13]

$$ {}^{\beta}\lim_{n\to\infty}\left|\frac{e^{\frac{3.2^{n+1}}{(n+1)!}}}{e^{\frac{3.2^{n}}{n!}}}\beta\right|_{\beta} = \lim_{n\to\infty}\left|\frac{e^{\frac{3.2^{n+1}}{(n+1)!}}}{e^{\frac{3.2^{n}}{n!}}}\beta\right|_{\beta} = \lim_{n\to\infty}\left|e^{\frac{\frac{3.2^{n+1}}{(n+1)!}}{\frac{3.2^{n}}{n!}}}\right|_{\beta} = \lim_{n\to\infty} e^{\left|\frac{\frac{3.2^{n+1}}{(n+1)!}}{\frac{3.2^{n}}{n!}}\right|} = \lim_{n\to\infty} e^{\frac{2}{n+1}} = e^{0} = \ddot{0} \,\ddot{<}\, \ddot{1}. $$

Therefore the series ${}_{\beta}\sum_{n=1}^{\infty} M_n$ is $\beta-$absolutely convergent. Thus ${}_{\beta}\sum_{n=1}^{\infty} M_n$ is $\beta-$convergent. Hence, by the *-Weierstrass M-criterion the series ${}_{\beta}\sum_{n=1}^{\infty} f_n(x)$ is *-uniform convergent on $\left[\frac{1}{2},2\right]$.

*2.3. *-UNIFORM CONVERGENCE AND *-CONTINUITY*

The most essential property related with *-uniform convergence, as expressed in following theorem, is its relation with *-continuous functions.

Theorem 6. If $f_k : A \subset \mathbb{R}(N)_{\alpha} \to \mathbb{R}(N)_{\beta}$ is *-continuous and if the *-function sequence (f_k) is *-uniform convergent to function f on the set A, then the function f is *-continuous on the set A. Namely,

$$ *-\lim_{k\to\infty}\left[*-\lim_{x\to x_0} f_k(x)\right] = *-\lim_{x\to x_0}\left[*-\lim_{k\to\infty} f_k(x)\right]. $$

Proof: Let take an arbitrary $x_0 \in A$. Since $f_k \xrightarrow{*} f$ $(*-uniform)$, for $\varepsilon \,\ddot{>}\, \ddot{0}$ there exists $k_0 \in N$ such that

$$ \left| f_k(x) \,\ddot{-}\, f(x)\right|_{\beta} \,\ddot{<}\, \frac{\varepsilon}{\ddot{3}}\beta $$

for $k > k_0(\varepsilon)$ on the set A. Furthermore, since f_k is *-continuous on the point x_0 for all $k \in \mathbb{N}$ there exists a number $\delta \,\dot{>}\, \dot{0}$ such that for $x \in A$

$$ \left| f_k(x) \,\ddot{-}\, f_k(x_0)\right|_{\beta} \,\ddot{<}\, \frac{\varepsilon}{\ddot{3}}\beta $$

whenever $\left|x - x_0\right|_{\alpha} \,\dot{<}\, \delta$. Therefore, we have

$$ \begin{aligned} \left| f(x) \,\ddot{-}\, f(x_0)\right|_{\beta} &= \left| f(x) \,\ddot{-}\, f_k(x_0) \,\ddot{+}\, f_k(x_0) \,\ddot{-}\, f_k(x) \,\ddot{+}\, f_k(x) \,\ddot{-}\, f(x_0)\right|_{\beta} \\ &\ddot{\leq} \left| f(x) \,\ddot{-}\, f_k(x)\right|_{\beta} \ddot{+} \left| f_k(x) \,\ddot{-}\, f_k(x_0)\right|_{\beta} \ddot{+} \left| f_k(x_0) \,\ddot{-}\, f(x_0)\right|_{\beta} \\ &\ddot{<} \frac{\varepsilon}{\ddot{3}}\beta \,\ddot{+}\, \frac{\varepsilon}{\ddot{3}}\beta \,\ddot{+}\, \frac{\varepsilon}{\ddot{3}}\beta = \varepsilon \end{aligned} $$

for $x \in A$. Hence the function f is *-continuous at the point x_0 and the function f is *-continuous on the set A since $x_0 \in A$ is arbitrary.

