ORIGINAL PAPER

TRAVELLING WAVE SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATIONS DESCRIBING NONLINEAR WAVE MOTION

NAWAB KHAN[1], M. YAQUB KHAN[1], KAMRAN AYUB[1],
QAZI MAHMOOD UL HASSAN[2]

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Abstract. *The solution of nonlinear mathematical models has much importance and in soliton theory their worth has increased. In present article, a research has been made of nonlinear Jimbo Miwa and Kadomtsev-Petviashvilli equations, to discuss behavior of these equations and to attain travelling wave solutions.* $Exp(-\varphi(\xi))$ *-expansion technique is used to construct soliton wave solutions. Wave transformation is applied to convert problem in the form of ordinary differential equation. The drawn-out novel type outcomes pay an essential role in the transportation of energy. It is noticed that under study approach is extremely dependable and it may be prolonged to further mathematical models signified mostly in nonlinear differential equations.*

Keywords: $Exp(-\varphi(\xi))$ *expansion technique; Travelling wave solution; Jimbo Miwa equation; Kadomtsev-Petviashvilli (KP) equation; Homogeneous principle*

1. INTRODUCTION

Recently solitary wave theory got some great improvements. Soliton wave occurrence enticed number of researchers for its comprehensive applications in engineering, and mathematical physics. Firstly J.S. Russell (by profession an engineer) contemplated the solitary wave in 1834. In the form of differential equations, various physical occurrences in nature are modeled. Vital effort for scientists is to get solution of such differential equations. To get soliton solutions, different attempts are made by scientists. Modeling of various physical, biochemical and biological occurrences are in the form of nonlinear PDEs. The vigorous attainment is the headway for exact soliton solutions of mathematically modelled differential equations. Different mathematical techniques are developed. For the observation of physical activities of problem exact solutions are vital. We have more applications and ability to examine the number of properties of mathematical model by utilizing the exact solution.

NLE equations play a very vital part in innumerable engineering and scientific arenas, such as, the heat flow, quantum mechanics, solid state physics, chemical kinematics, fluid mechanics, optical fibers, plasma physics, the wave proliferation phenomena, proliferation of shallow water waves etc.

Therefore, different techniques for finding exact solutions are used for a diversified field of partial differential equations like, The homogenious balance technique [1-2], Hirota's

[1] Riphah International University, Department of Mathematics & Statistics, Islamabad, Pakistan. E-mail: kamranayub88@gmail.com
[2] University of Wah, Department of Mathematics, Wah Cantt., Pakistan.

bilinear approach [3-4], Auxiliary equivalence technique[5], Trial task technique [6], Jacobi elliptic task system [7], tanh-function technique [8-9] by Malfliet, (1992), HPM [10], and method of sine- cosine [11-12], truncated Painleve expansion technique [13], variational iteration method (VIM) [14-16], Exp-function technique [17-18], (G'/G)-expansion approach [19-27], Exact soliton solution [28-30]. On exact solution some novel results and computational methods involved to travelling-wave transformation, see the references, [31-39].

In this work, our elementary incentive is the application of much reliable and effective technique known in literature by $\exp(-\varphi(\xi))$-expansion technique to attain soliton like solutions of nonlinear differential equations. The applications of under study nonlinear equations are very vast. Additionally, such type of equation found in different physical phenomenon related to fluid mechanic, astrophysics, solid state physics, chemical kinematics, ion acoustic waves in plasma and nonlinear optics etc. The straightforward emphasis of our technique is that the obtained solutions of differential equations are expressed in the form of a polynomial in $\exp(-\varphi(\xi))$, where $\varphi(\xi)$ must satisfies the ordinary differential equation.

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \psi \exp(\varphi(\xi)) + \omega, \tag{1}$$

where $\xi = x + y + z - Vt.$

The degree of the polynomial is find by homogenous principle. By balancing the highest order derivative involved with nonlinear term, we attain set of algebraic equations. These algebraic equations are solved to obtain coefficients of the polynomial. Under study article is divided in different parts. In next section, we give the analysis of method used to attain soliton wave solutions. Third part is devoted to application of $\exp(-\varphi(\xi))$- expansion technique. Last section, results and discussion are given to draw some conclusions.

