PARALLEL TRANSPORTS WITH RESPECT TO FRENET AND DARBOUX FRAMES IN THE GALILEAN SPACE

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Abstract. *In this paper, we defined Fermi-Walker derivative in Galilean space* $\mathbb{G}^3$ *Fermi-Walker transport and non-rotating frame by using Fermi-Walker derivative are given in* $\mathbb{G}^3$*. Being conditions of Fermi-Walker transport and non-rotating frame are investigated along any curve for Frenet frame and Darboux frame.*

Keywords: *Fermi-Walker derivative, Fermi-Walker transport, Darboux frame, Non-rotating frame, Galilean space.*

1. INTRODUCTION

The universe is perceptible through observation. A relativistic observer g needs reference frames, for measurements of durations and of ("geo") metric quantities: the proper time ("proper clock") is given by its own canonical parameter running on an interval of the real numbers axis; the restspaces are refered to "fixed" directions, maintained by gyroscopes focused toward "fixed" celestial bodies.

The choice of an appropriate reference frame is a fundamental and controversial problem in astronomy [1]: one needs a "center" and several "fixed" directions. In a general relativistic setting, if g is freely falling, its restspaces are transported through Levi-Civita parallelism, so a fix spacelike direction has, by definition, a null covariant derivative [2, 3]. If g is not freely falling i.e. for accelerated observes, the restspace are not transported by the Levi-Civita parallelism, anymore. In this case, in order to define "constant" directions, another parallelism is used: the Fermi-Walker transport which is an isometry between the tangent space along g [4-8]. Fermi-Walker transport is a process used to define a coordinate system or reference frame in general relativity. All the curvatures in the reference frame in due to the presence of mass-energy density. These curvatures are not arbitrary spin or rotation of the frame.

There are various transport laws. Along a given curve, parallel transport of a tensor is defined as that its covariant derivative is zero. In a similar way, Fermi-Walker transport of a tensor along a curve is defined as that its Fermi derivative is zero [9]. If the given curve is geodesic, parallel and Fermi-Walker transport coincide. In general, this is not the case, i.e. these transports don't coincide [10].

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As stated in [11] "A Fermi-Walker transported set of tetrad fields is the best approximation to a non-rotating reference frame in the sense of Newtonian mechanics", important phenomena like the gyroscopic precession [11] can be characterized by the relative rotational acceleration of a Frenet-Serret frame with respect to a Fermi-Walker transported frame.

Parallel vector fields have important applications in differential geometry, physics and especially in robotic kinematics. The tangent vector of the curve is parallel along the curve if and only if $\nabla_T T = 0$ in Euclidean space. In this case, the curve is a geodesic in Euclidean space $\mathbb{E}^n$. Similarly, the curve which is on the surface is a geodesic if and only if $\overline{\nabla}_T T = 0$. Namely, the tangent vector is parallel along the curve on the surface. All the straight lines are geodesic curves in Euclidean space. I wonder if all the curves will be geodesic in Euclidean space? The answer to this is hidden in the connection which is obtained by using Fermi-Walker derivative. Indeed, the solution of $\widetilde{\nabla}_T T = 0$, is provided for all curves in $\mathbb{E}^n$. Accordingly, the curves and the lines are the same. That is, the curves behave like the lines with respect to Fermi-Walker connection which is a affine connection.

In [12, 13], Fermi-Walker derivative along any space curve was identified and was given physical properties in $\mathbb{E}^3$. In [14], Fermi-Walker derivative, Fermi-Walker transport and non-rotating frame are analyzed for Bishop, Darboux and Frenet frames along the curve in Euclidean space. In [15, 16], we have shown Fermi-Walker and modified Fermi-Walker derivative, and non-rotating frame are being conditions are analyzed in Minkowski space $\mathbb{E}_1^3$.