Corollary 3. Let the functions $f_k : A \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be *-continuous and let the function $f : A \to \mathbb{R}(N)_\beta$ be given. If ${}_\beta\sum_{k=1}^{\infty} f_k = f\left(*-uniform\right)$, then the function f is *-continuous on the set A.

Example 10. If $f(x) = {}^*\sin x = {}_\beta\sum_{k=1}^{\infty}\left(\left(\ddot{0} \dot{-} \ddot{1}\right)^{k_\beta} \ddot{\times} \dfrac{\iota(x)^{(2k+1)_\beta}}{\left(\ddot{2} \ddot{\times} \ddot{k} \ddot{+} \ddot{1}\right)!_\beta}\beta\right)$, then the function f is *-continuous on space $\mathbb{R}(N)_\alpha$.

Solution 10. By corollary 3, we need to show that the partial sums of series *-converges uniformly to the function ${}^*\sin x$. Since $n!_\beta = \ddot{1} \ddot{\times} \ddot{2} \ddot{\times} ... \ddot{\times} n$, we have

$$\left|s_k\left(x\right) \dot{-} {}^*\sin x\right|_\beta = \left|{}_\beta\sum_{n=k+1}^{\infty}\left(\left(\ddot{0} \dot{-} \ddot{1}\right)^{n_\beta} \ddot{\times} \frac{\iota\left(x\right)^{(2n+1)_\beta}}{\left(\left(\ddot{2} \ddot{\times} \ddot{n}\right) \ddot{+} \ddot{1}\right)!_\beta}\beta\right)\right|_\beta \ddot{\leq}_\beta \sum_{n=k+1}^{\infty}\left(\frac{\iota\left(a\right)^{(2n+1)_\beta}}{\left(\left(\ddot{2} \ddot{\times} \ddot{n}\right) \ddot{+} \ddot{1}\right)!_\beta}\beta\right),$$

where $a \dot{>} \dot{0}$ and $|x|_\alpha \dot{<} a$. Thus $s_k\left(x\right) \xrightarrow{*} {}^*\sin x\left(*-uniform\right)$. Since, by corollary 3, the function ${}^*\sin x$ is *-continuous on $[\dot{0} \dot{-} a, a\dot{]}$ and since a is arbitrary, the function ${}^*\sin x$ is *-continuous on $\mathbb{R}(N)_\alpha$.

Example 11. Let $f_n(x) = \iota\left(x\right)^{n_\beta}$, $\dot{0} \dot{\leq} x \dot{\leq} \dot{1}$. Then (f_n) is not *-uniform convergent.

Solution 11. It is easy to see that (f_n) *-converges pointwise to the function $f\left(x\right) = \begin{cases} \ddot{0} & , \quad x \neq \dot{1} \\ \ddot{1} & , \quad x = \dot{1} \end{cases}$. Since f is not *-continuous, by theorem 6, we get (f_n) is not *-uniform convergent.

*2.4. *-UNIFORM CONVERGENCE AND *-INTEGRAL*

Theorem 7. Let the functions $f_k : \dot{[}a,b\dot{]} \to \mathbb{R}(N)_\beta$ be *-continuous on $\dot{[}a,b\dot{]}$ for all $k \in \mathbb{N}$ and let $f_k \xrightarrow{*} f\left(*-uniform\right)$ on $\dot{[}a,b\dot{]}$. Then the function f is *-continuous on $\dot{[}a,b\dot{]}$ and

$$*-\lim_{k\to\infty} {}^*\int_a^b f_k\left(x\right)d^*x = {}^*\int_a^b f\left(x\right)d^*x$$

$$\left(\text{or } *-\lim_{k\to\infty} {}^{*}\!\!\int_a^b f_k(x)\,d^*x = {}^{*}\!\!\int_a^b \left(*-\lim_{k\to\infty} f_k(x)\right) d^*x\right).$$