2. ANALYSIS OF TECHNIQUE

In this section, we discuss the algorithm of $\exp(-\varphi(\xi))$-expansion technique for obtaining soliton solutions of nonlinear partial differential equations.
Assume nonlinear partial differential equation in the general form:

$$P(s, s_x, s_y, s_z, s_{xx}, s_{xy}, s_{xz}, \ldots) = 0. \tag{2}$$

Here $s(x, y, z, t)$ is the function to be determined, P is a polynomial in $s(x, y, z, t)$ and its partial derivatives. We work on following steps to obtain solution by using the $\exp(-\varphi(\xi))$ -expansion technique.

Step 1: Invoking the transformation

$$s(x, y, z, t) = s(\xi), \xi = x + y + z - Vt. \tag{3}$$

where V is the wave speed. Using transformation given in (3), we convert equation (2) in the form of an ODE.

$$Q(s, s', s'', s''', \ldots) = 0. \tag{4}$$

where prime signifies the derivative w.r.t. ξ. If possible, integrate equation (4) one or more times, which yields a constant(s) of integration.

Step 2: Suppose the solution of (4) can be expressed by a polynomial in the $\exp(-\varphi(\xi))$ as:

$$s(\xi) = \beta_n (\exp(-\varphi(\xi))))^n + \beta_{n-1}(\exp(-\varphi(\xi))))^{n-1} + \cdots \tag{5}$$

where $\beta, \beta_{n-1} \ldots$ and V are constants to be determined later such that $\beta_n \neq 0$ and $\varphi(\xi)$ satisfies equation (1).

Step 3: The positive integer n can be determined by using the homogeneous balance principle. We balance the highest order linear and nonlinear terms involving in (4). Our solutions now depend on the parameters involved in equation (1).

Case 1**:** $\omega^2 - 4\psi > 0$ and $\psi \neq 0$.

$$\varphi(\xi) = \ln\{1/2\omega(-\sqrt{\psi^2 - 4\omega})\tanh(\tfrac{\sqrt{\psi^2-4\omega}}{2}(\xi + d_1)) - \psi)\}. \tag{6}$$

where d_1 is a constant of integration.

Case 2: $\omega^2 - 4\psi < 0$ and $\psi \neq 0$.

$$\varphi(\xi) = \ln\{1/2\omega(-\sqrt{4\omega - \psi^2})\tanh(\tfrac{\sqrt{4\omega-\psi^2}}{2}(\xi + d_1)) - \psi)\}. \tag{7}$$

Case 3: $\psi \neq 0$ and $\omega = 0$.

$$\varphi(\xi) = -\ln\left\{\frac{\omega}{\exp(\omega(\xi+d_1))-1}\right\} \tag{8}$$

Case 4: $\omega^2 - 4\psi = 0$ and $\psi \neq 0$, $\omega \neq 0$.

$$\varphi(\xi) = \ln\left\{\frac{2(\omega(\xi+d_1))+2}{\omega^2(\xi+d_1)}\right\} \tag{9}$$

Case 5: $\psi = 0$ and $\omega = 0$.

$$\varphi(\xi) = \ln(\xi + d_1). \tag{10}$$

Step 4: Inserting (5)into (4) and using (1) the left hand side is converted into a polynomial in $\exp(-\varphi(\xi))$. Equating each coefficient of this polynomial to zero, we obtain a set of algebraic equations for $\beta, \ldots, V, \psi$ and ω.

Step 5: With the help of Maple software solving the algebraic equations obtained in step 4, we obtain the values for the constants $\beta, \ldots, V, \psi$ and ω. Replacing the values of $\beta, \ldots, V$, and the general solution of (1) into solution (5), we obtain some useful traveling wave solutions of (2).

3. APPLICATIONS OF THE TECHNIQUE

In this section, we apply $\exp(-\varphi(\xi))$-expansion technique for nonlinear partial differential equations such as the $(3+1)$ dimensional Jimbo Miwa equation and Kadomtsev-Petviashvilli equation.

