ORIGINAL PAPER

ON THE HARMONIC EVOLUTE OF QUASI NORMAL SURFACES

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Abstract. *In this paper, we study a harmonic evolute surface of quasi normal surface associated with quasi frame. We construct quasi normal surface with first and second fundamental forms. Moreover, we determine harmonic evolute surface of quasi normal surface by using these fundamental forms. Finally, we obtain some new results about these new surfaces.*

Keywords: *Bonnet Surfaces, Curvatures, Fundamental Forms, Normal Spherical Image.*

1. INTRODUCTION

The application of surfaces, which is a subject of differential geometry, to physics, and engineering is endless. It has been an area of impressive information whereby ruled surfaces which are generated by the motion of a straight line along a curve has been studied extensively by many researchers in both Euclidean and Minkowski spaces [1-4] after it was initially discovered by Gaspard Mongea. Besides ruled surfaces, many researchers have studied the harmonic evolute of various surfaces in detail for many years, which are ruled surfaces, timelike ruled surfaces, quasi tangent ruled surfaces, helicoid surfaces, B-scrolls with constant mean curvature to name a few [5-8]. As a follow up to these existing studies, we have explored harmonic evolute surfaces of the ruled surfaces generated by quasi normal vector.

This paper consists of three sections. Imperative knowledge on the differential geometric construction of the frames in the 3-dimensional Euclidean space that is, Serret Frenet frame, quasi frame and the relation between these frames are examined within the first section. In order to obtain harmonic evolute of ruled surfaces, the mean curvature of the surfaces is given by calculating the first and the second fundamental forms. In the following section, we identify the harmonic evolute surface of quasi normal surface associated with quasi frame. The necessary and sufficient conditions of how the quasi normal surface and its harmonic evolute surface can or can not be a Bonnet surface are given. Furthermore, the quasi normal surface of a helix and a harmonic evolute of the quasi normal surface are depicted.

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2. MATERIALS AND METHODS

By the way of design and style, this model is kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Frenet-Serret frame is established by way of subsequent equations for a presented framework [9]

$$\begin{bmatrix} \nabla_{\mathbf{t}}\mathbf{t} \\ \nabla_{\mathbf{t}}\mathbf{n} \\ \nabla_{\mathbf{t}}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where $\kappa = \|\mathbf{t}\|$ and τ are the curvature and torsion of γ, respectively.

After Bishop in 1975 showed that there is more than one way to frame a curve in [10], Yilmaz and Turgut introduced second type of Bishop frame in [14]. Besides them, Dede et. al. in defined quasi frame in [11]. The quasi frame of a regular curve γ is given by

$$\mathbf{t_q} = \mathbf{t}, \mathbf{n_q} = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b_q} = \mathbf{t_q} \wedge \mathbf{n_q},$$

where $\mathbf{k}$ is the projection vector.

For simplicity, we have chosen the projection vector $\mathbf{k} = (0,0,1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel, the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0,1,0)$ or $\mathbf{k} = (1,0,0)$.

If the angle between the quasi normal vector $\mathbf{n_q}$ and the normal vector $\mathbf{n}$ is chosen as ψ, then the following relation is obtained between the quasi and FS frame.

$$\begin{aligned} \mathbf{t_q} &= \mathbf{t}, \\ \mathbf{n_q} &= \cos\psi\mathbf{n} + \sin\psi\mathbf{b}, \\ \mathbf{b_q} &= -\sin\psi\mathbf{n} + \cos\psi\mathbf{b}. \end{aligned}$$

Therefore, by using the equations $(1-3)$ the variation of parallel adapted quasi frame is obtained by

$$\begin{aligned} \nabla_{\mathbf{t_q}}\mathbf{t_q} &= \kappa_1\mathbf{n_q} + \kappa_2\mathbf{b_q}, \\ \nabla_{\mathbf{t_q}}\mathbf{n_q} &= -\kappa_1\mathbf{t_q} + \kappa_3\mathbf{b_q}, \\ \nabla_{\mathbf{t_q}}\mathbf{b_q} &= -\kappa_2\mathbf{t_q} - \kappa_3\mathbf{n_q}, \end{aligned}$$

where

$$\kappa_1 = \kappa\cos\psi, \quad \kappa_2 = -\kappa\sin\psi, \quad \kappa_3 = \psi' + \tau,$$

and the vector products of the quasi vectors are given by

$$\mathbf{t_q} \times \mathbf{n_q} = \mathbf{b_q}, \mathbf{n_q} \times \mathbf{b_q} = \mathbf{t_q}, \mathbf{b_q} \times \mathbf{t_q} = \mathbf{n_q}.$$

