ORIGINAL PAPER

THE TAN$\left(\frac{F(z)}{2}\right)$-EXPANSION METHOD FOR THE SOME TRAVELING WAVE SOLUTIONS OF THE (2+1)-DIMENSIONAL BURGERS EQUATION

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Abstract. *In this paper, we implemented a tan*$\left(\frac{F(z)}{2}\right)$*-Expansion Method for traveling wave solutions of (2+1)-dimensional Burgers equation.*

Keywords: *(2+1)-dimensional Burgers equation, tan*$\left(\frac{F(z)}{2}\right)$*-Expansion Method, Traveling wave solutions.*

1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics [1, 2]. Many analytical methods have been found in literature [3-8]. Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Using these methods, partial differential equations are transformed into ordinary differential equations. These nonlinear partial differential equations are solved with the help of ordinary differential equations. These methods are given in [9-16].

We used the tan$\left(\frac{F(z)}{2}\right)$-Expansion Method for find the traveling wave solutions of the (2+1)-dimensional Burgers equation. This method is presented by Manafian and Lakestain [17].

2. ANALYSIS OF METHOD

Let's introduce the method briefly. Consider a general partial differential equation of four variables,

$$\varphi\left(v, v_t, v_y, v_x, v_{xx}, \ldots\right) = 0. \tag{1}$$

Using the wave variable $(x, y, t) = v(z)$, $z = (x + \alpha y - kt)$, the equation (1) turns into an ordinary differential equation,

$$\varphi' = (v', v'', v''', \ldots) = 0. \tag{2}$$

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here k, α are constants. With this conversion, we obtain a nonlinear ordinary differential equation for $v(z)$. We can express the solution of equation (2) as below,

$$v(x,y,t) = v(z) = \sum_{i=0}^{m} a_i \left[p + tan\left(\frac{F(z)}{2}\right)\right]^i + \sum_{i=1}^{m} b_i \left[p + tan\left(\frac{F(z)}{2}\right)\right]^{-i}, \ a_i \neq 0\ , \ b_i \neq 0 \qquad (3)$$

here m is a positive integer and is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation, the coefficients a_i $(0 \leq i \leq$ $m,$ $bi1\leq i\leq m,$ are constant. If we write these solutions in equation (2), we obtain a system of algebraic equations for $tan\left(\frac{F(z)}{2}\right)^i, cot\left(\frac{F(z)}{2}\right)^i$ after, if the coefficients of $tan\left(\frac{F(z)}{2}\right)^i, cot\left(\frac{F(z)}{2}\right)^i$ are equal to zero, we can find the $k, \alpha, p, a_0, a_1, b_1, \dots, a_m, b_m$ constants, where $F = F(z)$ satisfies the first order nonlinear ODE:

$$F'(z) = asin\big(F(z)\big) + bcos\big(F(z)\big) + c \qquad (4)$$

and a, b and c are constants. The expressed the solutions by Manafian and Lakestain [25].

Example 1. We consider the (2+1)-dimensional Burgers equation,

$$u_t - uu_x - u_{xx} - u_{yy} = 0 \qquad (5)$$

Let us consider the traveling wave solutions $(x, y, t) = u(z)$, $z = (x + \alpha y - kt)$, then equation (5) becomes

$$-ku' - uu' - u'' - \alpha^2 u'' = 0 \qquad (6)$$

When balancing u'' with uu' then gives $m = 1$. The solution is as follows.

$$u(z) = a_0 + a_1 \left[p + tan\left(\frac{F(z)}{2}\right)\right] + b_1 \left[p + tan\left(\frac{F(z)}{2}\right)\right]^{-1} \qquad (7)$$

(7) is substituted in equation (6), a system of algebraic equations for $k, \alpha, p, a_0, a_1, b_1$ are obtained. İf the system is solved, the coefficients are found as

Case 1

$$a = -bp + cp, b - c \neq 0, b_1 = \frac{ba_1 + ca_1 + bp^2 a_1 - cp^2 a_1}{b - c}, \alpha = \sqrt{\frac{-b + c + a_1}{b - c}}, k = -a_0 \qquad (8)$$