Proof: By theorem 6, the function f is *-continuous on the interval $[\dot{a},\dot{b}]$. Therefore, the function $f_k \dot{-} f$ is *-continuous on the interval $[\dot{a},\dot{b}]$ and hence is *-integrable on $[\dot{a},\dot{b}]$. Let $\varepsilon \ddot{>} \ddot{0}$ be given. Then, there exists a number $k_0(\varepsilon)\in\mathbb{N}$ such that

$$\left| f_k(x) \dot{-} f(x)\right|_\beta \ddot{<} \frac{\varepsilon}{\iota(b)\dot{-}\iota(a)}\beta$$

on $[\dot{a},\dot{b}]$ for $k > k_0$. Thus, we get

$$\begin{aligned}
\left| {}^{*}\!\!\int_a^b f_k(x)\,d^*x \dot{-} {}^{*}\!\!\int_a^b f(x)\,d^*x \right|_\beta &= \left| \beta\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(f_k(\alpha(x))\right)dx \right) \dot{-} \beta\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(f(\alpha(x))\right)dx \right) \right|_\beta \\
&= \left| \beta\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(f_k(\alpha(x))\right)dx - \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(f(\alpha(x))\right)dx \right) \right|_\beta \\
&= \beta\left(\left| \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(f_k(\alpha(x))\right)dx - \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \beta^{-1}\left(f(\alpha(x))\right)dx \right| \right) \\
&= \beta\left(\left| \int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \left[\beta^{-1}\left(f_k(\alpha(x))\right) - \beta^{-1}\left(f(\alpha(x))\right)\right]dx \right| \right) \\
&\ddot{\leq} \beta\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \left|\beta^{-1}\left(f_k(\alpha(x))\right) - \beta^{-1}\left(f(\alpha(x))\right)\right| dx \right) \\
&\ddot{<} \beta\left(\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} \frac{\beta^{-1}(\varepsilon)}{\alpha^{-1}(b)-\alpha^{-1}(a)}\,dx \right) \\
&= \beta\left(\frac{\beta^{-1}(\varepsilon)}{\alpha^{-1}(b)-\alpha^{-1}(a)}.\left(\alpha^{-1}(b)-\alpha^{-1}(a)\right) \right) \\
&= \beta\left(\beta^{-1}(\varepsilon)\right) \\
&= \varepsilon
\end{aligned}$$

for $k > k_0$. Namely, $*-\lim_{k\to\infty} {}^{*}\!\!\int_a^b f_k(x)\,d^*x = {}^{*}\!\!\int_a^b f(x)\,d^*x$.

Corollary 4. If the functions $f_k : [\dot{a},\dot{b}] \to \mathbb{R}(N)_\beta$ are *-continuous on $[\dot{a},\dot{b}]$ and ${}_\beta\sum_{k=1}^{\infty} f_k(x) = f(x)$ $(*-uniform)$, then the function f is *-continuous on $[\dot{a},\dot{b}]$ and

$$ {}_{\beta}\sum_{k=1}^{\infty}\left({}^{*}\int_{a}^{b} f_k(x)\,d^*x \right) = {}^{*}\int_{a}^{b} f(x)\,d^*x $$

$$ \left(or\ {}_{\beta}\sum_{k=1}^{\infty}\left({}^{*}\int_{a}^{b} f_k(x)\,d^*x \right) = {}^{*}\int_{a}^{b}\left({}_{\beta}\sum_{k=1}^{\infty} f_k(x)\,d^*x \right)\right). $$

Proof: Let $s_n(x) = {}_{\beta}\sum_{k=1}^{n} f_k(x)$. By hypothesis, the sequence $s_n(x)$ *-converges uniformly to the function f. Then, by theorem 7, we have

$$ *-\lim_{k\to\infty} {}^{*}\int_{a}^{b} s_k(x)\,d^*x = {}^{*}\int_{a}^{b} f(x)\,d^*x \ \text{ or } \ {}_{\beta}\sum_{k=1}^{\infty}\left({}^{*}\int_{a}^{b} f_k(x)\,d^*x \right) = {}^{*}\int_{a}^{b}\left({}_{\beta}\sum_{k=1}^{\infty} f_k(x)\,d^*x \right). $$

Example 12. Let $f_k : [\dot{0},\dot{1}] \to \mathbb{R}(N)_{\beta}$, where $f_k(x) = \iota(x)^{2_{\beta}} \dot{-} \frac{\iota(x)}{\ddot{k}}\beta$. Then we investigate $*-\lim_{k\to\infty} {}^{*}\int_{\dot{0}}^{\dot{1}} f_k(x)\,d^*x$.