Let n be the standard unit normal vector field on a surface ϕ defined by

$$n = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|},$$

where $\phi_s = \partial\phi/\partial s, \phi_t = \partial\phi/\partial t,$ respectively. Then, the first fundamental form $\mathbf{I}$ and the second fundamental form $\mathbf{II}$ of a surface ϕ are defined by

$$\begin{aligned} \mathbf{I} &= Eds^2 + 2Fdsdt + Gdt^2, \\ \mathbf{II} &= eds^2 + 2fdsdt + gdt^2, \end{aligned}$$

where

$$\begin{aligned} E &= \langle \phi_s, \phi_s \rangle, F = \langle \phi_s, \phi_t \rangle, G = \langle \phi_t, \phi_t \rangle, \\ e &= \langle \phi_{ss}, n \rangle, f = \langle \phi_{st}, n \rangle, g = \langle \phi_{tt}, n \rangle. \end{aligned}$$

respectively [12,13].

The mean curvature H is defined as:

$$H = \frac{\mathrm{Eg} - 2\mathrm{Ef} + \mathrm{Ge}}{2(EG - F^2)}.$$

Theorem 2.1. The surface is minimal if and only if it has vanishing mean curvature [9,13].

3. HARMONIC EVOLUTE SURFACES OF QUASI NORMAL SURFACES

In this section, we aim to explore harmonic evolute surface of quasi normal surface associated with quasi frame when the mean curvature does not vanish.

Firstly, we construct quasi normal surface of a quasi curve as

$$\phi^{\mathbf{n_q}}(s,t) = \alpha + t\mathbf{n_q}.$$

Definition 3.1. If $\mathbf{E} = \mathbf{G}$, $\mathbf{F} = 0$, $\mathbf{f} = c \neq 0$ $(\mathrm{c} = \mathrm{const.})$ *are satisfied then the surface is called A-net on a surface [9].*

Theorem 3.2. *A surface to be a Bonnet surface if and only if surface has an A-net, [9].*

Theorem 3.3. *Let $\phi^{\mathbf{n_q}}$ be a quasi normal surface of a quasi curve in space. $\phi^{\mathbf{n_q}}$ is minimal iff*

$$t^2\kappa_3(\kappa_2\kappa_3 + (\kappa_1)_s) + (1 - t\kappa_1)^2\kappa_2 + (1 - t\kappa_1)t(\kappa_3)_s = 0.$$

Proof. From the definition of quasi normal surface, we have

$$\begin{aligned}
\phi_s^{\mathbf{n_q}} &= (1-t\kappa_1)\mathbf{t_q} + t\kappa_3\mathbf{b_q}, \\
\phi_t^{\mathbf{n_q}} &= \mathbf{n_q}.
\end{aligned}$$

By using this field, the coefficients of the first fundamental form are given

$$\begin{aligned}
E &= (1-t\kappa_1)^2 + t^2\kappa_3^2, \\
F &= 0, G = 1.
\end{aligned}$$

The second partial derivatives of $\phi^{\mathbf{n_q}}$ are expressed as follows:

$$\begin{aligned}
\phi_{ss}^{\mathbf{n_q}} &= -t((\kappa_1)_s + \kappa_2\kappa_3)\mathbf{t_q} + (\kappa_1(1-t\kappa_1) - t\kappa_3^2)\mathbf{n_q} + (\kappa_2(1-t\kappa_1) + t(\kappa_3)_s)\mathbf{b_q}, \\
\phi_{ts}^{\mathbf{n_q}} &= -\kappa_1\mathbf{t_q} + \kappa_3\mathbf{b_q}, \\
\phi_{tt}^{\mathbf{n_q}} &= 0.
\end{aligned}$$