The nonlinear Jimbo Miwa equation governing the isomonodromic deformation of meromorphic linear systems. It has vast applications in many fields of physical sciences. These are natural reductions of the Ernst equation and thus provide solutions to the Einstein field equations of general relativity, also give solutions of the Einstein equations in terms of theta functions. They have applications in the work in mirror symmetry. They are used to explain properties of shock wave formation for the dispersion less limit of the KdV equation.

$$s_{xxxy} + 3s_y s_{xx} + 3s_x s_{xy} + 2s_{ty} - 3s_{xz} = 0. \tag{11}$$

Introducing a transformation as $\xi = x + y + z - Vt$,

$$s^{iv} + 6s's'' - 2s''V - 3s'' = 0. \tag{12}$$

On integrating, we have

$$A - 2Vs' + s''' + 3(s')^2 - 3s' = 0. \tag{13}$$

where prime signifies the derivative w.r.t ξ.

By balancing the highest order linear and nonlinear terms, we attain value of n.

$$n = 1.$$

Equation (5) reduces to

$$s(\xi) = \beta_0 + \beta_1 e^{(-\varphi(\xi))}. \tag{14}$$

where β_0 and β_1 are the constants.

By inserting (14) into (13), we attain a polynomial in $e^{-\varphi(\xi)}$. After equating each coefficient of polynomial to zero, a set of algebraic equations are attained as:

$$-8\beta_1\psi\omega + 6\beta_1^2\psi\omega - \beta_1\omega^3 + 3\beta_1\omega + 2V\beta_1\omega = 0,$$

$$2V\beta_1 + 3\beta_1 - 8\beta_1\psi - 7\beta_1\omega^2 + 3\beta_1^2\omega^2 + 6\beta_1^2\psi = 0,$$

$$-12\beta_1\omega + 6\beta_1^2\omega = 0,$$

$$-6\beta_1 + 3\beta_1^2 = 0,$$

$$A + 2V\beta_1\psi - \beta_1\psi\omega^2 + 3\beta_1^2\psi^2 - 2\beta_1\psi^2 + 3\beta_1\psi = 0.$$

After solving the simultaneous algebraic equations, we obtain the following solution set

$$V = -2\psi + \tfrac{1}{2}\omega^2 - \tfrac{3}{2}, A = 0, \beta_0 = \beta_0, \beta_1 = 2. \qquad (15)$$

where ω and ψ are arbitrary constants.

By using (15) into (14), we obtain

$$s = \beta_0 + 2e^{(-\varphi(\xi))}. \qquad (16)$$

where $\xi = x + y + z - Vt.$

Substituting the solutions of (1) into (16), we get five cases of traveling wave solutions for the Jimbo Miwa equation.

Case 1. When $\omega^2 - 4\psi > 0$ and $\psi \neq 0$ we attain the hyperbolic function solution:

$$S_1 = \beta_0 + \frac{4\psi}{-\sqrt{\omega^2-4\psi}\tanh(\frac{1}{2}\sqrt{\omega^2-4\psi}(\xi+d_1))-\omega}.$$

Case 2. When $\omega^2 - 4\psi < 0$ and $\psi \neq 0$ we attain the trigonometric solution:

$$S_2 = \beta_0 + \frac{4\psi}{-\omega+\sqrt{-\omega^2+4\psi}\tanh(\frac{1}{2}\sqrt{-\omega^2+4\psi}(\xi+d_1))}.$$