Solution 12. Since $f_k(x) \xrightarrow{*} \iota(x)^{2_{\beta}}$ $(*-uniform)$, we get

$$ *-\lim_{k\to\infty} {}^{*}\int_{\dot{0}}^{\dot{1}} f_k(x)d^*x = {}^{*}\int_{\dot{0}}^{\dot{1}} f(x)d^*x = {}^{*}\int_{\dot{0}}^{\dot{1}} \iota(x)^{2_{\beta}}\,d^*x = \beta\left(\int_{0}^{1} \beta^{-1}\left[\iota(\alpha(x))^{2_{\beta}} \right] dx \right) = \beta\left(\int_{0}^{1} x^2 dx \right) $$

$$ = \beta\left(\frac{1}{3}\right) $$

$$ = \frac{\ddot{1}}{\ddot{3}}\beta $$

Example 13. Let $x \in \mathbb{R}(N)_{\alpha}$ and let $f(x) = {}_{\beta}\sum_{k=1}^{\infty}\left[\left(\ddot{0} \dot{-} \ddot{1}\right)^{(k+1)_{\beta}} \ddot{\times} \frac{\iota(x)^{(2k-1)_{\beta}}}{\left((\ddot{2}\ddot{\times}\ddot{k}) \dot{-} \ddot{1}\right)!_{\beta}}\beta \right]$. Then we investigate the *-integral ${}^{*}\int_{\dot{0}}^{x} f(t)\,d^*t$.

Solution 13. Let $a \in \mathbb{R}(N)_{\alpha}$ such that $|x|_{\alpha} \dot{\leq} a \dot{<} \dot{+}\infty$ and $\lim_{x\to+\infty} \alpha(x) = \dot{+}\infty$. Then, we have

$$ \left(\ddot{0} \dot{-} \ddot{1}\right)^{(k+1)_{\beta}} \ddot{\times} \frac{\iota(x)^{(2k-1)_{\beta}}}{\left((\ddot{2}\ddot{\times}\ddot{k}) \dot{-} \ddot{1}\right)!_{\beta}}\beta \ddot{\leq} \frac{\iota(a)^{(2k-1)_{\beta}}}{\left((\ddot{2}\ddot{\times}\ddot{k}) \dot{-} \ddot{1}\right)!_{\beta}}\beta $$

for all $k \in \mathbb{N}$. If we apply $\beta-$rate test[13] for the series ${}_{\beta}\sum_{k=1}^{\infty}\left[\frac{\iota(a)^{(2k-1)_\beta}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{-}\ddot{1})!_\beta}\beta\right]$, then we get

$$
\begin{aligned}
{}^{\beta}\lim_{k\to\infty}\left|\frac{\frac{\iota(a)^{(2k+1)_\beta}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{+}\ddot{1})!_\beta}\beta}{\frac{\iota(a)^{(2k-1)_\beta}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{-}\ddot{1})!_\beta}\beta}\beta\right|_\beta &= {}^{\beta}\lim_{k\to\infty}\left|\frac{\iota(a)^{2_\beta}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{+}\ddot{1})\ddot{\times}(\ddot{2}\ddot{\times}\ddot{k})}\beta\right|_\beta \\
&= \iota(a)^{2_\beta}\ddot{\times}\,{}^{\beta}\lim_{k\to\infty}\left|\frac{\ddot{1}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{+}\ddot{1})\ddot{\times}(\ddot{2}\ddot{\times}\ddot{k})}\beta\right|_\beta \\
&= \iota(a)^{2_\beta}\ddot{\times}\,{}^{\beta}\lim_{k\to\infty}\frac{\ddot{1}}{(\ddot{2}\ddot{\times}\ddot{k})}\beta\,\ddot{-}\,{}^{\beta}\lim_{k\to\infty}\frac{\ddot{1}}{(\ddot{2}\ddot{\times}\ddot{k})\ddot{+}\ddot{1}}\beta \\
&= \iota(a)^{2_\beta}\ddot{\times}(\ddot{0}\ddot{-}\ddot{0}) \\
&= \ddot{0} \\
&\ddot{<}\ddot{1}.
\end{aligned}
$$

