ORIGINAL PAPER

NUMERICAL SOLUTION OF INTEGRAL EQUATION USING GALERKIN METHOD WITH HERMITE, CHEBYSHEV & ORTHOGONAL POLYNOMIALS

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Abstract.*In this paper we introduce new approaches for numerical solution of Volterra integral equation of second kind. This numerical solution based on Galerkin method by using Hermite, Chebyshev & orthogonal polynomials. All calculation performed by MATLAB 13 versions. Many examples are given for comparative study of numerical solution of integral equations with estimated error. This comparative study of numerical solution of linear and nonlinear integral equation is more effective & better accuracy other than existing method****.***

Keywords: *Numerical solutions, Galerkin method, hermite polynomial, Chebyshev polynomial and orthogonal polynomial.*

1. INTRODUCTION

Several problems of mathematical physics can be initiated in integral equations. These equations are found to be formulated as mathematical problems such as ordinary differential equations and partial differential equations. A lot of mathematical models used in various applied problems of physics, biology, engineering, chemistry and in the other areas are transformed into integral equations, namely linear and nonlinear Volterra integral equations of the first and second kind. Hence study of integral equations and methods were frequently used in many applications and solving them [1-5] (e.g., Dirichletproblems, potential theory, Able test, electrostatics, diffusion problem and heat transfer problems). Consequently, most conventional analytical equations are solved, developed and implemented in digital computer was introduced. Accordingly considerations are solved with accurate solution, using less computation time, implement and give a compact solution form. By using various polynomials some researchers developed numerical methods for the Volterra integral equation for example an augmented Galerkin method was used for first kind Fredholm integral equations [1], solution of Volterra integral equation was used for various polynomials [3]. Bernstain's approximations were usedon a new approach to the numerical solution of Volterra integrals [6], these polynomials are used for solving Fredhom integral equations of second kind [7], Bernstein polynomials were used for the solution of second order linear and first order nonlinear differential equations [8]. Solve first and second kind linear and nonlinear Volterra integral equation were solved by collection method [9-13].We use the technique of Galerkin weighted residual method for providing numerical solutions for the Volterra integral equations based on Hermite, Chebyshev and orthogonal polynomials basis. To solve the

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linear Volterra integral equation of second kinds are regular as well as weakly singular kernals, the formulation is derived.

2. MATERIALS AND METHODS

2.1. ORTHOGONAL POLYNOMIALS

Orthogonal polynomials includes classes of polynomials $\{\emptyset_n(x)\}$ which were defined over a range [a, b] that obey an orthogonality relation

$$\int_a^b w(x)\emptyset_m(x)\emptyset_n(x)dx = g_m\delta_{mn} \tag{1}$$

where $w(x)$ is a weighting function and δ_{mn} is the Kronecker delta and moments is defined as:

$$\delta_{\mathrm{mn}} = \begin{cases} 0, & \mathrm{m} \neq \mathrm{n} \\ 1, & \mathrm{m} = \mathrm{n} \end{cases}$$

$$\mu = \int_a^b w(x)x^m dx, \;\; m = 0,1,2,3,4, \ldots. \tag{2}$$

Then the integral equations is denoted as inner product of polynomials $\emptyset_{\mathrm{m}}$ and $\emptyset_{\mathrm{n}}$.

$$\langle\emptyset_m, \emptyset_n\rangle = \int_a^b w(x)\emptyset_m(x)\,\emptyset_n(x)dx \tag{3}$$

For the orthogonality we have

$$\langle\emptyset_m, \emptyset_n\rangle = \int_a^b w(x)\emptyset_m(x)\,\emptyset_n(x)dx = 0\;, m \neq n, [-1,1] \tag{4}$$

and if $\delta_{mn} = 1$, then the polynomials are not only orthogonal but orthonormal as well.
Here, we consider a weight function $w(x) = 1 + x^2$ in the interval $[a, b] \cong [-1,1]$.
In this construction the function$\emptyset_m$, m=1, 2, 3,……
Then the approximant:

$$\breve{u}(x) = \sum_{r=0}^{n} a_r\emptyset_r(x) \cong u(x)(5)$$

2.2. MATHEMATICAL FORMULATION OF INTEGRAL EQUATION

We are using additional property for construction of the basis function as follows:

$$\emptyset_n(1) = 1$$

where

$$\emptyset_n(x) = \sum_{r=0}^{n} C_r^{(n)}\, x^r \tag{6}$$

Then equation (6) obeys orthogonality property (4). Then the six orthogonal polynomials $\emptyset_r;\ \ r \geq 7$ valid in [-1,1] are given below:

$$\emptyset_0(x) = 1$$

$$\emptyset_1(x) = x$$

$$\emptyset_2(x) = \frac{1}{3}(5x^2 - 2)$$

$$\emptyset_3(x) = \frac{1}{5}(14x^2 - 9x)$$

$$\emptyset_4(x) = \frac{1}{648}(3213x^4 - 2898x^2 + 333)$$

$$\emptyset_5(x) = \frac{1}{136}(1221x^5 - 1410x^3 + 325x)$$

$$\emptyset_6(x) = \frac{1}{1064}(17589x^6 - 24750x^4 + 8685x^2 - 460)$$

2.3. HERMITE POLYNOMIALS

The general form of the Hermite polynomials of $\mathrm{n^{th}}$ degree is given by:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) n = 0,1,2,3, \ldots (7)$$

Using MATLAB 13 version for calculation, the first few Hermite polynomials are given below:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^2 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

2.4. CHEBYSHEV POLYNOMIALS

The general form of the Chebyshev polynomials of nth degree is given by:

$$T_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{r!}{(2r)!(n-2r)!} (1-x^2)^r x^{n-2r} \quad (8)$$

$$[n/2] = \begin{cases} n/2 \; if n even \\ n+1/2 \; if n is odd \end{cases}$$

Accordingly by using MATLAB 13 version, the first few Chebyshev polynomials are given below:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

2.5. FORMULATION OF INTEGRAL EQUATION IN MATHEMATICAL FORM

Let us suppose that the Volterra integral equation (VIE) of the first kind

$$\int_a^x k(x,t)y(t)dt = f(x) \quad a \leq x \leq b \quad (9)$$

where$y(t)$ is a unknown functions which has to be determined, *k(x,t)* is the kernel function of continuous or discontinuous and square integrals function, *f(x)* being the known function satisfying*f(a)=0.*

Now assume the following to find an approximate solution $\tilde{y}(x)$ of (9) by using Galerkin method:

$$\tilde{y}(x) = \sum_{r=0}^{n} a_r P_r(x) \quad (10)$$

whereP_r(x) are Hermite, Chebyshev, Orthogonal Polynomials of degree r and a_r is unknown parameters, to be determined and n is number of piecewise polynomials. The approximate solution $\widetilde{\emptyset}(x)$ is not producing an identically zero function but a function is called residual function. Putting (10) into (9), we get the residual function as:

$$R(x) = \sum_{r=0}^{n} a_r \int_a^x k(x,t) P_r\,(t) dt - f(x), a \le x \le b \tag{11}$$

Now Galerkin equations (10) corresponding to the approximation (11), given by:

$$\int_a^b R(x) P_s\,(x) dx = 0 \tag{12}$$

with doing minor simplification, we obtain equation (11) and (12) as:

$$\sum_{r=0}^{n} a_r \int_a^b \left[\int_a^x k(x,t) P_r(t) dt\right] P_s\,(x) dx = \int_a^b P_s(x) f(x) dx, \qquad s = 0,1,2, \tag{13}$$

Equation (13) is written in the matrix form as

$$DC = B \tag{14}$$

where the elements of D, C, and B are $a_r, d_{r,s}$andb_s, respectively, given below:

$$a_r = [a_1, a_2, a_3, \dots \dots \dots a_n]^T \tag{15}$$

$$d_{r,s} = \int_a^b \left[\int_a^x k(x,t) P_r(t) dt\right] P_s(x) dx, \qquad r, s = 0,1,2, \dots \dots \;..n \tag{16}$$

$$b_s = \int_a^b P_s(x) f(x) dx, \qquad s = 0,1,2, \dots n \tag{17}$$

After solving the equation (15) and putting these value in equation (10) we can determined the unknown parametera_r, we get the approximate solution $\breve{u}(x)$ of the integral equation (3).

If we are putting Volterra integral equation of the second kind we have :

$$y(x) + \mu \int_a^x k(x,t) y(t) dt = f(x), \qquad a \le x \le b \tag{18}$$

where*f(x)* and *y(x)* is the known function and unknown function respectively which has to be determined and μ is a constant, *k(x,t)* is the kernal function, continuous or discontinuous.Then applying the same procedure as describe above,we obtainthe matrix forms

$$DA = B \tag{19}$$

where the elements of the matrix A, D and B are $a_r, d_{r,s}$and b_s respectively, given by:

$$a_r = [a_1, a_2, a_3, \dots \dots \dots a_n]^T \tag{20}$$

$$d_{r,s} = \int_a^b \left[P_r(x) + \mu \int_a^x k(x,t) P_r(t) dt\right] P_s(x) dx, \qquad r, s = 0,1,2, \dots n \tag{21}$$

$$b_s = \int_a^b P_s(x) f(x) dx, \qquad s = 0,1,2, \dots n \tag{22}$$

Now a_i is the unknown parameter and solving equation (20) and putting these values of the parameter in (10), we get the approximate solution $\check{y}(x)$ of the integral equation (18), Then the absolute error of the formula can be defined as:

$$\text{Absolute Error} = |y(x) - \tilde{y}(x)| \tag{24}$$

The formulation for nonlinear integral equation will be discussed by somenumerical problems as below.