In [10], Fermi-Walker derivative is redefined in dual space $\mathbb{D}^3$. Fermi-Walker transport and non-rotating frame being conditions are analyzed along the dual curves in dual space $\mathbb{D}^3$. The notion of Fermi-Walker derivative, it shows us one method, which is used for defining "constant" direction, that may contain lots of condition to have Fermi-Walker transport or non-rotating frame. The condition of Fermi-Walker transport depends on a solution that contains differential equation system which is not always easy to find the answer. Therefore, it is important to analyze this concept. In this paper, Fermi-Walker derivative, Fermi-Walker transport and non-rotating frame concepts are defined along any curve and the notions have been analyzed for both isotropic and non-isotropic vector fields.

We have investigated Fermi-Walker derivative and geometric applications in various spaces like Euclidean, Lorentz and Dual space up to now [10, 14, 15]. The Fermi-Walker derivative which is defined in $\mathbb{G}^3$ is different from them so far since it is examined for both isotropic and non-isotropic vector fields.

Firstly, Fermi-Walker derivative is redefined for any isotropic vector fields along a curve which is in Galilean space. We have proved Fermi-Walker derivative that is defined for the isotropic vector fields coincides with Fermi derivative which is defined in any surface. We have shown Fermi-Walker derivative which is defined for any non-isotropic vector fields is not coincides with derivative of the vector fields. Being Fermi-Walker transport conditions are examined for any isotropic and non-isotropic vector fields. We have shown that if the curve is a line or a planar curve which is not a line then the non-zero isotropic vector field is Fermi-Walker transported. We have obtained that Frenet frame is not a non-rotating frame if the curve is not a line.

Then, similar investigations have been made for any isotropic and non-isotropic vector fields with respect to the Darboux frame in Galilean space. We have proved while the curve is a line the Darboux frame is a non-rotating frame.

2. PRELIMINARIES

In non-homogeneous coordinates the group of motion of 3-dimensional Galilean Geometry (i.e. the group of isometries of $\mathbb{G}^3$) has the form define:

$$\begin{aligned}
\overline{x} &= a_1 + x, \\
\overline{y} &= a_2 + a_3 x + y\cos j + z\sin j, \\
\overline{z} &= a_4 + a_5 x - y\sin j + z\cos j,
\end{aligned} \tag{1}$$

where a_1, a_2, a_3, a_4, a_5, and j are real numbers [17].

If the first component of a vector is not zero, then the vector is called as non-isotropic, otherwise it is called isotropic vector [17].

The scalar product of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{G}^3$ is defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \begin{cases} v_1 w_1, & \text{if } v_1 \neq 0 \text{ or } w_1 \neq 0 \\ v_2 w_2 + v_3 w_3, & \text{if } v_1 = 0 \text{ and } w_1 = 0. \end{cases} \tag{2}$$

If $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, then $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

The norm of $\mathbf{w}$ is defined by $\|\mathbf{w}\|_G = \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}$. Also, the Galilean cross product of two vectors defined by

$$\mathbf{v} \times_G \mathbf{w} = \begin{vmatrix} 0 & \mathbf{e_2} & \mathbf{e_3} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{3}$$

for $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ [18].

Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}^3$ be a curve parameterized by arc length (we abbreviate as p.b.a.l) with curvature $k > 0$ and torsion t. If a is a unit speed curve,

$$a(x) = \left(x, y(x), z(x) \right),$$

then the Frenet frame fields are given by

$$\begin{aligned}
T(x) &= \alpha'(x), \\
N(x) &= \frac{\alpha''(x)}{\|\alpha''(x)\|_G}, \\
B(x) &= T(x) \times_G N(x) = \frac{1}{\kappa(x)}\left(0, -z''(x), y''(x) \right),
\end{aligned} \tag{4}$$

where $k(x)$ and $t(x)$ are defined by

$$\kappa(x) = \| \alpha''(x) \|_G, \ \tau(x) = \frac{\det(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)}.$$

The vectors T, N and B are called the vectors of the tangent, the principal normal and the binormal vector field, respectively [18]. Therefore, the Frenet-Serret formulae can be written as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & t \\ 0 & -t & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{5}$$

Theorem 2.1. For any curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}^3$, we call $D(x) = t(x)T(x) + k(x)B(x)$ a *Darboux vector* of a [19]. By using the Darboux vector, Frenet-Serret formulas can be rewritten as follows:

$$\begin{aligned} T\ell(x) &= D(x) \ ´_G \ T(x) \\ N\ell(x) &= D(x) \ ´_G \ N(x) \\ B\ell(x) &= D(x) \ ´_G \ B(x). \end{aligned} \tag{6}$$

We define a vector $\widetilde{D}(x) = \left(\frac{\tau}{\kappa}\right)(x)T(x) + B(x)$ and we call it a modified Darboux vector along a [19]. For more on Galilean Geometry, one can refer to [17, 18, 20, 21] and references there in.

