***ORIGINAL PAPER***

# BI-PERIODIC BALANCING NUMBERS

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***Abstract.*** *In this paper, we introduce a new generalization of the balancing numbers which we call bi-periodic balancing numbers as*

$$b_n = \begin{cases} 6cb_{n-1} - b_{n-2}, & if\ n\ is\ even \\ 6db_{n-1} - b_{n-2}, & if\ n\ is\ odd \end{cases}, n \geq 2$$

*with initial conditions $b_0 = 0, b_1 = 1$. We find the generating function for this sequence and produce a Binet's formula.*

***Keywords:*** *Balancing numbers, $k-$balancing numbers, Bi-periodic balancing numbers, Binet formula, Generating function, Cassini identity, Catalan identity*

## 1. INTRODUCTION

There are many studies on integer sequences such as Fibonacci, Lucas, Jacobsthal and their applications [1-4]. Another well-known sequence is balancing numbers which satisfies the recurrence relation

$$b_n = 6b_{n-1} - b_{n-2}, \quad n \geq 2$$

with initial conditions $b_0 = 0, b_1 = 1$. Balancing numbers was firstly mentioned by Behera and Panda in [5]. Moreover, many researchers worked on balancing numbers and its applications [6-8].

In many studies, authors worked on the generalizations of integer sequences in different ways. Among these studies, the most interesting generalization is bi-periodic Fibonacci sequence which was produced by Edson and Yayenie [9]. The bi-periodic Fibonacci sequence was defined as

$$q_n = \begin{cases} aq_{n-1} - q_{n-2}, & if\ n\ is\ even \\ bq_{n-1} - q_{n-2}, & if\ n\ is\ odd \end{cases}, n \geq 2$$

with initial conditions $q_0 = 0, q_1 = 1$. Then, the generating function for the bi-periodic Fibonacci sequence was obtained as

$$F(x) = \frac{x(1 + ax - x^2)}{1 - (\mathrm{ab} + 2)x^2 + x^4}.$$

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Moreover, the authors gave the Binet formula for the bi-periodic Fibonacci sequence as

$$q_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right)\left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right)$$

where $\lfloor a \rfloor$ is the floor function of $a$ and $\xi(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ is the parity function and $\alpha$ and $\beta$ are the roots of quadratic equation

$$x^2 - abx - ab = 0.$$

In view of this generalization, Yayenie [10] made some studies on bi-periodic fibonacci sequence and Bilgici [11] defined the bi-periodic Lucas sequence and obtained some identities using the Binet formula of the bi-periodic Lucas sequence. Also, Tasci and Kizilirmak worked on the periods of bi-periodic Fibonacci and bi-periodic Lucas numbers in [12]. Lastly, Uygun and Owusu defined the bi-periodic Jacobsthal sequence with the similar way [13].

## 2. MAIN RESULTS

**Definition 2.1.** For any two non-zero real numbers $c$ and $d$, the bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ is defined recursively by

$$b_0 = 0, b_1 = 1, b_n = \begin{cases} 6cb_{n-1} - b_{n-2}, & if\ n\ is\ even \\ 6db_{n-1} - b_{n-2}, & if\ n\ is\ odd \end{cases}, n \geq 2.$$

When $c = d = 1$ , we have the classic balancing numbers. If we set $= d = k$ , for any positive number, we get the $k$ −balancing numbers. The first five elements of the bi-periodic balancing numbers are

$$b_0 = 0, b_1 = 1, b_2 = 6c, b_3 = 36cd - 1, b_4 = 216c^2d - 12c\ .$$

The quadratic equation for the bi-periodic balancing numbers is defined as

$$x^2 - 36cdx + 36cd = 0$$

with the roots

$$\alpha = 18cd + 6\sqrt{9c^2d^2 - cd}\ and\ \beta = 18cd - 6\sqrt{9c^2d^2 - cd}. \tag{2.1}$$

