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CONSTRUCTION OF SYMMETRIC FUNCTIONS OF GENERALIZED TRIBONACCI NUMBERS

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Abstract. *In this paper, we will recover the generating functions of some generalized Tribonacci numbers. The technic used her is based on the theory of the so called symmetric functions.*

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1. INTRODUCTION

The generalized Tribonacci sequence $\{W_n\}_{n\in \mathrm{N}}$ is defined by a third order recurrance sequence [10]

$$\begin{cases} W_n = aW_{n-1} + bW_{n-2} + cW_{n-3}, \\ W_0 = \alpha,\ W_1 = \beta,\ W_2 = \gamma. \end{cases}$$

where α, β and γ are arbitrary integers and a, b and c are real numbers.

This sequence has been studied by many authors, see for example [6-10]. Note that generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, third order Jacobsthal and third order Jacobsthal-Lucas, Padovan, Perrin, Padovan-Perrin, and Narayana.

By [10], as $\{W_n\}$ is a third order recurrance sequence (difference equation), it's characteristic equation is $x^3 - ax^2 - bx - c = 0$, whose roots are r_1 , r_2 and r_3 such that

$$\begin{cases} r_1 = \dfrac{a}{3} + A + B, \\ r_2 = \dfrac{a}{3} + \omega A + \omega^2 B, \\ r_3 = \dfrac{a}{3} + \omega^2 A + \omega B. \end{cases}$$

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where

$$\begin{cases} A = \left(\dfrac{a^3}{27} + \dfrac{ab}{6} + \dfrac{c}{2} + \sqrt{\Delta} \right)^{1/3}, \\ B = \left(\dfrac{a^3}{27} + \dfrac{ab}{6} + \dfrac{c}{2} - \sqrt{\Delta} \right)^{1/3}, \\ \Delta = \dfrac{a^3 c}{27} - \dfrac{a^2 b^2}{108} + \dfrac{abc}{6} - \dfrac{b^3}{27} + \dfrac{c^2}{4}. \end{cases}$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$.

Note that the following identities are verified

$$\begin{cases} r_1 + r_2 + r_3 = a, \\ r_1 r_2 + r_1 r_3 + r_2 r_3 = -b, \\ r_1 r_2 r_3 = c. \end{cases}$$

From now on, we assume that $\Delta > 0$, so r_1 is real and r_2 and r_3 are two conjugate complex. In this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers *n*, using Binet's formula

$$W_n = \frac{R}{(r_1 - r_2)(r_1 - r_3)} r_1^n + \frac{S}{(r_1 - r_2)(r_2 - r_3)} r_2^n - \frac{T}{(r_1 - r_3)(r_2 - r_3)} r_3^n$$

Where

$$\begin{cases} R = \gamma - (r_2 + r_3)\beta + r_2 r_3 \alpha, \\ S = \gamma - (r_1 + r_3)\beta + r_1 r_3 \alpha, \\ T = \gamma - (r_1 + r_2)\beta + r_1 r_2 \alpha. \end{cases}$$

In fact, the well-known sequences below are special cases of the generalized Tribonacci sequence:

- Putting $a = b = c = 1$ and $\alpha = 1, \beta = 1, \gamma = 2$ reduces to **Tribonacci** sequence known as

$$\begin{cases} T_n = T_{n-1} + T_{n-2} + T_{n-3}, \\ T_0 = T_1 = 1, T_2 = 2, \\ \{T_n\} = \{1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, ...\}. \end{cases}$$

- Substituting $a = b = c = 1$ and $\alpha = 3,\ \beta = 1,\ \gamma = 3$ yields **Tribonacci-Lucas** sequence given by

$$\begin{cases} K_n = K_{n-1} + K_{n-2} + K_{n-3}, \\ K_0 = 3, K_1 = 1, K_2 = 3, \\ \{K_n\} = \{3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, ...\}. \end{cases}$$

- Taking $a = 0,\ b = c = 1$ and $\alpha = \beta = \gamma = 1$ gives **Padovan** sequence given by

$$\begin{cases} P_n = P_{n-2} + P_{n-3}, \\ P_0 = P_1 = P_2 = 1 \\ \{P_n\} = \{1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, ...\}. \end{cases}$$

- Taking $a = 0,\ b = 2,\ c = 1$ and $\alpha = \beta = \gamma = 1$ gives **Pell-Padovan** sequence given by

$$\begin{cases} PP_n = 2PP_{n-2} + PP_{n-3}, \\ PP_0 = PP_1 = PP_2 = 1, \\ \{PP_n\} = \{1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, ...\}. \end{cases}$$