Definition 2.2. X is any vector field and a is unit-speed any curve in Galile space, then

$$\widetilde{\nabla}_T X = \nabla_T X - \langle T, X \rangle A + \langle A, X \rangle T \tag{7}$$

defined as $\widetilde{\nabla}_T X$ derivative is called Fermi-Walker derivative in Galilean space $\mathbb{G}^3$. Here T is the tangent vector field of a and $A = \nabla_T T$.

Definition 2.3. In Galilean space $\mathbb{G}^3$, let $\alpha : I \subset \mathbb{R} \to \mathbb{G}^3$ be a curve and X be any vector field along the curve a. If the Fermi-Walker derivative of the vector field X vanishes, i.e., if $\widetilde{\nabla}_T X = 0$, then X is called the Fermi-Walker transported vector field along the curve.

Definition 2.4. Let a unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}^3$ together with orthonormal vector field U, V, W along a be given. If the Fermi-Walker derivative of the vector field vanish, then $\{U, V, W\}$ is called non-rotating frame.

3. FRENET FRAME AND FERMI-WALKER DERIVATIVE

In this section, Fermi-Walker derivative, Fermi-Walker transport and non-rotating frame concepts have been investigated along any curve which is in Galilean space. Fermi-Walker derivative has been redefined along a curve for both isotropic and non-isotropic vector fields. The vector fields which are Fermi-Walker transported are analyzed along any curve in $\mathbb{G}^3$. Then; we show that the Frenet frame whether it is a non-rotating frame or not.

Lemma 3.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}^3$ be a curve in Galilean space $\mathbb{G}^3$ and X is any vector field along the curve $a(x)$, Fermi-Walker derivative can be expressed as

i) If X is an isotropic vector field along the $a(x)$, then Fermi-Walker derivative of X is given by

$$\widetilde{\nabla}_T X = \nabla_T X + \kappa \langle N, X \rangle T.$$

ii) If X is a non-isotropic vector field along the $a(x)$, then Fermi-Walker derivative of X is given by

$$\widetilde{\nabla}_T X = \nabla_T X - \kappa \langle T, X \rangle N.$$

Proof. Using definition 2.2 and the equation 2, the above equations are obtained.

Corollary 3.2. Let X be an isotropic vector field along the curve a. Then, the Fermi-Walker derivative coincides with Fermi derivative.

Corollary 3.3. Let X be a non-zero isotropic vector field along the curve $a(x)$ which is not a line. Fermi-Walker derivative coincides with derivative of X if and only if the vector field X is linearly dependent with the binormal vector field B.

Proof. Using lemma 3.1(i), $X = mN + lB$ and $\widetilde{\nabla}_T X = \nabla_T X + \mu \kappa T$ is obtained. Therefore, $\widetilde{\nabla}_T X = \nabla_T X$ iff $m = 0$.

Corollary 3.4. Let X be a non-isotropic vector field along the curve $a(x)$. Then, Fermi-Walker derivative is not coincide with derivative of X.

Theorem 3.5. Let a be a curve in $\mathbb{G}^3, X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ be any non-isotropic vector field along a. The vector field X is Fermi-Walker transported along the curve a if and only if

$$l_1(x) = \text{const.},$$

$$l_2(x) = c_1 \cos\left(\int_0^x t(t)dt\right) + c_2 \sin\left(\int_0^x t(t)dt\right)$$

$$l_3(x) = c_2 \cos\left(\int_0^x t(t)dt\right) - c_1 \sin\left(\int_0^x t(t)dt\right)$$

where c_1, c_2 are constants of integration and $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of arc length parameter x.