**Lemma 2.2.** The bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ satisfies the following properties:

$$b_{2n} = (36cd - 2)b_{2n-2} - b_{2n-4}$$

$$b_{2n+1} = (36cd - 2)b_{2n-1} - b_{2n-3}.$$

*Proof.* Using the recurrence relation for the bi-periodic balancing numbers we can obtain

$$\begin{aligned} b_{2n} &= 6cb_{2n-1} - b_{2n-2} \\ &= 6c(6db_{2n-2} - b_{2n-3}) - b_{2n-2} \\ &= (36cd - 1)b_{2n-2} - 6cb_{2n-3} \\ &= (36cd - 1)b_{2n-2} - (b_{2n-2} + b_{2n-4}) \\ &= (36cd - 2)b_{2n-2} - b_{2n-4} \end{aligned}$$

$$\begin{aligned} b_{2n+1} &= 6db_{2n} - b_{2n-1} \\ &= 6d(6cb_{2n-1} - b_{2n-2}) - b_{2n-1} \\ &= (36cd - 1)b_{2n-1} - 6db_{2n-2} \\ &= (36cd - 1)b_{2n-2} - (b_{2n-1} + b_{2n-3}) \\ &= (36cd - 2)b_{2n-1} - b_{2n-3}. \end{aligned}$$

**Lemma 2.3.** The roots $\alpha$ and $\beta$ defined in (2.1) satisfies the following properties:

$$(\alpha - 1)(\beta - 1) = 1$$

$$\alpha\beta = 36cd \qquad \alpha + \beta = 36cd$$

$$\alpha - 1 = \frac{\alpha^2}{36cd} \qquad 6\beta - 1 = \frac{\beta^2}{36cd}$$

$$(\alpha - 1)\beta = \alpha \qquad (\beta - 1)\alpha = \beta$$

*Proof.* By using the definitions of $\alpha$ and $\beta$ defined in (2.1), the properties can easily be proved.

**Theorem 2.4.** The generating function for the bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ is

$$B(x) = \frac{x(1 + 6cx + x^2)}{1 - (36\mathrm{cd} - 2)x^2 + x^4}.$$

.

*Proof.* The formal power series representation of the generating function for $\{b_n\}_{n=0}^{\infty}$ is

$$B(x) = \mathrm{b}_0 + \mathrm{b}_1 x + \mathrm{b}_2 x^2 + \cdots + \mathrm{b}_r x^r + \cdots = \sum_{m=0}^{\infty} b_m x^m.$$

By multiplying this series by 6dx and $x^2$ respectively, we can get the following series;

$$6\mathrm{dx}B(x) = 6\mathrm{db}_0 x + 6\mathrm{db}_1 x^2 + 6d\mathrm{b}_2 x^3 + \cdots + 6\mathrm{db}_r x^{r+1} + \cdots = \sum_{m=1}^{\infty} 6db_{m-1} x^m$$

and

$$x^2B(x) = \mathrm{b}_0x^2 + \mathrm{b}_1x^3 + \mathrm{b}_2x^4 + \cdots + \mathrm{b}_rx^{r+2} + \cdots = \sum_{m=2}^{\infty} b_{m-2}x^m.$$

Therefore, we can write

$$(1 - 6\mathrm{dx} + x^2)B(x) = \mathrm{b}_0 + \mathrm{b}_1x - 6d\mathrm{b}_0x + \sum_{m=2}^{\infty}(b_m - 6d\mathrm{b}_{m-1} + b_{m-2})x^m. \quad (2.2)$$

Since $b_{2m+1} = 6db_{2m} - b_{2m-1}$ and $b_0 = 0, b_1 = 1$ equation (2.2) reduces to

$$(1 - 6\mathrm{dx} + x^2)B(x) = x + \sum_{m=1}^{\infty}(b_{2m} - 6d\mathrm{b}_{2m-1} + b_{2m-2})x^{2m}.$$

Since $b_{2m} = 6cb_{2m-1} - b_{2m-2}$, we get

$$(1 - 6\mathrm{dx} + x^2)B(x) = x + \sum_{m=1}^{\infty} 6(c-d)\mathrm{b}_{2m-1}x^{2m}$$
$$= x + 6(c-d)x\sum_{m=1}^{\infty} \mathrm{b}_{2m-1}x^{2m-1}.$$