So, an algebraic calculus shows that

$$\phi_s^{\mathbf{n_q}} \times \phi_t^{\mathbf{n_q}} = -t\kappa_3\mathbf{t_q} + (1-t\kappa_1)\mathbf{b_q}.$$

Moreover, by the definition of the unit normal vector, we have

$$n^{\mathbf{n_q}} = \frac{-t\kappa_3}{\sqrt{t^2\kappa_3^2 + (1-t\kappa_1)^2}}\mathbf{t_q} - \frac{(1-t\kappa_1)}{\sqrt{t^2\kappa_3^2 + (1-t\kappa_1)^2}}\mathbf{b_q}.$$

Therefore, the coefficients of the second fundamental form are given

$$\begin{aligned}
e &= \frac{1}{\sqrt{t^2\kappa_3^2 + (1-t\kappa_1)^2}}(t^2\kappa_3(\kappa_2\kappa_3 + (\kappa_1)_s) + (1-t\kappa_1)^2\kappa_2 + (1-t\kappa_1)t(\kappa_3)_s), \\
f &= \frac{\kappa_3}{\sqrt{t^2\kappa_3^2 + (1-t\kappa_1)^2}}, \\
g &= 0.
\end{aligned}$$

The mean curvature of $\phi^{\mathbf{n_q}}$ is presented

$$H^{\mathbf{n_q}} = \frac{1}{2(t^2\kappa_3^2 + (1-t\kappa_1)^2)^{\frac{3}{2}}}(t^2\kappa_3(\kappa_2\kappa_3 + (\kappa_1)_s) + (1-t\kappa_1)^2\kappa_2 + (1-t\kappa_1)t(\kappa_3)_s).$$

This completes the proof.

Theorem 3.4. *Let $\phi^{\mathbf{n_q}}$ be a quasi normal surface of a quasi curve in space. $\phi^{\mathbf{n_q}}$ is not a Bonnet surface.*

Assume that, $\phi^{\mathbf{n_q}}$ is not minimal. Then, a harmonic evolute surface of the quasi normal surface is given by

$$\phi^h(s,t) = \phi^{\mathbf{n_q}}(s,t) + \frac{1}{H^{\mathbf{n_q}}} n^{\mathbf{n_q}}.$$

Theorem 3.5. *A harmonic evolute surface of $\phi^{\mathbf{n_q}}$ is given by*

$$\phi^h(s,t) = \alpha + t\mathbf{n_q} - \frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2 + (1-t\kappa_1)^2}}\mathbf{t_q} + \frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2 + (1-t\kappa_1)^2}}\mathbf{b_q}.$$

Theorem 3.6. *A harmonic evolute surface of $\phi^{\mathbf{n_q}}$ is a Bonnet surface if and only if*

$$\begin{aligned}
&((1-tx_1)(1-\frac{\kappa_2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}) - (\frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)^2 \\
&+(\frac{\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2 + (tx_3 - \frac{t\kappa_2\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
&+(\frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)^2 = 1 + (\frac{-\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2 \\
&+(\frac{\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2,
\end{aligned}$$

$$\begin{aligned}
&\frac{-\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}((1-tx_1)(1-\frac{\kappa_2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}) \\
&-(\frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s) - \frac{\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} - \frac{\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}(tx_3 \\
&-\frac{t\kappa_2\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} + (\frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\pi}[(\frac{x_1\kappa_3}{H^{\mathrm{n_q}2}(t^2\kappa_3^2+(1-t\kappa_1)^2)} - (tx_3 - \frac{t\kappa_2\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
&+(\frac{1-t\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s) - ((\kappa_1 + (\frac{\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s
\end{aligned}$$

$$+(\frac{\kappa_1 x_2}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}))) + (x_3 - \frac{x_2\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$

$$-(\frac{\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)(1-tx_1)(1-\frac{\kappa_2}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})$$