Proof. By the lemma 3.1. ii),

$$\widetilde{\nabla}_T X = \left(\frac{d\lambda_1}{dx}\right)T + \left(\frac{d\lambda_2}{dx} - \tau\lambda_3\right)N + \left(\frac{d\lambda_3}{dx} + \tau\lambda_2\right)B$$

is obtained. X is Fermi-Walker transported along the curve iff

$$\frac{d\lambda_1}{dx} = 0,$$

$$\frac{d\lambda_2}{dx} - \tau\lambda_3 = 0,$$

$$\frac{d\lambda_3}{dx} + \tau\lambda_2 = 0.$$

From the solution of the equation system,

$$\lambda_1 = const.,$$

$$\lambda_2 = c_1\cos\left(\int_1^x \tau(t)dt\right) + c_2\sin\left(\int_1^x \tau(t)dt\right),$$

$$\lambda_3 = c_2\cos\left(\int_1^x \tau(t)dt\right) - c_1\sin\left(\int_1^x \tau(t)dt\right).$$

The rest is obvious.

Corollary 3.6 Let $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ be any non-isotropic vector field along α and the parameters λ_i are constants. The vector field X is Fermi-Walker transported if and only if the curve α is the planar curve or $\lambda_2 = \lambda_3 = 0$.

Proof. Using Theorem 3.5 and " $\lambda_i = const.,$

$$\widetilde{\nabla}_T X = \tau\left(-\lambda_3 N + \lambda_2 B\right)$$

is obtained. Therefore, the proof is clear.

Theorem 3.7. Let α be a curve in $\mathbb{G}^3$ and $X = \lambda_2 N + \lambda_3 B$ be any non-zero isotropic vector field along α. The vector field X is Fermi-Walker transported along the curve α if and only if

$$\lambda_2 k = 0$$

$$\lambda_2 = c_1 \cos\left(\int_1^x t(t)dt\right) + c_2 \sin\left(\int_1^x t(t)dt\right)$$

$$\lambda_3 = c_2 \cos\left(\int_1^x t(t)dt\right) - c_1 \sin\left(\int_1^x t(t)dt\right)$$

where c_1, c_2 are constants of integration and λ_2, λ_3 are continuously differentiable functions of arc length parameter x.

Proof. By the lemma 3.1. i),

$$\widetilde{\nabla}_T X = \left(\lambda_2 \kappa\right) T + \left(\frac{d\lambda_2}{dx} - \tau\lambda_3\right) N + \left(\frac{d\lambda_3}{dx} + \tau\lambda_2\right) B$$

is obtained. X is Fermi-Walker transported along the curve iff

$$\lambda_2 k = 0,$$

$$\frac{d\lambda_2}{dx} - t\lambda_3 = 0,$$

$$\frac{d\lambda_3}{dx} + t\lambda_2 = 0.$$

This is equivalent to

$$\lambda_2 k = 0,$$

$$\lambda_2 = c_1 \cos\left(\int_1^x t(t)dt\right) + c_2 \sin\left(\int_1^x t(t)dt\right),$$

$$\lambda_3 = c_2 \cos\left(\int_1^t t(t)dt\right) - c_1 \sin\left(\int_1^x t(t)dt\right).$$

The rest is obvious.

Corollary 3.8. Let a be a curve in $\mathbb{G}^3$ and $X = \lambda_2 N + \lambda_3 B$ be any non-zero isotropic vector field along a.

i) If a is a line in Galilean space, then the vector field X is Fermi-Walker transported.

ii) If a is a planar curve which is not a line, and $\lambda_2 = 0$ then the vector field X is Fermi-Walker transported.

Corollary 3.9. Let $\{T, N, B\}$ be the Frenet frame of a. The $\{T, N, B\}$ is a non-rotating frame along the curve if and only if the curve is a line. Otherwise, the Frenet frame is not a non-rotating frame along the curve in Galilean space.