Case 2

$$a_1 = 0, (-b - c + 2ap + bp^2 - cp^2) b_1 \neq 0, \alpha = \sqrt{\frac{b + c - 2ap - bp^2 + cp^2 - b_1}{-b - c + 2ap + bp^2 - cp^2}},$$

$$k = a + bp - cp + a\alpha^2 + bp\alpha^2 - cp\alpha^2 - a_0, \qquad (9)$$

with the help of the Mathematica program. After these operations, the solutions of equation (5) for case 1 and case 2 are as follows:

Solutions for Case 1

Solution 1

$$a^2+b^2-c^2<0 \; and \; b-c\neq 0,$$

$$u_1(x,y,t)=\frac{(b-c)\sqrt{-a^2-b^2+c^2}a_0-2(a^2+b^2-c^2)\text{Cot}\left[\sqrt{-a^2-b^2+c^2}\left(x+\sqrt{\frac{c-b+a_1}{b-c}}y+a_0t\right)\right]a_1}{(b-c)\sqrt{-a^2-b^2+c^2}} \tag{10}$$

Solution 2

$$a^2+b^2-c^2>0 \; and \; b-c\neq 0,$$

$$u_2(x,y,t)=\frac{(b-c)\sqrt{a^2+b^2-c^2}a_0+2(a^2+b^2-c^2)\text{Coth}\left[\sqrt{a^2+b^2-c^2}\left(x+\sqrt{\frac{c-b+a_1}{b-c}}y+a_0t\right)\right]a_1}{(b-c)\sqrt{a^2+b^2-c^2}} \tag{11}$$

Solution 3

$$a^2+b^2-c^2>0 \; and \; b\neq 0, c=0,$$

$$u_3(x,y,t)=\frac{b\sqrt{a^2+b^2}a_0+2(a^2+b^2)\text{Coth}\left[\sqrt{a^2+b^2}\left(x+\sqrt{\frac{a_1}{b}-1}\,y+a_0t\right)\right]a_1}{b\sqrt{a^2+b^2}} \tag{12}$$

Solution 4

$$a^2+b^2-c^2<0 \; and \; c\neq 0, b=0,$$

$$u_4(x,y,t)=\frac{c\sqrt{c^2-a^2}a_0+2(a^2-c^2)\text{Cot}\left[\sqrt{c^2-a^2}\left(x+\sqrt{-\frac{c+a_1}{c}}\,y+a_0t\right)\right]a_1}{c\sqrt{c^2-a^2}} \tag{13}$$

Solution 5

$$a=c=wa \; and \; b=-wa$$

$$u_5(x,y,t)=\frac{\left(-1+e^{2aw\left(x+\sqrt{-1-\frac{a_1}{2aw}}y+a_0t\right)}\right)a_0-\left(-1+e^{2aw\left(x+\sqrt{-1-\frac{a_1}{2aw}}y+a_0t\right)}\right)a_1}{\left(-1+e^{2aw\left(x+\sqrt{-1-\frac{a_1}{2aw}}y+a_0t\right)}\right)} \tag{14}$$

Solution 6

$$c = a,$$

$$u_6(x,y,t) =$$

$$a_0 - \frac{b^2 a_1}{(a-b)(b-a)\left(\frac{a}{a-b} - \frac{-1+(a+b)e^{b\left(x+\sqrt{\frac{a-b+a_1}{b-a}}y+a_0 t\right)}}{-1+(a-b)e^{b\left(x+\sqrt{\frac{a-b+a_1}{b-a}}y+a_0 t\right)}}\right)} + \left(\frac{a}{a-b} - \frac{-1+(a+b)e^{b\left(x+\sqrt{\frac{a-b+a_1}{b-a}}y+a_0 t\right)}}{-1+(a-b)e^{b\left(x+\sqrt{\frac{a-b+a_1}{b-a}}y+a_0 t\right)}}\right)a_1 \quad (15)$$

Solution 7

$$a = c,$$

$$u_7(x,y,t) =$$

$$a_0 + \frac{b^2 a_1}{(b-c)^2\left(\frac{c}{c-b} + \frac{1+(b+c)e^{b\left(x+\sqrt{\frac{c-b+a_1}{b-c}}y+a_0 t\right)}}{-1+(b-c)e^{b\left(x+\sqrt{\frac{c-b+a_1}{b-c}}y+a_0 t\right)}}\right)} + \left(\frac{c}{c-b} + \frac{1+(b+c)e^{b\left(x+\sqrt{\frac{c-b+a_1}{b-c}}y+a_0 t\right)}}{-1+(b-c)e^{b\left(x+\sqrt{\frac{c-b+a_1}{b-c}}y+a_0 t\right)}}\right)a_1 \quad (16)$$