Now we define $b(x)$ as

$$b(x) = \sum_{m=1}^{\infty} b_{2m-1}t^{2m-1}.$$

By applying the same way as above, we get

$$(1 - (36\mathrm{cd} - 2)\mathrm{x}^2 + x^4)b(x)$$
$$= \sum_{m=1}^{\infty} b_{2m-1}x^{2m-1} - (36\mathrm{cd} - 2)\sum_{m=2}^{\infty} b_{2m-3}x^{2m-1} + \sum_{m=3}^{\infty} b_{2m-5}x^{2m-1}$$
$$= \mathrm{b}_1x + \mathrm{b}_3x^3 - (36\mathrm{cd} - 2)\mathrm{b}_1x^3$$
$$+ \sum_{m=3}^{\infty}(b_{2m-1} - (36\mathrm{cd} - 2)b_{2m-3} + b_{2m-5})x^{2m-1}.$$

Lemma (2.2) implies that $b_{2m-1} - (36\mathrm{cd} - 2)b_{2m-3} + b_{2m-5} = 0$, so replacing this in the above expansion gives

$$(1 - (36\mathrm{cd} - 2)\mathrm{x}^2 + x^4)b(x) = x + x^3 + 0.$$

Therefore we hold

$$b(x) = \frac{x + x^3}{(1 - (36\mathrm{cd} - 2)\mathrm{x}^2 + x^4)}.$$

Substituting $b(x)$ in $B(x)$ we obtain

$$(1-6\mathrm{dx}+x^2)B(x)=\ x+6(c-d)x\left(\frac{x+x^3}{(1-(36\mathrm{cd}-2)\mathrm{x}^2+x^4)}\right).$$

Simplifying this, we have the generating function for the bi-periodic balancing numbers as

$$B(x)=\frac{x(1+6cx+x^2)}{1-(36\mathrm{cd}-2)x^2+x^4}.$$

**Theorem 2.3.** We can express the terms of the bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ by using the Binet formula:

$$b_m=\frac{(6c)^{1-\xi(m)}}{(36cd)^{\lfloor\frac{m}{2}\rfloor}}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\ \right)$$

where $\lfloor c\rfloor$ is the floor function of $c$ and $\xi(m)=m-2\left\lfloor\frac{m}{2}\right\rfloor$ is the parity function.

*Proof.* We know that the generating function for the bi-periodic balancing numbers $\{b_n\}_{n=0}^{\infty}$ is given by

$$B(x)=\frac{x(1+6cx+x^2)}{1-(36\mathrm{cd}-2)x^2+x^4}.$$

Using the partial fraction decomposition, $B(x)$ can be written as

$$B(x)=\frac{1}{\alpha-\beta}\left[\frac{6c(\alpha-1)+\alpha x}{x^2-(\alpha-1)}-\frac{6c(\beta-1)+\beta x}{x^2-(\beta-1)}\right]$$

Since the Maclaurin series expansion of the function

$$\frac{A-Bz}{z^2-C}$$

is expressed as

$$\frac{A-Bz}{z^2-C}=\sum_{n=0}^{\infty}BC^{-n-1}z^{2n+1}-\sum_{n=0}^{\infty}AC^{-n-1}z^{2n}$$

the generating function $B(x)$ can be written as

$$B(x)=\frac{1}{\alpha-\beta}\left[\sum_{m=0}^{\infty}\frac{\beta(\alpha-1)^{m+1}-\alpha(\beta-1)^{m+1}}{(\alpha-1)^{m+1}(\beta-1)^{m+1}}x^{2m+1}\right]$$
$$+\frac{6c}{\alpha-\beta}\left[\sum_{m=0}^{\infty}\frac{(\beta-1)(\alpha-1)^{m+1}-(\alpha-1)(\beta-1)^{m+1}}{(\alpha-1)^{m+1}(\beta-1)^{m+1}}x^{2m}\right].$$