Case 3. When $\psi = 0$ and $\omega \neq 0$, we obtain the exponential solution:

$$S_3 = \beta_0 + \frac{2\omega}{\exp(\omega(\xi+d_1))-1}.$$

Case 4. When $\omega^2 - 4\psi = 0$, $\psi = 0$ and $\omega \neq 0$, we attain the solution:

$$S_4 = \beta_0 - \frac{\psi^2(\xi+d_1)}{(\omega(\xi+d_1)+2}.$$

Case 5. When $\psi = 0$ and $\omega = 0$ we attain the rational function solution:

$$S_5 = \beta_0 + \frac{2}{\xi+d_1}.$$

where $\xi = x + y + z - (-2\psi + \frac{1}{2}\omega^2 - \frac{3}{2})t.$

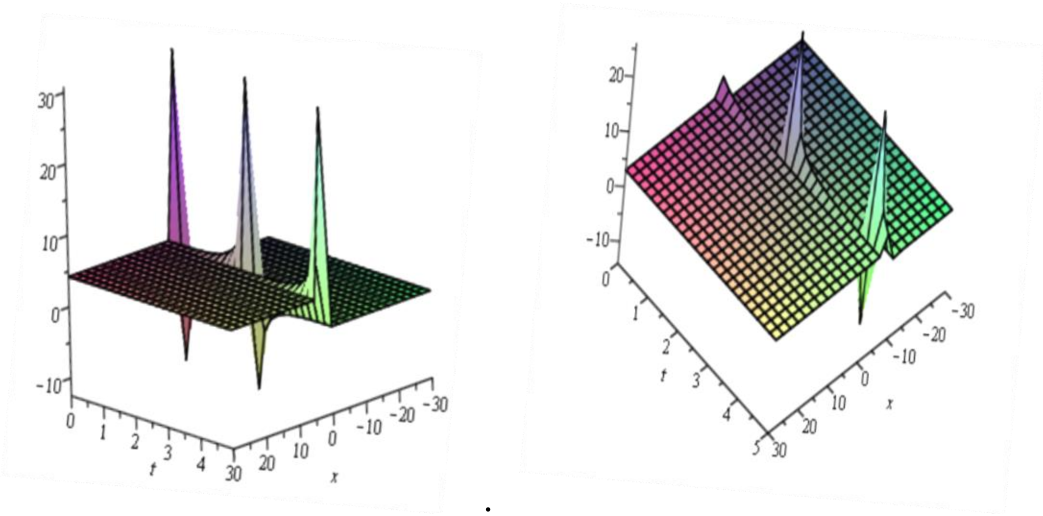

Figure 1. Trigonometric and exponential functions solution respectively for different values of parameters and for $0 \le t \le 5,\ -30 \le x \le 30.$

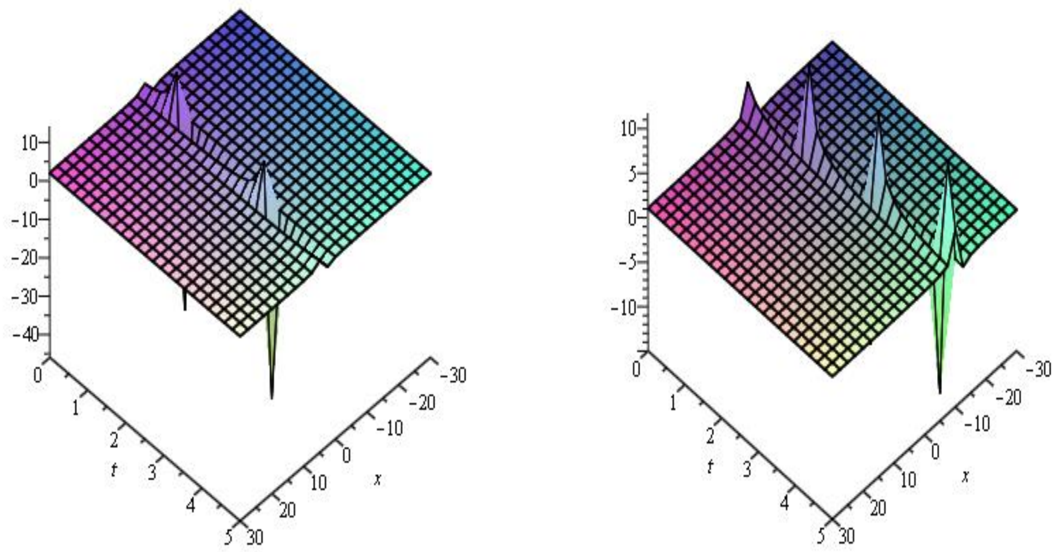

Figure 2. Rational and hyperbolic functions solution for different values of parameters and for $0 \le t \le 5,\ -30 \le x \le 30.$