Thus, this series is β-convergent and by *-Weierstrass M-criterion, the series

$$
{}_{\beta}\sum_{k=1}^{\infty}\left[(\ddot{0}\ddot{-}\ddot{1})^{(k+1)_\beta}\ddot{\times}\frac{\iota(x)^{(2k-1)_\beta}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{-}\ddot{1})!_\beta}\beta\right]
$$

is *-uniform convergent on $[\dot{0}\dot{-}a, a\dot{]}$. Then, by virtue of corollary 4, the series is term by term *-integrable and

$$
\begin{aligned}
{}^{*}\int_{\dot{0}}^{x} f(t)d^{*}t &= {}_{\beta}\sum_{k=1}^{\infty}\left[{}^{*}\int_{\dot{0}}^{x}\left((\ddot{0}\ddot{-}\ddot{1})^{(k+1)_\beta}\ddot{\times}\frac{\iota(t)^{(2k-1)_\beta}}{((\ddot{2}\ddot{\times}\ddot{k})\ddot{-}\ddot{1})!_\beta}\beta\right)d^{*}t\right] \\
&= {}_{\beta}\sum_{k=1}^{\infty}\left[(\ddot{0}\ddot{-}\ddot{1})^{(k+1)_\beta}\ddot{\times}\frac{\iota(x)^{(2k)_\beta}}{(\ddot{2}\ddot{\times}\ddot{k})!_\beta}\beta\right].
\end{aligned}
$$

Example 14. Let $(f_n(x))$ be *-uniform convergent on $\dot{0}\dot{\leq}x\dot{\leq}\dot{1}$ and let f_n be *-differentiable. The *-derivative sequence $\left(\overset{*}{D}f_n\right)(x)$ is not necessary to be *-uniform convergent.

Solution 14. Let sequence $f_n(x)=\frac{{}^{*}\sin(\dot{n}^{2_\alpha}\dot{\times}x)}{\iota(\dot{n})}\beta$ be given. The sequence $f_n(x)$ is *-

uniform convergent to function $f=\ddot{0}$. Then we have $\left(\overset{*}{D} f_n\right)(x)=\iota(\dot{n})\ddot{\times}\, {}^{*}\cos\left(\dot{n}^{2_\alpha}\dot{\times} x\right)$ since

$$\begin{aligned}\overset{*}{D}\left({}^{*}\sin x\right) &= \beta\left(D\left[\beta^{-1}\left({}^{*}\sin\alpha(t)\right)\right]\right)\\ &= \beta\left(D\left[\beta^{-1}\left(\beta\left[\sin\alpha^{-1}\left(\alpha(t)\right)\right]\right)\right]\right)\\ &= \beta\left(D(\sin t)\right)\\ &= \beta(\cos t)\\ &= \beta\left(\cos\alpha^{-1}(x)\right)\\ &= {}^{*}\cos x\end{aligned}$$

where $x=\alpha(t)$. However, the *-derivative sequence $\left(\overset{*}{D} f_n\right)(x)$ is not even *-pointwise convergent. Because $\left(\overset{*}{D} f_n\right)(x)=\iota(\dot{n})$ at the point $x=\dot{0}$.

*2.5. *-UNIFORM CONVERGENCE AND *-DERIVATIVE*

We know that all *-uniform convergent *-function sequences or series can not be term by term *-differentiable. Therefore, we need additional conditions to *-uniform convergence for term by term *-differentiability.

Theorem 8. Let the *-derivatives of the functions $f_k:[\dot{a},\dot{b}]\subset R(N)_\alpha\to R(N)_\beta$ exist on $[\dot{a},\dot{b}]$ and let they be *-continuous. Additionally, let

1. $f_k \xrightarrow{*} f\,(*-\textit{pointwise})$
2. $\overset{*}{D} f_k \xrightarrow{*} g\,(*-\textit{uniform})$.