By using the properties in Lemma 2.3, we get

$$B(x)=\sum_{m=0}^{\infty}\left(\frac{1}{36cd}\right)^{m+1}\left(\frac{\beta\alpha^{2m+2}-\alpha\beta^{2m+2}}{\alpha-\beta}\right)x^{2m+1}$$
$$+\sum_{m=0}^{\infty}6c\left(\frac{1}{36cd}\right)^{m+1}\left(\frac{(\beta-1)\alpha^{2m+2}-(\alpha-1)\beta^{2m+2}}{\alpha-\beta}\right)x^{2m}$$
$$=\sum_{m=0}^{\infty}\left(\frac{1}{36cd}\right)^{m}\left(\frac{\alpha^{2m+1}-\beta^{2m+1}}{\alpha-\beta}\right)x^{2m+1}+\sum_{m=0}^{\infty}6c\left(\frac{1}{36cd}\right)^{m}\left(\frac{\alpha^{2m}-\beta^{2m}}{\alpha-\beta}\right)x^{2m}.$$

By the help of the parity function $\xi(m)$, the above expansion is simplified as

$$B(x)=\frac{(6c)^{1-\xi(m)}}{(36cd)^{\lfloor\frac{m}{2}\rfloor}}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)x^m.$$

Therefore, for all $m\geq 0$, we get

$$b_m=\frac{(6c)^{1-\xi(m)}}{(36cd)^{\lfloor\frac{m}{2}\rfloor}}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)$$

**Theorem 2.4. (Catalan's Identity)** For any two nonnegative integer $n$ and $r$, with $r\leq n$, we have

$$c^{\xi(n-r)}d^{1-\xi(n-r)}b_{n-r}b_{n+r}-c^{\xi(n)}d^{1-\xi(n)}{b_n}^2=-c^{\xi(r)}d^{1-\xi(r)}{b_r}^2.$$

*Proof.* Using the Binet formula, we obtain

$$c^{\xi(n-r)}d^{1-\xi(n-r)}b_{n-r}b_{n+r}$$
$$=c^{\xi(n-r)}d^{1-\xi(n-r)}\left(\frac{(6c)^{1-\xi(n-r)}}{(36cd)^{\lfloor\frac{n-r}{2}\rfloor}}\right)\left(\frac{(6c)^{1-\xi(n+r)}}{(36cd)^{\lfloor\frac{n+r}{2}\rfloor}}\right)\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}$$
$$=\left(\frac{(6c)^{2-\xi(n-r)}d^{1-\xi(n-r)}}{(36cd)^{n-\xi(n-r)}}\right)\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}$$
$$=\left(\frac{c}{(36cd)^{n-1}}\right)\left[\frac{\alpha^{2n}-(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

and

$$c^{\xi(n)}d^{1-\xi(n)}{b_n}^2=c^{\xi(n)}d^{1-\xi(n)}\left(\frac{(6c)^{2-2\xi(n)}}{(36cd)^{2\lfloor\frac{n}{2}\rfloor}}\right)\left[\frac{\alpha^{2n}-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}\right]$$
$$=\left(\frac{c}{(36cd)^{2\lfloor\frac{n}{2}\rfloor+\xi(n)-1}}\right)\left[\frac{\alpha^{2n}-2(\alpha\beta)^n+\beta^{2n}}{(\alpha-\beta)^2}\right]$$

$$= \left(\frac{c}{(36cd)^{n-1}}\right)\left[\frac{\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}}{(\alpha-\beta)^2}\right].$$

Therefore,

$$\begin{aligned}
c^{\xi(n-r)}d^{1-\xi(n-r)}b_{n-r}b_{n+r} - c^{\xi(n)}d^{1-\xi(n)}{b_n}^2 \\
&= \left(\frac{c}{(36cd)^{n-1}}\right)\left[\frac{2(\alpha\beta)^n - (\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})}{(\alpha-\beta)^2}\right] \\
&= \left(\frac{-c}{(36cd)^{n-1}}\right)(\alpha\beta)^{n-r}\left[\frac{\alpha^{2r} - 2\alpha^r\beta^r + \beta^{2r}}{(\alpha-\beta)^2}\right] \\
&= \left(\frac{-c}{(36cd)^{n-1}}\right)(36cd)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha-\beta}\right)^2 \\
&= \left(\frac{-c}{(36cd)^{r-1}}\right)\frac{(36cd)^{2\left\lfloor\frac{r}{2}\right\rfloor}}{(6c)^{2-2\xi(r)}}{b_r}^2 \\
&= -c(6c)^{2\xi(r)-2}(36cd)^{1-\xi(r)}{b_r}^2 \\
&= -c^{\xi(r)}d^{1-\xi(r)}{b_r}^2.
\end{aligned}$$