3. NUMERICAL EXAMPLES

In this paper, we consider some linear and nonlinear Volterra integral equations with regular and weakly kernels, as the exact solutions are available in the literature. For all the examples we used, solutions are obtained by the proposed method and thus compared with the exact solutions using two piecewise polynomials, namely orthogonal polynomials, Hermite polynomials and Chebyshev polynomials. Thus the convergences of each linear Volterra integral equations can be calculated by:

$$E = |\tilde{y}_{r+1}(x) - \tilde{y}_r(x)| < \delta$$

where$\tilde{y}_r(x)$ denotes the approximate solution by the proposed method using the n^{th} degree polynomials approximation and δ varies from 10^{-6} (orthogonl polynomials) for$n \geq 10$, 10^{-7}(Hermite polynomials) and 10^{-6} (Chebyshev polynomials) for $n \geq 10$.

4. RESUTS

Example 1: Consider the Volterra integral equations of second kinds

$$y(x) + \int_0^x 3^{x-t}\, y(t)dt = 3^x x (25)$$

$$0 \leq x \leq 1$$

The exact solution is $y(x) = 3^x(1 - e^{-x})$. Results have been shown in Table 1 for n= 4.The maximum absolute errors obtain in the order of 10^{-3} for n= 4 are shown in Table 1.

Example 2: Consider the first kind Abel integral equation [4] is given by:

$$\int_0^x \frac{1}{\sqrt{(x-t)}} y(t)dt = x^i, \qquad 0 \leq x \leq 1 \tag{26}$$

where r is any positive number and in this equation(26) is Volterra integral equation with weak singuarity. The exact solution of the integral equation is given by:

$$y(x) = \frac{2^{2i-1}}{\pi} r \frac{(\Gamma i)^2}{\Gamma 2i} x^{i-\frac{1}{2}}$$

If one numerical example i is chosen as $i = 5$ (integral value) and $\frac{3}{2}$ (non inegral value). For $i = 5$ the exact solution is $(x) = \frac{1280}{315\pi} x^{9/2}$. Numericalw results are given below in Table 2 for $n = 10$. The absolute error are obtained in the order of 10^{-8} for both polynomials basis while the absolute errors were obtained in order of 10^{-7} for $n = 10$ (degrre of Bersstein's polynomials) by Bhattacharya and Mandal [4]. For $i = \frac{3}{2}$ the exact solution is $y(x) = \frac{3}{4}x$. Using Hermite, Chebyshev polynomials and orthogonal polynomials and the formula derived in the equation (19) for $n \geq 1$. And we get approximate solution is $\tilde{u}(x) = \frac{3}{4}x$. And this is a exact solutions, the absolute errors were obtained in the order 10^{-16} for $n = 5$ degrre of Bersstein's polynomials) by Bhattacharya and Mandal [4].

Table 1. Compute Absolute Error of example 1.

		Hermite Polynomials		**Chebyshev Polynomials**		**Orthogonal Polynomials**	
x	**Exact Solutions**	**Approximate Solutions**	**Absolute Error**	**Approximate Solutions**	**Absolute Error**	**Approximate Solutions**	**Absolute Error**
0.1	0.106232124	0.038429652	0.067802472	0.038429894	0.067802472	0.038429632	0.067802492
0.2	0.225812734	0.208459056	0.017353678	0.208459094	0.017353678	0.208459054	0.017353680
0.3	0.360363578	0.383550147	0.023186569	0.383550247	0.023186569	0.383550200	0.023186622
0.4	0.511612439	0.563702634	0.052090195	0.563702675	0.052090195	0.563702644	0.052090205
0.5	0.681508965	0.748916777	0.067407812	0.748916399	0.067407812	0.748916789	0.067407824
0.6	0.872229254	0.939192499	0.066963245	0.939192688	0.066963245	0.939192411	0.066963157
0.7	1.086202485	1.134529565	0.048327080	1.134529544	0.048327080	1.134529541	0.048327056
0.8	1.326139624	1.334928266	0.008788642	1.334928332	.0087886420	1.334928269	0.087886450
0.9	1.595066847	1.540388511	0.054678336	1.540388745	0.054678336	1.540388538	0.054678309
1.0	1.896361785	1.750910388	0.145451497	1.750910395	0.145451497	1.750910382	0.145451403

Table 2. Compute Absolute Error of example 2.