$$-(\frac{t\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s - (\frac{\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2) = \rho,$$

where ρ *is constant.*

Proof. Now, we obtain the derivative formulas

$$\begin{aligned}
\phi_s^h(s,t) &= ((1-tx_1)(1-\frac{\kappa_2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}) - (\frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)\mathbf{t_q} \\
&\quad - \frac{\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}\mathbf{n_q} + (tx_3 - \frac{t\kappa_2\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
&\quad + (\frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)\mathbf{b_q}, \\
\phi_t^h(s,t) &= \frac{-\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}\mathbf{t_q} + \mathbf{n_q} - \frac{\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}\mathbf{b_q}.
\end{aligned}$$

Then, it is easy to see that

$$\begin{aligned}
E^{\phi^h} &= ((1-tx_1)(1-\frac{\kappa_2}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}) - (\frac{t\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)^2 \\
&\quad + (\frac{\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2 + (tx_3 - \frac{t\kappa_2\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
&\quad + (\frac{1-t\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)^2,
\end{aligned}$$

$$\begin{aligned}
F^{\phi^h} &= \frac{-\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}((1-tx_1)(1-\frac{\kappa_2}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}) \\
&\quad - (\frac{t\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s) - \frac{\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
&\quad - \frac{\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}(tx_3 - \frac{t\kappa_2\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
&\quad + (\frac{1-t\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s),
\end{aligned}$$

$$G^{\phi^h} = 1+\left(\frac{-\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}\right)^2+\left(\frac{\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}\right)^2.$$

We instantly calculate

$$\begin{aligned}
\phi^h_{ss}(s,t) = {} & (((1-tx_1)(1-\frac{\kappa_2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}))_s-(\frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_{ss} \\
& +\frac{\kappa_1\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}-x_2(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
& +(\frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s))\mathbf{t_q}+(x_1(1-tx_1)(1-\frac{\kappa_2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}) \\
& -(\frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s-(\frac{\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s \\
& +x_3(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}+(\frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)\mathbf{n_q} \\
& +(x_2((1-tx_1)(1-\frac{\kappa_2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})-(\frac{t\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s) \\
& -\frac{\kappa_3^2}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}+(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s \\
& +(\frac{1-t\kappa_1}{H^{\mathbf{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_{ss})\mathbf{b_q},
\end{aligned}$$

$$\phi^h_{tt}(s,t) = (\frac{-\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_t\mathrm{t_q}+(\frac{-\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_t\mathrm{b_q},$$

$$\begin{aligned}
\phi^h_{ts}(s,t) = {} & -(\kappa_1+(\frac{\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s+(\frac{\kappa_1x_2}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}))\mathrm{t_q} \\
& +(x_3-\frac{x_2\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}-(\frac{\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)\mathrm{b_q}.
\end{aligned}$$

With the help of the obtained equations, we express

$$\begin{aligned}
n_h^{\mathrm{n_q}} = {} & \frac{1}{\pi}[(\frac{x_1\kappa_3}{H^{\mathrm{n_q}2}(t^2\kappa_3^2+(1-t\kappa_1)^2)}-(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}} \\
& +(\frac{1-t\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)\mathrm{t_q}+((\frac{\kappa_1}{H^{\mathrm{n_q}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})((1-tx_1)(1
\end{aligned}$$

$$-\frac{\kappa_2}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})-(\frac{t\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)$$
$$+\frac{\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$
$$+(\frac{1-t\kappa_1}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s))\mathrm{n}_\mathrm{q}+(1-tx_1)(1-\frac{\kappa_2}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})$$
$$-(\frac{t\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s-(\frac{\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2)\mathrm{b}_\mathrm{q}].$$