Consider the Kadomtsev-Petviashvilli equation, which explains nonlinear wave motion and is completely integrable equation. It is a generalization to two spatial dimensions, x and y, of the one-dimensional Korteweg–de Vries equation. Kadomtsev-Petviashvilli (KP) equation is the mathematical modelling of the solitons in shallow water. The KP equation can be utilized to model water waves of long wavelength, with weakly nonlinear restoring forces and frequency dispersion. It can also be used to model waves in ferromagnetic media.

$$(s_t + 6ss_x + s_{xxx})_x + \gamma s_{yy} = 0. \tag{17}$$

Introducing a transformation as $\xi = x + y - Vt,$

$$-Vs'' + 6s's'' + 6(s')^2 + s^{iv} + \gamma s'' = 0. \tag{18}$$

On integrating twice, we have,

$$E + F + 3s^2 + \gamma s - Vs + s'' = 0. \tag{19}$$

where prime signifies the derivative w.r.t ξ.

By balancing the highest order linear and nonlinear terms, we attain value of n.

$$n = 2.$$

Equation (5) reduces to

$$s(\xi) = \beta_0 + \beta_1 e^{(-\varphi(\xi))} + \beta_2 (e^{(-\varphi(\xi))})^2. \tag{20}$$

where β_0, β_1 and β_2 are the constants.

We insert (20) into (19) and get a polynomial in $e^{-\varphi(\xi)}$. After equating each coefficient of polynomial to zero, we attain a set of algebraic equations as follows:

$$6\beta - 2\psi\omega + \gamma\beta_1 + \beta_1\omega^2 + 2\beta_1\omega + 6\beta_0\beta_1 - V\beta_1 = 0,$$

$$8\beta_2\psi + 6\beta_0\beta_2 + 3\beta_1\omega - V\beta_2 + 4\beta_2\omega^2 + \gamma\beta_2 + 3\beta_1^2 = 0,$$

$$10\beta_2\psi + 6\beta_1\beta_2 + 2\beta_1 = 0,$$

$$6\beta_2 + 3\beta_2^2 = 0,$$

$$E + F + 3\beta_0^2 + \gamma\beta_0 - V\beta_0 + \beta_1\psi\omega + 2\beta_2\psi^2 = 0.$$

After solving simultaneous algebraic equations, the following solution set is obtained.

$$\begin{aligned} &V = 8\psi + 6\beta_0 + \omega^2 + \gamma, E = 3\beta_0^2 + 8\beta_0\psi + \beta_0\omega^2 + 2\omega^2\psi + 4\psi^2 - \gamma, \\ &F = F, \beta_0 = \beta_0, \beta_1 = -2\omega, \beta_2 = -2. \end{aligned} \tag{21}$$

where ψ and ω are arbitrary constants.

By using (21) into (20), we get

$$s(\xi) = \beta_0 - 2\omega e^{(-\varphi(\xi))} - 2(e^{(-\varphi(\xi))})^2. \tag{22}$$

where $\xi = x + y - Vt.$

Substituting the solutions of (1) into (22), following five cases are obtained.

Case 1. When $\omega^2 - 4\psi > 0$ and $\psi \neq 0$, we get the hyperbolic function solution:

$$s_1 = \beta_0 - 4\frac{\psi\omega}{-\sqrt{\omega^2-4\psi}\,\tanh(\frac{1}{2}\sqrt{\omega^2-4\psi}\,(\xi+d_1)-\omega} - \frac{8\psi^2}{-\omega+\sqrt{-\omega^2+4\psi}\,\tanh(\frac{1}{2}\sqrt{-\omega^2+4\psi}\,(\xi+d_1)-\omega)^2}.$$