		Hermite Polynomials		**Chebyshev Polynomials**		**Orthogonal Polynomials**	
x	**Exact Solutions**	**Approximate Solutions**	**Absolute Error**	**Approximate Solutions**	**Absolute Error**	**Approximate Solutions**	**Absolute Error**
0.0	0.000000000000	0.000000130065	1.3006526E-007	0.000000130066	1.3006525E-007	0.000000130068	1.3006526E-007
0.1	0.000040902471	0.000040923323	2.0853202E-008	0.000040923324	2.0853201E-008	0.000040923325	2.0853202E-008
0.2	0.000925517263	0.000925505766	1.1496714E-008	0.000925505767	1.1496715E-008	0.000925505767	1.1496714E-008
0.3	0.005738457763	0.005738474293	1. 653120 E-008	0.005738474293	1. 653121E-008	0.005738474295	1. 653120 E-008
0.4	0.020942065044	0.020942045606	1.9438018 E-008	0.020942045607	1.9438019E-008	0.020942045608	1.9438018 E-008
0.5	0.057162940701	0.057162953563	1.2854656E-008	0.057162953564	1.2854656E-008	0.057162953565	1.2854656E-008
0.6	0.129846476719	0.129846477741	1.0217471 E-008	0.129846477742	1.0217470E-008	0.129846477741	1.0217471 E-008
0.7	0.259830855488	0.259830841672	1.3815856E-0008	0.259830841673	1.3815855E-0008	0.259830841673	1.3815856E-0008
0.8	0.473864838556	0.473864859781	2.1224498E-008	0.473864859781	2.1224498E-008	0.473864859782	2.1224498E-008
0.9	0.805083332553	0.805083306076	2.6477660E-008	0.805083306077	2.6477661E-008	0.805083306077	2.6477660E-008
1.0	1.293449696239	1.293449495167	2.0107182E-007	1.293449495167	2.0107183E-007	1.293449495168	2.0107182E-007

Example 3: Consider the second Abel's linear Volterra integral equation of the form

$$y(x) = e^x + \int_0^x y\,(t)dt, \qquad 0 \leq x \leq 1 \tag{27}$$

The exact solution is $y(x) = e^x(1 + x)$. Approximate results and analytical solutions are given below:

Table 3. Compute Absolute Error of example 3.

		Hermite Polynomials		Chebyshev Polynomials		Orthogonal Polynomials	
x	**Exact Solutions**	**Approximate Solutions**	**Absolute Error**	**Approximate Solutions**	**Absolute Error**	**Approximate Solutions**	**Absolute Error**
0.0	1.000000000	0.994533321	5.466679E-03	0.994533311	5.466689E-03	0.994533301	5.466699E-03
0.1	1.215688010	1.216793033	1.105023E-03	1.216793015	1.105005E-03	1.216793012	1.105002E-03
0.2	1.465683311	1.467968611	2.285300E-03	1.467968627	2.285316E-03	1.467968625	2.285314E-03
0.3	1.754816453	1.755664923	8.484700E-04	1.755664928	8.485398E-04	1.755664911	8.484580E-04
0.4	2.088554577	2.087487710	1.066867E-03	2.087487708	1.066869E-03	2.087487703	1.066874E-03
0.5	2.473081906	2.471042233	2.039673E-03	2.471042220	2.039686E-03	2.471042211	2.039695E-03
0.6	2.915390081	2.913934124	1.455957E-03	2.913934131	1.455950E-03	2.913934122	1.455959E-03
0.7	3.423379603	3.423768909	3.893060E-04	3.423768909	3.893060E-03	3.423768907	3.893040E-04
0.8	4.005973671	4.008152110	2.178439E-03	4.008152116	2.178445E-03	4.008152111	2.178440E-03
0.9	4.673245911	4.674689320	1.443409E-03	4.674689336	1.443425E-03	4.674689314	1.443403E-03
1.0	5.436563657	5.430986018	5.577639E-03	5.430986019	5.577638E-03	5.430986013	5.577644E-03

5. CONCLUSION

In this paper devloped new technique to find numerical solution of Volterra integral equation of second kind by Hermite, Chebyshev and orthogonal polynomials. The estimated solutions obtained by the Hermite, Chebyshev and orthogonal polynomials. All compared result more accurate and effective. This research gives new ideas in further research in the field of integral equation.

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