- Taking $a = 0,\ b = 1,\ c = 2$ and $\alpha = \beta = \gamma = 1$ gives **Jacobsthal-Padovan** sequence given by

$$\begin{cases} JP_n = JP_{n-2} + 2JP_{n-3}, \\ JP_0 = JP_1 = JP_2 = 1, \\ \{JP_n\} = \{1, 1, 1, 3, 3, 5, 9, 11, 19, 29, 41, ...\}. \end{cases}$$

- In the case when $a=0,\ b=c=1$ and $\alpha=3,\ \beta=0,\ \gamma=2$ it reduces to **Perin** sequence known as

$$\begin{cases} R_n = R_{n-2} + R_{n-3}, \\ R_0 = 3, R_1 = 0, R_2 = 2, \\ \{R_n\} = \{3,0,2,3,2,5,5,7,10,12,17,...\}. \end{cases}$$

- In the case when $a=0,\ b=2,\ c=1$ and $\alpha=3,\ \beta=0,\ \gamma=2$ it reduces to **Pell-Perin** sequence known as

$$\begin{cases} PR_n = 2PR_{n-2} + PR_{n-3}, \\ PR_0 = 3, PR_1 = 0, PR_2 = 2, \\ \{PR_n\} = \{3,0,2,3,4,8,11,20,30,51,80,...\}. \end{cases}$$

- In the case when $a=0,\ b=1,\ c=2$ and $\alpha=3,\ \beta=0,\ \gamma=2$ it reduces to **Jacobsthal-Perin** sequence known as

$$\begin{cases} JR_n = JR_{n-2} + 2JR_{n-3}, \\ JR_0 = 3, JR_1 = 0, JR_2 = 2, \\ \{JR_n\} = \{3,0,2,6,2,10,14,14,34,42,62,...\}. \end{cases}$$

- Putting $a=0,\ b=c=1$ and $\alpha=\beta=0$, $\gamma=1$ we obtain **Padovan-Perin** sequence known as

$$\begin{cases} S_n = S_{n-2} + S_{n-3}, \\ S_0 = S_1 = 0, S_2 = 1, \\ \{S_n\} = \{0,0,1,0,1,1,1,2,2,3,4,...\}. \end{cases}$$

- Substituting $a=1,\ b=0$, $c=1$ and $\alpha=0,\ \beta=\gamma=1$ yields **Narayana** sequence defined by.

$$\begin{cases} N_n = N_{n-1} + N_{n-3}, \\ N_0 = 0,\ N_1 = N_2 = 1, \\ \{N_n\} = \{0,1,1,1,2,3,4,6,9,13,19,...\}. \end{cases}$$

- Taking $a = b = 1, c = 2$ and $\alpha = 0, \beta = \gamma = 1$ we get **third order Jacobsthal** sequence given by

$$\begin{cases} J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}, \\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1, \\ \{J_n^{(3)}\} = \{0,1,1,2,5,9,18,37,73,146,293,...\} \end{cases}.$$

- Putting $a = b = 1, c = 2$ and $\alpha = 2, \beta = 1, \gamma = 5$ we obtain **third order Jacobsthal-Lucas** sequence known as.

$$\begin{cases} j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}, \\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5, \\ \{j_n^{(3)}\} = \{2,1,5,10,17,37,74,145,293,586,1169,...\}. \end{cases}$$

In order to determine generating functions of generalized Tribonacci numbers, we use analytical means and series manipulation methods. In the sequel, we derive new symmetric functions and some new properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in Section 3.

2. DEFINITIONS AND SOME PROPERTIES.

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 2.1. [5] Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_2, ..., a_n)$ is defined by

$$e_k(a_1, a_2, ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad (0 \le k \le n)$$

with $i_1, i_2, ..., i_n = 0$ or 1 .

Definition 2.2. [5] Let k and n be two positive integers and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k -th complete homogeneous symmetric function $h_k(a_1, a_2, ..., a_n)$ is defined by

$$h_k(a_1, a_{2,} ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n} \qquad (0 \le k \le n)$$

with $i_1, i_2, ..., i_n \ge 0$.

Remark 2.1. Set $e_0(a_1, a_{2,} ..., a_n) = 1$ and $h_0(a_1, a_{2,} ..., a_n) = 1$, by usual convention. For $k < 0,$ we set $e_k(a_1, a_{2,} ..., a_n) = 0$ and $h_k(a_1, a_{2,} ..., a_n) = 0$.