Case 2. When $\omega^2 - 4\psi < 0$ and $\psi \neq 0$, we get the trigonometric solution:

$$s_2 = \beta_0 - 4\frac{\psi\omega}{-\omega+\sqrt{-\omega^2+4\psi}\,\tanh(\frac{1}{2}\sqrt{\omega^2-4\psi}\,(\xi+d_1)-\omega} - \frac{8\psi^2}{-\omega+\sqrt{-\omega^2+4\psi}\,\tanh(\frac{1}{2}\sqrt{-\omega^2+4\psi}\,(\xi+d_1)-\omega)^2}.$$

Case 3. When $\psi = 0$ and $\omega \neq 0$, we get the exponential solution:

$$s_3 = \beta_0 - \frac{2\omega^2}{\exp(\omega(\xi + d_1) - 1)} - \frac{2\omega^2}{(\exp(\omega(\xi + d_1) - 1))^2}.$$

Case 4. When $\omega^2 - 4\psi = 0, \psi = 0$ and $\omega \neq 0$, we get the rational function solution:

$$s_4 = \beta_0 + \frac{(\omega^3(\xi + d_1))}{(\omega(\xi + d_1) + 2)} - \frac{1}{2}\frac{\omega^4(\xi + d_1)^2}{(\omega(\xi + d_1) + 2)^2}.$$

Case 5. When $\psi = 0$, $\omega = 0$, we get rational function solution:

$$s_5 = \beta_0 + \frac{2}{(\xi + d_1)^2}.$$

where as $\xi = x + y - (8\psi + 6\beta_0 + \omega^2 + \gamma)t$.

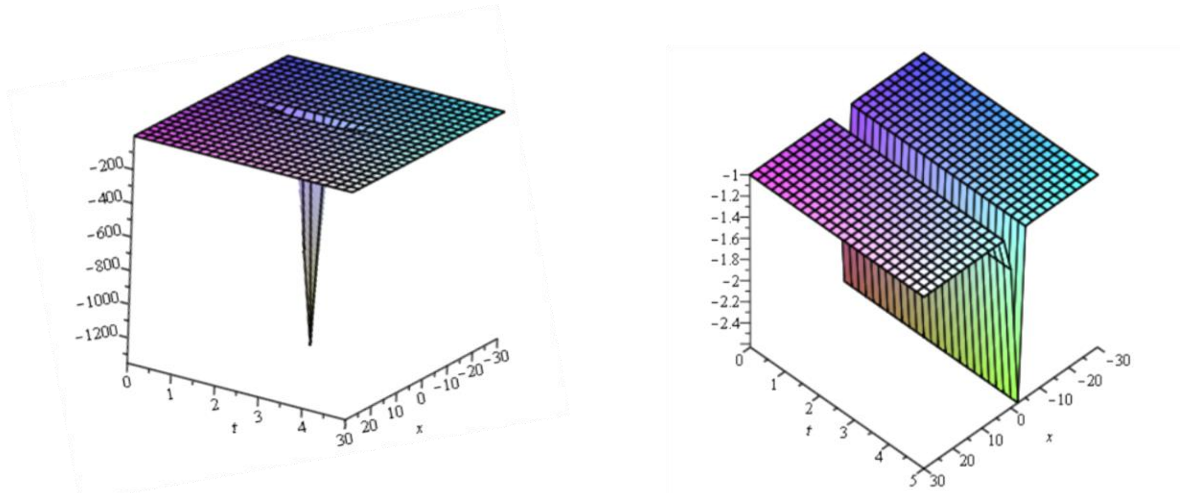

Figure 3. Hyperbolic and trigonometric functions travelling wave solution respectively for different values of parameters and for $0 \leq t \leq 5, -30 \leq x \leq 30$.