4. DARBOUX FRAME AND FERMI-WALKER DERIVATIVE IN $\mathbb{G}^3$

Frame fields constitute a very useful tool for studying curves and surfaces. However, the Frenet frame T, N, B of a is not useful to describe the geometry of surface M. Since N and B in general will be neither tangent nor perpendicular to M. Therefore, we require another frame of a for study the relation between the geometry of a and M. There is such a frame field that is called Darboux frame field of a with respect to M. The Darboux frame field consists of the triple of vector fields T, Q, n. The first and last vector fields of this frame T and n are a unit tangent vector field of a and unit normal vector field of M at the point $a(x)$ of a. Let $Q = n \times_G T$ be the tangential-normal.

Theorem 4.1. Let $\alpha : I \subset \mathbb{R} \to M \subset \mathbb{G}^3$ be a unit-speed curve, and let $\{T, Q, n\}$ be the Darboux frame field of a with respect to M. Then

$$\begin{bmatrix} T \\ Q \\ n \end{bmatrix}' = \begin{bmatrix} 0 & k_g & k_n \\ 0 & 0 & t_g \\ 0 & -t_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ n \end{bmatrix}. \tag{8}$$

where k_g and k_n give the tangential and normal component of the curvature vector, and these functions are called the geodesic and the normal curvature, respectively [22].

Proof. We have

$$\begin{aligned} T' &= (T' \times_G Q)Q + (T' \times_G n)n \\ &= (a'' \times_G Q)Q + (a'' \times_G n)n \\ &= k_g Q + k_n n. \end{aligned} \tag{9}$$

The other formulae are proved in a similar fashion. Also, equation (5) implies the important relations

$$k^2(x) = k_g^2(x) + k_n^2(x), \qquad tau(x) = -t_g(x) + \frac{k_g'(x)k_n(x) - k_g(x)k_n'(x)}{k_g^2(x) + k_n^2(x)}$$

where $k^2(x)$ and $t(x)$ are the square curvature and the torsion of a, respectively.

Lemma 4.2. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}^3$ be a curve in $\mathbb{G}^3$ and X is any vector field along the curve, Fermi-Walker derivative with respect to the Darboux frame can be expressed as

i) If X is an isotropic vector field along a, then Fermi-Walker derivative with respect to the Darboux frame of X is given by

$$\widetilde{\nabla}_T X = \nabla_T X + \left(\kappa_g \langle Q, X \rangle + \kappa_n \langle n, X \rangle \right) T.$$

ii) If X is a non-isotropic vector field along a in $\mathbb{G}^3$, then Fermi-Walker derivative with respect to the Darboux frame of X is given by

$$\widetilde{\nabla}_T X = \nabla_T X - \left(\kappa_g Q + \kappa_n n \right) \langle T, X \rangle.$$

Proof. Using definition 2.2 and the equation (2), the above equations are obtained.

Corollary 4.3. Let $X = \lambda_2 Q + \lambda_3 n$ be an isotropic vector field along the curve a. The Fermi-Walker derivative coincides with derivative of X iff $\lambda_2 k_g + \lambda_3 k_n = 0$.

Corollary 4.4. Let X be a non-isotropic vector field along the curve a. The Fermi-Walker derivative coincides with derivative of X iff a is a line in $\mathbb{G}^3$.

Proof. Using lemma 4.2(ii) and $X = \lambda_1 T + \lambda_2 Q + \lambda_3 n$,

$$\widetilde{\nabla}_T X = \nabla_T X - \lambda_1 (\kappa_g Q + \kappa_n n)\$$

is obtained. Since X is a non-isotropic, $\lambda_1 \neq 0$. Therefore, $\widetilde{\nabla}_T X = \nabla_T X$ iff $k_g Q + k_n n = 0$. Hence, $k_g = k_n = 0$. That is, the curve is a line in $\mathbb{G}^3$.

Theorem 4.5. Let a be a curve in $\mathbb{G}^3$, $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ be any non-isotropic vector field along a. The vector field X is Fermi-Walker transported along the curve a if and only if

$$
\begin{aligned}
\lambda_1(x) &= \text{const}, \\
\lambda_2(x) &= c_1 \cos\left(\int_1^x t_g(t)dt \right) + c_2 \sin\left(\int_1^x t_g(t)dt \right) \\
\lambda_3(x) &= c_2 \cos\left(\int_1^x t_g(t)dt \right) - c_1 \sin\left(\int_1^x t_g(t)dt \right)
\end{aligned}
$$

where c_1, c_2 are constants of integration and $\lambda_1, \lambda_2, \lambda_3$ are continuously differentiable functions of arc length parameter x.