Definition 2.3. [2] Let A and P be any two alphabets. We define $S_n(A-P)$ by the following form

$$\frac{\Pi_{p\varepsilon P}(1-pt)}{\Pi_{a\varepsilon A}(1-at)} = \sum_{n=0}^{\infty} S_n(A-P)t^n, \tag{2.1}$$

with the condition $S_n(A-P) = 0$ for $n < 0$.

Equation (2.1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A-P)t^n = \left(\sum_{n=0}^{\infty} S_n(A)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right), \tag{2.2}$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P) S_j(A).$$

Definition 2.4. [1] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(f) = \frac{f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - f(p_1, \cdots p_{i-1}, p_{i+1,} p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}.$$

Definition 2.5. [3] The symmetrizing operator $\delta_{e_1 e_2}^k$ is defined by

$$\delta_{p_1 p_2}^k(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathrm{N}.$$

3. GENERATING FUNCTION OF GENERALIZED TRIBONACCI NUMBERS.

The following proposition is one of the key tools of the proof of our main result which is already given its proof in [4].

Proposition 3.1. Given two alphabets $P=\{p_1,p_2\}$ and $A=\{a_1,a_2,a_3\}$, we have

$$\sum_{n=0}^{\infty} S_n(A)S_n(P)t^n = \frac{S_0(-A)-p_1p_2S_2(-A)t^2-p_1p_2S_3(-A)S_1(P)t^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)p_2^n t^n\right)}, \tag{3.1}$$

with $S_0(-A)=1, S_1(-A)=-(a_1+a_2+a_3),\ S_2(-A)=a_1a_2+a_1a_3+a_2a_3,\ S_3(-A)=-a_1a_2a_3.$

If $P=\{1,0\}$ in (3.1), the following lemmas allows us to obtain many generating functions of generalized Tribonacci numbers and some well-known numbers cited above, using a technique symmetric functions.

Lemma 3.1. Given an alphabet $A=\{a_1,a_2,a_3\}$, we have

$$\sum_{n=0}^{+\infty} S_n(a_1+a_2+a_3)t^n = \frac{1}{1+S_1(-A)t+S_2(-A)t^2+S_3(-A)t^3}. \tag{3.2}$$

Lemma 3.2. Given an alphabet $A=\{a_1,a_2,a_3\}$, we have

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+a_2+a_3)t^n = \frac{t}{1+S_1(-A)t+S_2(-A)t^2+S_3(-A)t^3}. \tag{3.3}$$

Lemma 3.3. Given an alphabet $A=\{a_1,a_2,a_3\}$, we have

$$\sum_{n=0}^{+\infty} S_{n-2}(a_1+a_2+a_3)t^n = \frac{t^2}{1+S_1(-A)t+S_2(-A)t^2+S_3(-A)t^3}. \tag{3.4}$$

Setting $\begin{cases} S_1(-A)=-a \\ S_2(-A)=-b \\ S_3(-A)=-c \end{cases}$ in (3.2), (3.3) and (3.4) this gives

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3)t^n = \frac{1}{1-at-bt^2-ct^3}. \tag{3.5}$$

and

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+a_2+a_3)t^n = \frac{t}{1-at-bt^2-ct^3}. \quad (3.6)$$

$$\sum_{n=0}^{\infty} S_{n-2}(a_1+a_2+a_3)t^n = \frac{t^2}{1-at-bt^2-ct^3}. \quad (3.7)$$

Multiplying the equation (3.5) by (α) and subtracting it from (3.6) by $(\alpha a-\beta)$ and also subtracting it from (3.7) by $(\alpha b+\beta a-\gamma)$ we obtain

$$\sum_{n=0}^{\infty} \begin{bmatrix} \alpha S_n(a_1+a_2+a_3)+(\beta-\alpha a)S_{n-1}(a_1+a_2+a_3) \\ +(\gamma-\alpha b-\beta a)S_{n-2}(a_1+a_2+a_3) \end{bmatrix} t^n = \frac{\alpha+(\beta-\alpha a)t+(\gamma-\alpha b-\beta a)t^2}{1-at-bt^2-ct^3}. \quad (3.8)$$

and we have the following new Proposition.

Proposition.3.2. For $n \in \mathbb{N}$, the new generating function of generalized Tribonacci numbers is given by

$$\sum_{n=0}^{+\infty} W_n t^n = \frac{\alpha+(\beta-\alpha a)t+(\gamma-\alpha b-\beta a)t^2}{1-at-bt^2-ct^3},$$

with $W_n = \alpha S_n(a_1+a_2+a_3)+(\beta-\alpha a)S_{n-1}(a_1+a_2+a_3)+(\gamma-\alpha b-\beta a)S_{n-2}(a_1+a_2+a_3)$.