Then, g is *-differentiable on $[\dot{a},\dot{b}]$ and
$\overset{*}{D} f=g$, namely $\overset{*}{D}\left(*-\lim\limits_{k\to\infty} f_k(x)\right)=*-\lim\limits_{k\to\infty}\left(\overset{*}{D} f_k\right)(x)$.

Proof: By theorem 6, g is *-continuous and hence *-integrable on $[\dot{a},\dot{b}]$. Therefore, by theorem 7 and second fundamental theorem of *-calculus, we have

$${}^{*}\int_a^x g(t)d^{*}t={}^{*}-\lim_{k\to\infty}{}^{*}\int_a^x\left({}^{*}Df_k\right)(t)d^{*}t={}^{*}-\lim_{k\to\infty}\left[f_k(x)\dot{-}f_k(a)\right]=f(x)\dot{-}f(a)$$

for $x\in[\dot{a},\dot{b}]$. Thus, we get $f(x)=f(a)\dot{+}\,{}^{*}\int_a^x g(t)d^{*}t$. In view of the first fundamental theorem of *-calculus, we obtain $\left(\overset{*}{D} f\right)(x) = g(x)$. This completes the proof.

Example 15. $\mathbb{R}(N)_\alpha = \mathbb{R}(N)_\beta = \mathbb{R}^+$ since $\alpha = \beta = \exp$ in bigeometric calculus. Let $f_k : \mathbb{R}^+ \to \mathbb{R}^+$, $f_k(x) = \frac{\iota(x)}{\ddot{k}}\beta$. Then we investigate $\overset{*}{D}\left(* - \lim_{k\to\infty} f_k(x)\right)$.

Solution 15. Since

$$f_k(x) = \frac{\iota(x)}{\ddot{k}}\beta = e^{\frac{\ln x}{k}} = x^{\frac{1}{k}} \text{ and } {}^{\beta}\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} x^{\frac{1}{k}} = \lim_{k\to\infty} e^{\frac{\ln x}{k}} = \lim_{k\to\infty} (e^{\frac{1}{k}})^{\ln x} = 1,$$

we have $f_k(x) \xrightarrow{*} 1 = \ddot{0}\,(* - pointwise)$. $\overline{f}_k(x) = \ln\left(e^x\right)^{\frac{1}{k}} = \frac{x}{k}$ since $f_k(x) = x^{\frac{1}{k}}$ [1]. From $D\overline{f}_k(x) = \frac{1}{k}$, we get $\overset{*}{D} f_k(x) = \beta(D\overline{f}_k(\overline{x})) = e^{\frac{1}{k}}$. Since $c_k = {}^{\beta}\sup\left\{\left|\overset{*}{D} f_k(x) \dot{-} 1\right|_\beta, x \in \mathbb{R}\right\} = e^{\frac{1}{k}}$ and ${}^{\beta}\lim_{k\to\infty} c_k = 1 = \ddot{0}$, $\overset{*}{D} f_k(x) \xrightarrow{*} 1(* - uniform)$. Hence, by the theorem 8, we get

$$\overset{*}{D}\left(* - \lim_{k\to\infty} f_k(x)\right) = \overset{*}{D}\left(* - \lim_{k\to\infty} x^{\frac{1}{k}}\right) = \lim_{k\to\infty} e^{\frac{1}{k}} = 1.$$

If the theorem 8 is used, then the following corollary is obtained.

Corollary 5. Let the derivatives $\overset{*}{D} f_k$ exists and continuous on $[\dot{a}, \dot{b}]$ where $f_k : [\dot{a}, \dot{b}] \to \mathbb{R}(N)_\beta$. Moreover, suppose that

1. ${}_{\beta}\sum_{k=1}^{\infty} f_k = f\ (* - pointwise)$ on $[\dot{a}, \dot{b}]$ and,
2. ${}_{\beta}\sum_{k=1}^{\infty} \overset{*}{D} f_k = h\ (* - uniform)$ on $[\dot{a}, \dot{b}]$.