Proof. By the lemma 4.2(ii), the proof is obvious.

Corollary 4.6. Let $X = \lambda_1 T + \lambda_2 N + \lambda_3 B$ be any non-isotropic vector field along a and the parameters λ_i are constants. The vector field X is Fermi-Walker transported iff the curve a is the line of curvature or $\lambda_2 = \lambda_3 = 0$.

Proof. Using lemma 4.2(ii) and $\forall \lambda_i = const.$,

$$\widetilde{\nabla}_T X = \tau_g \left(\lambda_2 n + -\lambda_3 Q \right)$$

is obtained. Using the above equation, the proof can be obtained.

Theorem 4.7. Let a be a curve in $\mathbb{G}^3, X = \lambda_2 Q + \lambda_3 n$ be any non-zero isotropic vector field along a. The vector field X is Fermi-Walker transported along the curve a if and only if

$$\lambda_2 \kappa_g + \lambda_3 \kappa_n = 0$$

$$\lambda_2 = c_1 \cos(\int_1^x \tau_g (t)dt) + c_2 \sin(\int_1^x \tau_g (t)dt)$$

$$\lambda_3 = c_2 \cos(\int_1^x \tau_g (t)dt) - c_1 \sin(\int_1^x \tau_g (t)dt)$$

where c_1, c_2 are constants of integration and λ_2, λ_3 are continuously differentiable functions of arc length parameter x.

Proof. By the lemma 4.2(i), the results are obvious.

Corollary 4.8. Let a be a curve in $\mathbb{G}^3, X = \lambda_2 N + \lambda_3 B$ be any non-zero isotropic vector field along a and the parameters λ_i are constants. The vector field X along the curve a in $\mathbb{G}^3$ is the Fermi-Walker transported iff

$$\left(\frac{k_g}{k_n} \right)' = 0.$$

Corollary 4.9. Let $\{T, Q, n\}$ be the Darboux frame of the curve a. $\{T, Q, n\}$ Darboux frame of the curve is a non-rotating frame if and only if the curve is a line. Otherwise, the Darboux frame is not a non-rotating frame along the curve in Galilean space.

5. CONCLUSIONS

The notion of Fermi-Walker derivative, it shows us one method, which is used for defining "constant" direction, that may contain lots of condition to have Fermi-Walker transport or non-rotating frame. The condition of Fermi-Walker transport depends on a solution that contains differential equation system which is not always easy to find the answer. Therefore, it is important to analyze this concept. In this paper, Fermi-Walker derivative, Fermi-Walker transport and non-rotating frame concepts are defined along any curve and the notions have been analyzed or both isotropic and non-isotropic vector fields.

We have investigated Fermi-Walker derivative and geometric applications in various spaces like Euclidean, Lorentz and Dual space up to now [10, 14, 15]. The Fermi-Walker derivative which is defined in $\mathbb{G}^3$ is different from them so far since it is examined for both isotropic and non-isotropic vector fields.

Firstly, Fermi-Walker derivative is redefined for any isotropic vector fields along a curve which is in Galilean space. We have proved Fermi-Walker derivative that is defined for the isotropic vector fields coincides with Fermi derivative which is defined in any surface. We have shown Fermi-Walker derivative which is defined for any non-isotropic vector fields is not coincides with derivative of the vector fields. Being Fermi-Walker transport conditions are examined for any isotropic and non-isotropic vector fields. We have shown that if the curve is a line or a planar curve which is not a line then the non-zero isotropic vector field is Fermi-Walker transported. We have obtained that Frenet frame is not a non-rotating frame if the curve is not a line.

Then, similar investigations have been made for any isotropic and non-isotropic vector fields with respect to the Darboux frame in Galilean space. We have proved while the curve is a line the Darboux frame is a non-rotating frame.

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