Solution 8

$$c = -a,$$

$$u_8(x,y,t) = \frac{(a+b)\left((a+b)^2 - e^{2b\left(x+\sqrt{-\frac{a+b-a_1}{a+b}}y+a_0 t\right)}\right)a_0 - 2b\left((a+b)^2 + e^{2b\left(x+\sqrt{-\frac{a+b-a_1}{a+b}}y+a_0 t\right)}\right)a_1}{(a+b)\left(a+b-e^{b\left(x+\sqrt{-\frac{a+b-a_1}{a+b}}y+a_0 t\right)}\right)\left(a+b+e^{b\left(x+\sqrt{-\frac{a+b-a_1}{a+b}}y+a_0 t\right)}\right)} \quad (17)$$

Solution 9

$$b = -c,$$

$$u_9(x,y,t) = \frac{\left(-c+c^3 e^{\left(2ax+\sqrt{-4-\frac{2a_1}{c}}ay+2aa_0 t\right)}\right)a_0 - a\left(1+c^2 e^{\left(2ax+\sqrt{-4-\frac{2a_1}{c}}ay+2aa_0 t\right)}\right)a_1}{c\left(-1+c^2 e^{\left(2ax+\sqrt{-4-\frac{2a_1}{c}}ay+2aa_0 t\right)}\right)} \quad (18)$$

Solutions for Case 2

Solution 1

$$a^2 + b^2 - c^2 < 0 \text{ and } b - c \neq 0,$$

$$u_{10}(x,y,t) = a_0 + \frac{(b-c)b_1}{a+bp-cp-\sqrt{-a^2-b^2+c^2}\,Tan\left[\frac{1}{2}\sqrt{-a^2-b^2+c^2}\left(x+\sqrt{-\frac{b+c-2ap-bp^2+cp^2-b_1}{b+c-2ap-bp^2+cp^2}}y+\left(\frac{(b+c-2ap-bp^2+cp^2)a_0-(a+bp-cp)b_1}{b+c-2ap-bp^2+cp^2}\right)t\right)\right]} \quad (19)$$

Solution 2

$$a^2 + b^2 - c^2 > 0 \text{ and } b - c \neq 0,$$

$$u_{11}(x,y,t) = a_0 + \frac{(b-c)b_1}{a+bp-cp-\sqrt{a^2+b^2-c^2}\,Tanh\left[\frac{1}{2}\sqrt{a^2+b^2-c^2}\left(x+\sqrt{-\frac{b+c-2ap-bp^2+cp^2-b_1}{b+c-2ap-bp^2+cp^2}}y+\left(\frac{(b+c-2ap-bp^2+cp^2)a_0-(a+bp-cp)b_1}{b+c-2ap-bp^2+cp^2}\right)t\right)\right]} \quad (20)$$

Solution 3

$$a^2 + b^2 - c^2 > 0 \text{ and } b \neq 0, c = 0,$$

$$u_{12}(x,y,t) = a_0 + \frac{b b_1}{a + bp + \sqrt{a^2 + b^2}\,Tanh\left[\frac{1}{2}\sqrt{a^2 + b^2}\left(x + \sqrt{-\frac{2ap + b(p^2-1) + b_1}{2ap + b(p^2-1)}}\,y + \left(\frac{(a+bp)b_1}{2ap + b(p^2-1)} + a_0\right)t\right)\right]} \quad (21)$$

Solution 4

$$a^2 + b^2 - c^2 < 0 \text{ and } c \neq 0, b = 0,$$

$$u_{13}(x,y,t) = a_0 + \frac{c b_1}{-a+cp+\sqrt{c^2-a^2}\,Tan\left[\frac{1}{2}\sqrt{c^2-a^2}\left(x+\sqrt{\frac{2ap-c-cp^2+b_1}{c-2ap+cp^2}}y+\left(\frac{(-a+cp)b_1}{c-2ap+cp^2}+a_0\right)t\right)\right]} \quad (22)$$