**Theorem 2.5. (Cassini's Identity)** For any nonnegative integer $n$, we have

$$c^{1-\xi(n)}d^{\xi(n)}b_{n-1}b_{n+1} - c^{\xi(n)}d^{1-\xi(n)}{b_n}^2 = -c.$$

*Proof.* In Catalan's identity, if we take $r = 1$ we get Cassini's identity.

**Theorem 2.6.** For any two nonnegative integers $m$ and $n$ with $m \geq n$, we have

$$c^{\xi(mn+m)}d^{\xi(mn+n)}b_m b_{n+1} - c^{\xi(mn+n)}d^{\xi(mn+m)}b_{m+1}b_n = c^{\xi(m-n)}b_{m-n}.$$

*Proof.* We first note that

$$\xi(m+1) + \xi(n) - 2\xi(mn+n) = \xi(m) + \xi(n+1) - 2\xi(mn+m) = 1 - \xi(m-n)$$

and

$$\xi(m-n) = \xi(mn+m) + \xi(mn+n).$$

By using the Binet formula and the above equalities, we get

$$c^{\xi(mn+m)}d^{\xi(mn+n)}b_m b_{n+1}$$
$$= \left(\frac{c(cd)^{-n}}{6^{m+n-1}(cd)^{\frac{m-n-\xi(m-n)}{2}}}\right)\left[\frac{\alpha^{m+n+1} + \beta^{m+n+1} - (\alpha\beta)^n(\beta\alpha^{m-n} + \alpha\beta^{m-n})}{(\alpha-\beta)^2}\right]$$

and

$$c^{\xi(mn+n)}d^{\xi(mn+m)}b_{m+1}b_n = \left(\frac{c(cd)^{-n}}{6^{m+n-1}(cd)^{\frac{m-n-\xi(m-n)}{2}}}\right)\left[\frac{\alpha^{m+n+1}+\beta^{m+n+1}-(\alpha\beta)^n(\alpha^{m-n+1}+\beta^{m-n+1})}{(\alpha-\beta)^2}\right]$$

Therefore,

$$\begin{aligned}
&c^{\xi(mn+m)}d^{\xi(mn+n)}b_m b_{n+1} - c^{\xi(mn+n)}d^{\xi(mn+m)}b_{m+1}b_n \\
&= \left(\frac{c(cd)^{-n}}{6^{m+n-1}(cd)^{\frac{m-n-\xi(m-n)}{2}}}\right)\left[\frac{-(\alpha\beta)^n(\beta\alpha^{m-n}+\alpha\beta^{m-n}-\alpha^{m-n+1}-\beta^{m-n+1})}{(\alpha-\beta)^2}\right] \\
&= -\left(\frac{c(cd)^{-n}}{6^{m+n-1}(cd)^{\left\lfloor\frac{m-n)}{2}\right\rfloor}}\right)(36cd)^n\left[\frac{\beta\alpha^{m-n}+\alpha\beta^{m-n}-\alpha^{m-n+1}-\beta^{m-n+1}}{(\alpha-\beta)^2}\right] \\
&= -\left(\frac{c}{6^{m-n-1}(cd)^{\left\lfloor\frac{m-n)}{2}\right\rfloor}}\right)\left[\frac{\alpha^{m-n}(\beta-\alpha)-\beta^{m-n}(\beta-\alpha)}{(\alpha-\beta)^2}\right] \\
&= \left(\frac{c}{6^{m-n-1}(cd)^{\left\lfloor\frac{m-n)}{2}\right\rfloor}}\right)\left(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right) \\
&= \left(\frac{c}{6^{m-n-1}(cd)^{\left\lfloor\frac{m-n)}{2}\right\rfloor}}\right)\left(\frac{(36cd)^{\left\lfloor\frac{m-n)}{2}\right\rfloor}}{(6c)^{1-\xi(m-n)}}\right)b_{m-n} \\
&= c^{\xi(m-n)}b_{m-n}.
\end{aligned}$$

