ORIGINAL PAPER

ON NORMAL FERMI-WALKER DERIVATIVE FOR FOCAL CURVES WITH MODIFIED FRAME

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Abstract.*In this paper, we study a new construction of curves by normal Fermi-Walker parallelism and derivative with modified frame. Finally, we give normal Fermi-Walker parallelism and derivative for a focal curve according to modified frame.*

Keywords: *Modified frame, normal Fermi Walker derivative-parallelism, Focal curve.*

1. INTRODUCTION

Curves in differential geometry are one of the important subject . They are defined in parametric represantation and their geometric properties are expressed with the aid of derivatives and integrals. One of the most important topics of curve analysis is the Frenet-Serret formulas which describe the kinematic properties that provide a coordinate system at each point of the curve, [1-3]. Maluf examined an orthogonal frame and gave a formula corresponding the Frenet-Serret equation [4]. Bukcu and Karacan studied modified orthogonal frame [5, 6]. The calculus involving flows of curves and some surfaces has been established by many researchers by using different methods [7-23].

Fermi Walker transport collection in tetrad areas is usually that greatest approximation to a nonrotating research framework on that feeling in Newtonian motion. It is literally recognized by means of a structure in gyroscopes. Fermi-Walker transported structures are essential on many research. A framework the fact that experiences linear and rotational speed might be explained by means of the Frenet fields [24-26].

The main goal this article is analyzed normal Fermi-Walker derivations along the focal curve according to modified frame.

2. MATERIALS AND METHODS

A γ curve in 3-space E^3 is a continuous mapping of class C^3 $\gamma : I \to E^3$, where I is a open interval on the real line. γ is as follows of the Frenet formula [25]:

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$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}$$

where s is the arc-length parameter, t, n, b, κ and τ are the tangent unit vector, the normal unit vector, the binormal unit vector, curvature and torsion respectively. Let $\gamma(s)$ be analytic curve, where s is present at some interval, and $\gamma(s)$ be analytic in s. We suppose that $<\gamma', \gamma'> \neq 0.$ Thus we can parametrize γ by its arc-length $s.$ We suppose that the curvature $\kappa(s)$ of γ is non-zero. Then we can give an orthogonal frame $\{T, N, B\}$ on γ as follows:

$$T = \frac{d\gamma}{ds}, N = \frac{dT}{ds}, B = T \times N.$$

The relations between those and classical Frenet frame $\{t, n, b\}$ are

$$\begin{aligned} T &= t \\ N &= \kappa n \\ B &= \kappa b, \end{aligned}$$

where $\kappa \neq 0.$ The above system can be written explicitly as follows [6]:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix},$$

where the dash notation means differantiation with respect to the arc-length s and

$$\tau(s) = \frac{\det(\gamma', \gamma'', \gamma''')}{\kappa^2}$$

is the torsion of γ. Furthermore $\{T, N, B\}$ is given by

$$\begin{aligned} &< T, T >= 1, < N, N >=< B, B >= \kappa^2 \\ &< T, N >=< T, B >=< N, B >= 0. \end{aligned}$$

Definition 1. X is any vector field and $\gamma(s)$ is unit speed any curve in space.

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} - < n, X > n' + < n', X > n$$

defined as $\frac{\tilde{D}X}{\tilde{D}s}$ derivative is called Normal Femi-Walker derivative. Here $n = \frac{t'}{\|t\|}$ is a normal vector of Frenet frame [24].

Definition 2. X is any vector field along the $\gamma(s)$ space curve. If the Normal Fermi-Walker derivative of the vector field X

$$\frac{\tilde{D}X}{\tilde{D}s} = 0$$

the vector field X along the curve is called normal Fermi-Walker parallel [24].

3. FOCAL CURVES WITH MODIFIED FRAME

Denoting the focal curve by C_γ^M according to modified orthogonal frame, we can write

$$C_\gamma^M = \gamma + c_1^M N + c_2^M B,$$

where the coefficients c_1^M and c_2^M are smooth functions of parameter of the curve γ, called the first and second focal curvatures of γ, respectively [1].