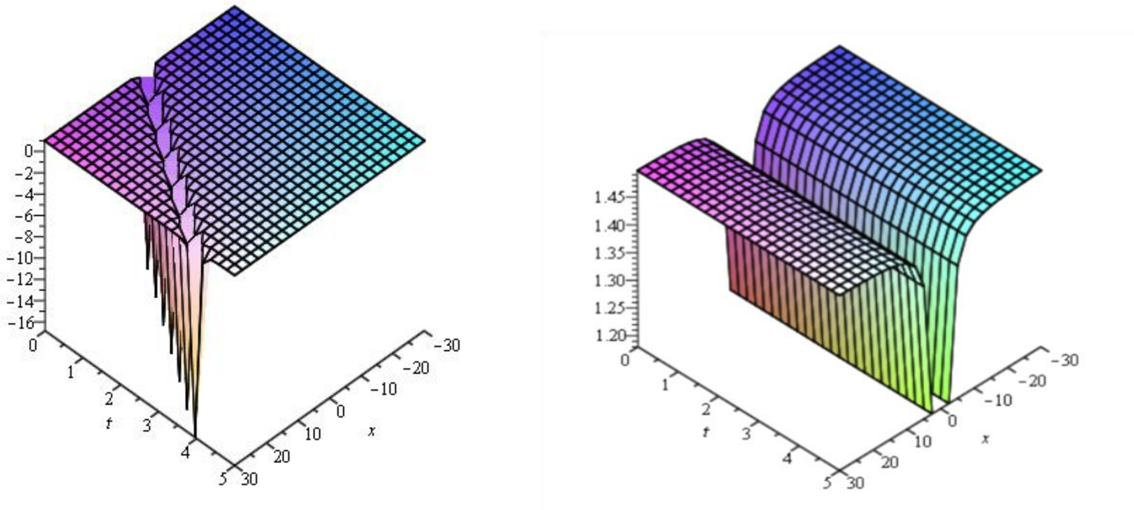

Figure 4. Exponential and rational function solutions for different values of parameters and for $0 \leq t \leq 5, -30 \leq x \leq 30$.

4. PHYSICAL DISCUSSION ON RESULTS

Soliton wave structure of non-linear partial differential equations via a reliable mathematical technique have been studied. It is noticed that, soliton is a wave which preserve its shape and speed. Mainly soliton wave is generated due to balance among the non-linear and dispersive effects. From graphical results it is noticed, an important feature of soliton waves is that these waves reserve its shape on interaction with same kind of waves. We have noticed that for positive values of velocity parameter soliton wave moves in right direction, and turns in left direction for negative values of velocity parameters. Some other behaviour of soliton waves are controlled by additional free parameters.

First we discuss graphical representation of nonlinear Jimbo Miwa equation. Fig. 1 shows trigonometric function travelling wave solution and exponential solution of Jimbo Miwa equation respectively for $\psi = -1, \omega = -1, d_1 = 1, \beta_0 = 1.$ Also Fig. 2 shows rational function solution and hyperbolic function solution for $\psi = -1, \omega = -1, d_1 = 1, \beta_0 = 1.$ At the end discuss graphical representation of nonlinear KP equation. Fig. 3 shows hyperbolic and trigonometric function travelling wave solution respectively for $\omega = -1, \psi = -1, c_1 = 1, \beta_0 = 1, \gamma = 1.$ Fig. 4 shows exponential and rational function solution for $\omega = -1, \psi = -1, c_1 = 1, \beta_0 = 1, \gamma = 1.$ Figures indicates graphical solutions for altered values of physical parameters. Since it is noticed that graphical representation of solution be not influenced by arbitrary parameters. We conclude that, different constraints being set as input to simulations.

5. CONCLUSION

In this paper, $\exp(-\varphi(\xi))$-expansion method is suggested and is applied successfully on well-known nonlinear partial differential equations; namely, the $(3+1)$ dimensional Jimbo Miwa equation and Kadomtsev-Petviashvilli equation. As a result, different types of the solutions i.e., trigonometric, hyperbolic and rational function solutions with numerous capricious parameters are revealed. The obtained solutions are more general with more parameters, and abundant new solutions are also attained by using the under study method. Computational work coupled with the graphical representation verifies the accuracy of the projected algorithms. Another important feature is that, there is no need of linearization, discretization or perturbation of terms for application of this technique. It is monitored that this technique is quite helpful, competent, also can be used for nonlinear physical problems. Computational work accompanied with the graphical presentation verifies the effectiveness of this technique.

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