**Teorem 2.7. (Sums Involving Binomial Coefficients)** For any nonnegative integer $n$, we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(36cd)^{\left\lfloor\frac{k}{2}\right\rfloor}(6c)^{\xi(k)}b_k = b_{2n}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(36cd)^{\left\lfloor\frac{k+1}{2}\right\rfloor}(6c)^{\xi(k+1)-1}b_{k+1} = b_{2n+1}.$$

*Proof.* We first note that, for any integer $k$

$$6c\frac{\alpha^k-\beta^k}{\alpha-\beta} = (36cd)^{\left\lfloor\frac{k}{2}\right\rfloor}(6c)^{\xi(k)}b_k.$$

Using this equality, we can get

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(36cd)^{\left\lfloor\frac{k}{2}\right\rfloor}(6c)^{\xi(k)}b_k
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}6c\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\\
&= \frac{6c}{\alpha-\beta}\left[\sum_{k=0}^{n}\binom{n}{k}\alpha^k(-1)^{n-k}-\sum_{k=0}^{n}\binom{n}{k}\beta^k(-1)^{n-k}\right]\\
&= \frac{6c}{\alpha-\beta}\left[(\alpha-1)^n-(\beta-1)^n\right]\\
&= \frac{6c}{\alpha-\beta}\left[\left(\frac{\alpha^2}{36cd}\right)^n-\left(\frac{\beta^2}{36cd}\right)^n\right]\\
&= \frac{6c}{(36cd)^n}\left(\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right)\\
&= b_{2n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(36cd)^{\left\lfloor\frac{k+1}{2}\right\rfloor}(6c)^{\xi(k+1)-1}b_{k+1}
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}\right)\\
&= \frac{1}{\alpha-\beta}\left[\alpha\sum_{k=0}^{n}\binom{n}{k}\alpha^k(-1)^{n-k}-\beta\sum_{k=0}^{n}\binom{n}{k}\beta^k(-1)^{n-k}\right]\\
&= \frac{1}{\alpha-\beta}\left[\alpha(\alpha-1)^n-\beta(\beta-1)^n\right]\\
&= \frac{1}{\alpha-\beta}\left[\alpha\left(\frac{\alpha^2}{36cd}\right)^n-\beta\left(\frac{\beta^2}{36cd}\right)^n\right]\\
&= \frac{1}{(36cd)^n}\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right)\\
&= b_{2n+1}.
\end{aligned}$$

**Teorem 2.8.** The nonnegative terms of the bi-periodic balancing numbers are defined in terms of the positive terms as

$$b_{-n} = -b_n.$$

*Proof.* By using the Binet formula

$$\begin{aligned}
b_{-n} &= \frac{(6c)^{1-\xi(-n)}}{(36cd)^{\left\lfloor\frac{-n}{2}\right\rfloor}}\left(\frac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta}\right)\\
&= \frac{(6c)^{1-\xi(n)}}{(36cd)^{\left\lfloor\frac{-n}{2}\right\rfloor}}\left(\frac{1/\alpha^{n}-1/\beta^{n}}{\alpha-\beta}\right)
\end{aligned}$$

$$= \frac{(6c)^{1-\xi(n)}}{(36cd)^{\left\lfloor\frac{-n}{2}\right\rfloor}} \frac{(\beta^n - \alpha^n)}{(36cd)^n(\alpha - \beta)}$$
$$= \frac{(6c)^{1-\xi(n)}}{(36cd)^{\left\lfloor\frac{n}{2}\right\rfloor}} \left(\frac{\beta^n - \alpha^n}{\alpha - \beta}\right)$$
$$= -b_n.$$

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