Theorem 3. $\gamma : I \to E^3$ is a differentiable curve and its focal curve given as

$$C_\gamma^M = \gamma + c_1^M N + c_2^M B.$$

Then, the coefficients c_i^M provide the following equations:

$$\begin{aligned} c_0^M &= 0, c_1^M = \frac{1}{\kappa^2}, c_2^M = \frac{(\kappa^{-2})'}{2\tau}, \\ c_1^{M'} &= -\frac{\kappa'}{\kappa} c_1^M + \tau c_2^M, \\ c_2^{M'} &= -\tau c_1^M - \frac{\kappa'}{\kappa} c_2^M. \end{aligned}$$

Corollary 4. Let curve γ in E^3 be (I,γ) neighbouring coordinate. γ is contained in a sphere if and only if

$$r^2 = (c_1^M)^2 + (c_2^M)^2 = \frac{1}{\kappa^2} + \left[\frac{(\kappa^{-2})'}{2\tau}\right]^2,$$

where r is the radius of the sphere [1].

Corollary 5. The curve γ according to modified frame is spherical if and only if $C_{\gamma}^{M'} = 0.$

4. MODIFIED FRAME AND NORMAL-FERMI WALKER DERIVATIVE

According to modified frame, normal Fermi Walker derivative of the vector field X is defined as follows [7]:

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} - < N, X > N' + < N', X > N.$$

Theorem 6. Let $\{T, N, B\}$ and X be modified frame and any vector field along the $\gamma(s)$ space curve, respectively . Normal Fermi-Walker derivative can be expressed along the curve on modified frame as follows:

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \kappa^2 (X \wedge B) + \tau (X \wedge T).$$

Proof:

$$\begin{aligned}
\frac{\tilde{D}X}{\tilde{D}s} &= \frac{dX}{ds} - < N, X > (T + \frac{\kappa'}{\kappa} N + \tau B) + < (-\kappa^2 T + \frac{\kappa'}{\kappa} N + \tau B), X > N, \\
\frac{\tilde{D}X}{\tilde{D}s} &= \frac{dX}{ds} + \kappa^2 < N, X > T - \frac{\kappa'}{\kappa} < N, X > N - \tau < N, X > B \\
-\kappa^2 &< T, X > N + \frac{\kappa'}{\kappa} < N, X > N + \tau < B, X > N \\
\frac{\tilde{D}X}{\tilde{D}s} &= \frac{dX}{ds} - \kappa^2 [< T, X > N - < N, X > T] - \tau < N, X > B - < B, X > N] \\
\frac{\tilde{D}X}{\tilde{D}s} &= \frac{dX}{ds} + \kappa^2 X \wedge B + \tau X \wedge T.
\end{aligned}$$

Theorem 7. Let $\gamma(s)$ be a unit-speed curve. Then $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ vector field according to ribbon frame along the curve is parallel to normal Fermi-Walker terms if and only if:

$$\begin{aligned}
\lambda_1 &= a, (a = \text{const.}), \\
\lambda_2 &= \frac{1}{\kappa} [-a \int (\kappa - \kappa^3) ds + c], \qquad (1) \\
\lambda_3 &= c\kappa^{-1},
\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions according to real parameter s and c is a integration constant.

Proof: $\Rightarrow$: The vector field X is parallel to the normal Fermi-Walker terms. Then

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \kappa^2 X \wedge B + \tau X \wedge T,$$

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{d\lambda_1}{ds}T + [\frac{d\lambda_2}{ds} + \frac{\kappa'}{\kappa}\lambda_2 + (1-\kappa^2)\lambda_1]N + [\frac{d\lambda_3}{ds} + \frac{\kappa'}{\kappa}\lambda_3]B$$

is obtained. X is parallel to the Fermi-Walker terms and $\frac{\tilde{D}X}{\tilde{D}s} = 0$ so,

$$\frac{d\lambda_1}{ds} = 0,$$

$$\frac{d\lambda_2}{ds} + \frac{\kappa'}{\kappa}\lambda_2 + (1-\kappa^2)\lambda_1 = 0,$$

$$\frac{d\lambda_3}{ds} + \frac{\kappa'}{\kappa}\lambda_3 = 0$$

is obtained. This the solution of the equation system is given as follows:

$$\lambda_1 = a, (a = \text{cons.}),$$
$$\lambda_2 = \frac{1}{\kappa}[-a\int(\kappa - \kappa^3)ds + c],$$
$$\lambda_3 = c\kappa^{-1}.$$

$\Leftarrow$: $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ vector field, from Eq.(1) and theorem 6,

$$\frac{\tilde{D}X}{\tilde{D}s} = 0$$

is obtained.