Then, $\overset{*}{D} f = h$ or $\overset{*}{D}\left({}_{\beta}\sum_{k=1}^{\infty} f_k(x)\right) = {}_{\beta}\sum_{k=1}^{\infty}\left(\overset{*}{D} f_k\right)(x)$ on $[\dot{a}, \dot{b}]$.

Example 16. Let the *-series ${}_{\beta}\sum_{k=1}^{\infty} \frac{{}^{*}\sin(\dot{k} \dot{\times} x)}{\ddot{2}^{k_\beta}}\beta$ be given on the set $\mathbb{R}(N)_\alpha$. Since $\left|\ddot{2}^{(-k)_\beta} \ddot{\times}\, {}^{*}\sin(\dot{k} \dot{\times} x)\right|_\beta \ddot{\le} \ddot{2}^{(-k)_\beta}$ for all $x \in \mathbb{R}(N)_\alpha$ and the series ${}_{\beta}\sum_{k=1}^{\infty} \ddot{2}^{(-k)_\beta} = {}_{\beta}\sum_{k=1}^{\infty} \frac{\ddot{1}}{\ddot{2}^{k_\beta}}\beta$ is $\beta -$convergent, by the *-Weierstrass M-criterion, the *-series ${}_{\beta}\sum_{k=1}^{\infty} \frac{{}^{*}\sin(\dot{k} \dot{\times} x)}{\ddot{2}^{k_\beta}}\beta$ is *-convergent uniformly. Let $f(x) = {}_{\beta}\sum_{k=1}^{\infty} \frac{{}^{*}\sin(\dot{k} \dot{\times} x)}{\ddot{2}^{k_\beta}}\beta$. Here, if $f_k(x) = \frac{{}^{*}\sin(\dot{k} \dot{\times} x)}{\ddot{2}^{k_\beta}}\beta$ and

$t=\alpha^{-1}(x)$ is written for $x\in\mathbb{R}(N)_{\alpha}$, then

$$\overset{*}{D}f_k(x)=\overset{*}{D}\left(\frac{{}^{*}\sin(\dot{k}\dot{\times}x)}{\ddot{2}^{k_\beta}}\beta\right)$$
$$=\overset{*}{D}\left(\beta\left[\frac{\sin\left(\alpha^{-1}(\dot{k}\dot{\times}x)\right)}{2^k}\right]\right)$$
$$=\beta\left(D\left[\beta^{-1}\left(\beta\left[\frac{\sin\left(\alpha^{-1}(\dot{k}\dot{\times}\alpha(x))\right)}{2^k}\right]\right)\right]\right)$$
$$=\beta\left(D\left[\frac{\sin(k.x)}{2^k}\right]\right)$$
$$=\beta\left(\frac{k}{2^k}.\cos(k.x)\right)$$
$$=\frac{\ddot{k}\ddot{\times}{}^{*}\cos(\dot{k}\dot{\times}x)}{\ddot{2}^{k_\beta}}\beta.$$

Since the *-derivative $\left[\overset{*}{D}f_k\right](x)$ is also *-continuous, $\left|\frac{\ddot{k}\ddot{\times}{}^{*}\cos(\dot{k}\dot{\times}x)}{\ddot{2}^{k_\beta}}\beta\right|_{\beta}\ddot{\leq}\frac{\ddot{k}}{\ddot{2}^{k_\beta}}\beta$ and the series ${}_{\beta}\sum_{k=1}^{\infty}\frac{\ddot{k}}{\ddot{2}^{k_\beta}}\beta$ is $\beta-$convergent, by *-Weierstrass M-criterion, the *-derivative series

$${}_{\beta}\sum_{k=1}^{\infty}\frac{\ddot{k}\ddot{\times}{}^{*}\cos(\dot{k}\dot{\times}x)}{\ddot{2}^{k_\beta}}\beta$$

is also *-uniform convergent on $\mathbb{R}(N)_{\alpha}$. Thus, by virtue of corollary 5, we get

$$\left[\overset{*}{D}f\right](x)={}_{\beta}\sum_{k=1}^{\infty}\frac{\ddot{k}\ddot{\times}{}^{*}\cos(\dot{k}\dot{\times}x)}{\ddot{2}^{k_\beta}}\beta.$$

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