where $\pi=\left|\phi_s^h\wedge\phi_t^h\right|.$

Then

$$e^{\phi^h} = \frac{1}{\pi}[(\frac{x_1\kappa_3}{H^{\mathrm{nq}\,2}(t^2\kappa_3^2+(1-t\kappa_1)^2)}-(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$
$$+(\frac{1-t\kappa_1}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)(((1-tx_1)(1-\frac{\kappa_2}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}))_s$$
$$-(\frac{t\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_{ss}+\frac{\kappa_1\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$
$$-x_2(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}+(\frac{1-t\kappa_1}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)$$
$$+((x_1(1-tx_1)(1-\frac{\kappa_2}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})-(\frac{t\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s$$
$$-(\frac{\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s+x_3(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$
$$+(\frac{1-t\kappa_1}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)+((\frac{\kappa_1}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})((1-tx_1)(1$$
$$-\frac{\kappa_2}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})-(\frac{t\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)$$
$$+\frac{\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}(tx_3-\frac{t\kappa_2\kappa_3}{H^{\mathrm{nq}}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$

$$+(\frac{1-t\kappa_1}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s))+((x_2((1-tx_1)(1$$

$$-\frac{\kappa_2}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})-(\frac{t\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)$$

$$-\frac{\kappa_3^2}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}+(tx_3-\frac{t\kappa_2\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s$$

$$+(\frac{1-t\kappa_1}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_{ss})(1-tx_1)(1-\frac{\kappa_2}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})$$

$$-(\frac{t\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s-(\frac{\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2)$$

$$f^{\phi h} = \frac{1}{\pi}[(\frac{x_1\kappa_3}{H^{n_q 2}(t^2\kappa_3^2+(1-t\kappa_1)^2)}-(tx_3-\frac{t\kappa_2\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$

$$+(\frac{1-t\kappa_1}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)(-(\kappa_1+(\frac{\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s$$

$$+(\frac{\kappa_1 x_2}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})))+(x_3-\frac{x_2\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$

$$-(\frac{\kappa_1}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)(1-tx_1)(1-\frac{\kappa_2}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})$$

$$-(\frac{t\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s-(\frac{\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2)$$

$$g^{\phi h} = \frac{1}{\pi}[(\frac{x_1\kappa_3}{H^{n_q 2}(t^2\kappa_3^2+(1-t\kappa_1)^2)}-(tx_3-\frac{t\kappa_2\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}}$$

$$+(\frac{1-t\kappa_1}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s)(\frac{-\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_t$$

$$+(\frac{-\kappa_1}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_t(1-tx_1)(1-\frac{\kappa_2}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})$$

$$-(\frac{t\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})_s-(\frac{\kappa_3}{H^{n_q}\sqrt{t^2\kappa_3^2+(1-t\kappa_1)^2}})^2)$$

Consequently, using the definition of a Bonnet surface, we have proved the theorem.

Application to helix

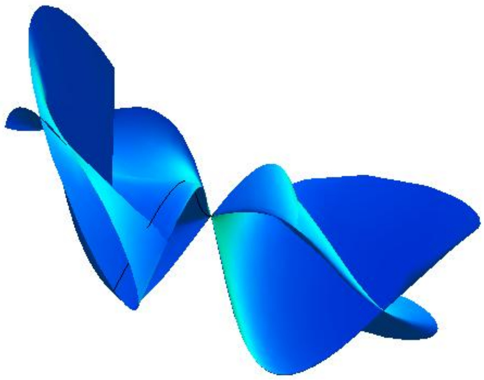

Figure 1. The quasi normal surface of a helix.

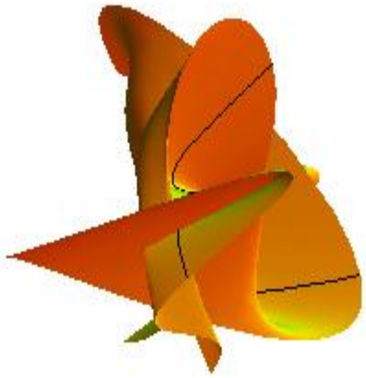

Figure 2. A harmonic evolute surface of ϕ^{n_q} .

4. CONCLUSION

In conclusion, a harmonic evolute surface of quasi normal surface associated with quasi frame was studied throughout this paper. Initially, quasi normal surface was constructed. We then established harmonic evolute surface of quasi normal surface by using quasi frame and mean curvature. Finally, our studies have enabled us to gain new results about these surfaces.

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