5. FOCAL CURVE WITH MODIFIED FRAME AND NORMAL-FERMI WALKER DERIVATIVE

Denoting the focal curve by C_γ^M according to modified frame, we can write

$$C_\gamma^M = \gamma + c_1^M N + c_2^M B,$$

where the coefficients c_1^M, c_2^M are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Frenet vectors of the focal curve C_α^{R} according to modified frame are $\{T_F, N_F, B_F\}$ and

$$T_F' = \kappa_F N_F$$

$$N_F' = -\kappa_F T_F + \tau_F B_F$$

$$B_F' = -\tau_F N_F.$$

Theorem 8. Let $\{T_F, N_F, B_F\}$ and X be modified frame and any vector field along the $\gamma(s)$ space curve, respectively . Normal Fermi-Walker derivative can be expressed along the curve on modified frame as follows:

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \kappa_F^2 X \wedge B_F + \tau_F X \wedge T_F.$$

Proof:

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} - < N_F, X > (T_F + \frac{\kappa_F'}{\kappa_F} N_F + \tau_F B_F) + < (-\kappa_F^2 T_F + \frac{\kappa_F'}{\kappa_F} N_F + \tau_F B_F), X > N_F,$$

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \kappa_F^2 < N_F, X > T_F - \frac{\kappa_F'}{\kappa_F} < N_F, X > N_F - \tau_F < N_F, X > B_F$$

$$-\kappa_F^2 < T_F, X > N_F + \frac{\kappa_F'}{\kappa_F} < N_F, X > N_F + \tau_F < B_F, X > N_F$$

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} - \kappa_F^2 [< T_F, X > N_F - < N_F, X > T_F] - \tau_F [< N_F, X > B_F - < B_F, X > N_F]$$

$$\frac{\tilde{D}X}{\tilde{D}s} = \frac{dX}{ds} + \kappa_F^2 X \wedge B_F + \tau_F X \wedge T_F.$$

Theorem 9. Let C_γ^M be a unit-speed focal curve. Then $X = \lambda_1 T_F + \lambda_2 N_F + \lambda_3 B_F$ vector field according to Frenet frame along the focal curve is parallel to normal Fermi-Walker terms if and only if:

$$\begin{aligned} \lambda_1 &= a, (a = \text{cons.}), \\ \lambda_2 &= \frac{1}{\kappa_F}[-a\int(\kappa_F - \kappa_F^3)ds + c], \qquad (2) \\ \lambda_3 &= c\kappa_F^{-1}, \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions according to real parameter s.

Proof: $\Rightarrow$: The vector field X is parallel to the normal Fermi-Walker terms. Then

$$\frac{d\lambda_1}{ds} = 0$$
$$\frac{d\lambda_2}{ds} + \frac{\kappa_F'}{\kappa_F}\lambda_2 + (1-\kappa_F^2)\lambda_1 = 0$$
$$\frac{d\lambda_3}{ds} + \frac{\kappa_F'}{\kappa_F}\lambda_3 = 0$$

is obtained. This the solution of the equation system is given as follows:

$$\lambda_1 = a, (a = \text{const.}),$$

$$\lambda_2 = \frac{1}{\kappa_F}[-a\int(\kappa_F - \kappa_F^3)ds + c]$$

$$\lambda_3 = c\kappa_F^{-1},$$

where c is a integration constant.

$\Leftarrow$: $X = \lambda_1 T_F + \lambda_2 N_F + \lambda_3 B_F$ vector field, from Eq.(2) and theorem 6,

$$\frac{\tilde{D}X}{\tilde{D}s} = 0$$

is obtained.

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