Solution 5

$$a = c = wa \text{ and } b = -wa,$$

$$u_{14}(x,y,t) = a_0 + \frac{\left(-1 + e^{aw\left(x+\frac{\sqrt{2a(p-1)pw-b_1}}{\sqrt{2}\sqrt{-a(p-1)pw}}y+\left(\frac{2(p-1)pa_0+(2p-1)b_1}{2(p-1)p}\right)t\right)}\right)}{e^{aw\left(x+\frac{\sqrt{2a(p-1)pw-b_1}}{\sqrt{2}\sqrt{-a(p-1)pw}}y+\left(\frac{2(p-1)pa_0+(2p-1)b_1}{2(p-1)p}\right)t\right)}(p-1) - p} \tag{23}$$

Solution 6

$$c = a,$$

$$u_{15}(x,y,t) = a_0 + \frac{b_1}{-\frac{-1+(a+b)e^{b\left(x+\frac{\sqrt{b+a(p-1)^2-bp^2-b_1}}{\sqrt{(a+b-ap+bp)(p-1)}}y+\left(\frac{(b+a(p-1)^2-bp^2)a_0+(a(p-1)-bp)b_1}{b+a(p-1)^2-bp^2}\right)t\right)}}{-1+(a-b)e^{b\left(x+\frac{\sqrt{b+a(p-1)^2-bp^2-b_1}}{\sqrt{(a+b-ap+bp)(p-1)}}y+\left(\frac{(b+a(p-1)^2-bp^2)a_0+(a(p-1)-bp)b_1}{b+a(p-1)^2-bp^2}\right)t\right)}}+p} \tag{24}$$

Solution 7

$$a = c,$$

$$u_{16}(x,y,t) = a_0 + \frac{b_1}{\frac{1+(b+c)e^{b\left(x+\frac{\sqrt{-(p-1)(b+c+bp-pc)-b_1}}{\sqrt{(p-1)(b+c+bp-pc)}}y+\left(\frac{(p-1)(b+c+bp-pc)a_0+(c+bp-cp)b_1}{(p-1)(b+c+bp-pc)}\right)t\right)}}{-1+(b-c)e^{b\left(x+\frac{\sqrt{-(p-1)(b+c+bp-pc)-b_1}}{\sqrt{(p-1)(b+c+bp-pc)}}y+\left(\frac{(p-1)(b+c+bp-pc)a_0+(c+bp-cp)b_1}{(p-1)(b+c+bp-pc)}\right)t\right)}}+p} \tag{25}$$

Solution 8

$$c = -a,$$

$$u_{17}(x,y,t) = a_0 + \frac{\left(-a-b+e^{b\left(x+\frac{\sqrt{-(p+1)(a-b+bp+ap)-b_1}}{\sqrt{(p+1)(b(p-1)+a(p+1))}}y+\left(\frac{(p+1)(a-b+ap+bp)a_0+(a+ap+bp)b_1}{(p+1)(b(p-1)+a(p+1))}\right)t\right)}\right)b_1}{b-bp-a(p+1)+e^{b\left(x+\frac{\sqrt{-(p+1)(a-b+bp+ap)-b_1}}{\sqrt{(p+1)(b(p-1)+a(p+1))}}y+\left(\frac{(p+1)(a-b+ap+bp)a_0+(a+ap+bp)b_1}{(p+1)(b(p-1)+a(p+1))}\right)t\right)}(p+1)} \quad (26)$$

Solution 9

$$b = -c,$$

$$u_{18}(x,y,t) = a_0 + \frac{b_1}{-\frac{ae^{a\left(x+\frac{\sqrt{-2ap+2cp^2-b_1}}{\sqrt{2}\sqrt{p(a-pc)}}y+\left(\frac{2p(a-pc)a_0+(a-2pc)b_1}{2p(a-pc)}\right)t\right)}}{-1+ce^{a\left(x+\frac{\sqrt{-2ap+2cp^2-b_1}}{\sqrt{2}\sqrt{p(a-pc)}}y+\left(\frac{2p(a-pc)a_0+(a-2pc)b_1}{2p(a-pc)}\right)t\right)}}+p} \quad (27)$$

4. CONCLUSION

It was used the $\tan\left(\frac{F(z)}{2}\right)$ expansion method for find traveling wave solutions of (2+1)-dimensional Burgers equation. This method has been successfully applied to solve some nonlinear wave equations and can be used to many other nonlinear equations or coupled ones.

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