Accordingly, we conclude the following Corollaries.

Corollary 3.1. For $n \in \mathbb{N}$, the generating function of Tribonacci numbers is given by

$$\sum_{n=0}^{\infty} T_n t^n = \frac{1}{1-t-t^2-t^3}, \text{ with } T_n = S_n(a_1+a_2+a_3).$$

Corollary 3.2. For $n \in \mathbb{N}$, the generating function of Tribonacci-Lucas numbers is given by

$$\sum_{n=0}^{\infty} K_n t^n = \frac{3-2t-t^2}{1-t-t^2-t^3}, \text{ with } K_n = 3S_n(a_1+a_2+a_3)-2S_{n-1}(a_1+a_2+a_3)-S_{n-2}(a_1+a_2+a_3).$$

Corollary 3.3. For $n \in \mathbb{N}$, the generating function of Padovan numbers P_n is given by

$$\sum_{n=0}^{\infty} P_n t^n = \frac{1+t}{1-t^2-t^3}, \text{ with } P_n = S_n(a_1+a_2+a_3)+S_{n-1}(a_1+a_2+a_3)$$

Corollary 3.4. For $n \in \mathbb{N}$, the generating function of Pell-Padovan numbers is given by

$$\sum_{n=0}^{\infty} PP_n t^n = \frac{1+t-t^2}{1-2t^2-t^3},$$

with $PP_n = S_n(a_1+a_2+a_3)+S_{n-1}(a_1+a_2+a_3)-S_{n-2}(a_1+a_2+a_3)$

Corollary 3.5. For $n \in \mathbb{N}$, the generating function of Jacobsthal-Padovan numbers is given by

$$\sum_{n=0}^{\infty} JP_n t^n = \frac{1+t}{1-t^2-2t^3},$$

with $JP_n = (S_n(a_1+a_2+a_3)+S_{n-1}(a_1+a_2+a_3))$.

Corollary 3.6. For $n \in \mathrm{N}$, the generating functions of Perin and Pell-Perin numbers are given by

$$\sum_{n=0}^{\infty} R_n t^n = \frac{3-t^2}{1-t^2-t^3}, \text{ with } R_n = 3S_n(a_1+a_2+a_3)-S_{n-2}(a_1+a_2+a_3).$$

$$\sum_{n=0}^{\infty} PR_n t^n = \frac{3-4t^2}{1-t^2-2t^3}, \text{ with } PR_n = 3S_n(a_1+a_2+a_3)-4S_{n-2}(a_1+a_2+a_3).$$

Corollary 3.7. For $n \in \mathbb{N}$, the generating function of Jacobsthal-Perin numbers is given by

$$\sum_{n=0}^{\infty} JR_n t^n = \frac{3-t^2}{1-t^2-2t^3}, \text{ with } JR_n = 3S_n(a_1+a_2+a_3)-S_{n-2}(a_1+a_2+a_3).$$

Corollary 3.8. For $n \in \mathbb{N}$, the generating function of Padovan-Perin numbers S_n is given by

$$\sum_{n=0}^{\infty} S_n t^n = \frac{t^2}{1-t^2-t^3}, \text{ with } S_n = S_{n-2}(a_1+a_2+a_3).$$

Corollary 3.9. For $n \in \mathbb{N}$, the generating function of Narayana numbers is given by

$$\sum_{n=0}^{\infty} N_n t^n = \frac{t}{1-t-t^3}, \text{ with } N_n = S_{n-1}(a_1+a_2+a_3).$$

Corollary 3.10. For $n \in \mathbb{N}$, the generating functions of third order Jacobsthal and third order Jacobsthal-Lucas numbers are given by

$$\sum_{n=0}^{\infty} J_n^{(3)} t^n = \frac{t}{1-t-t^2-2t^3}, \text{with } J_n^{(3)} = S_{n-1}(a_1+a_2+a_3).$$

$$\sum_{n=0}^{\infty} j_n^{(3)} t^n = \frac{2-t+2t^2}{1-t-t^2-2t^3},$$

with $j_n^{(3)} = 2S_n(a_1+a_2+a_3)-S_{n-1}(a_1+a_2+a_3)+2S_{n-2}(a_1+a_2+a_3)$

4. CONCLUSION

In this paper, by making use of Eq. (3.1), we have derived some new generating functions for generalized Tribonacci numbers. It would be interesting to apply the methods shown in the paper to families of other special numbers.

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