ORIGINAL PAPER

SOME CHARACTERIZATIONS OF $\bar{M}$-GEODESIC SPRAY AND $\bar{M}$-INTEGRAL CURVE IN DUAL SPACE

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Abstract. *In this study, firstly, $\bar{M}$- vector field Z on M, $\bar{M}$- integral curve of Z and $\bar{M}$-geodesic spray concepts are given. Along this study, $\bar{M}$ is a Riemann manifold and M is a hypersurface of $\bar{M}$ in dual space. Secondly, "The natural lift $\bar{\alpha}$ of the curve α is an $\bar{M}$-integral curve of $\bar{M}$- geodesic spray Z if and only if α is an $\bar{M}$- geodesic on M" is proved in dual space.*

Keywords: *$\bar{M}$- vector field,$\bar{M}$- integral curve,$\bar{M}$- geodesic spray, dual space.*

1. INTRODUCTION

In differential geometry, properties of curve and surface theory have substantial roles. There are a lot of studies about the concepts of curves and surfaces in literature [1-3]. In these works, the geometric properties, basic definitions and theorems are given.

In geometry, if the metric changes, then studied manifold varies. One of these manifolds is Riemannian manifold that is accepted Riemannian metric for calculations. Hypersurfaces and curves associated with $\bar{M}$- vector field were studied in Riemannian manifold by N. S. Agashe in [4].

The starting point of this study is the maintheorem that proved by J.A.Thorpe "The natural lift $\bar{\alpha}$of the curve $\alpha$is an M- integral curve of the M geodesic spray if and only if α is an geodesic on M" [5].

The theorem with defining the concepts of $\bar{M}$-geodetic spray and $\bar{M}$-integral curve that "The natural lift $\bar{\alpha}$ of the curve α is an $\bar{M}$- integral curve of the $\bar{M}$ geodesic spray if and only if α is an $\bar{M}$-geodesic on M" were given by M. Çalışkan and A. I. Sivridağ in [6].

The same theorem was given by M. Çalışkan and E.Karaca in dual space [7]. In that study, the concepts of geodesic spray and integral curve were presented in dual space.

In this study, firstly, $\bar{M}$- vector field Z on M, $\bar{M}$- integral curve of Z, and $\bar{M}$- geodesic spray concepts are given. Finally, the theorem proved by Çalışkan and Sivridağ that "The natural lift $\bar{\alpha}$ of the curve α is an $\bar{M}$- integral curve of the $\bar{M}$ geodesic spray if and only if α is an $\bar{M}$-geodesic on M" was proved in dual space.

2. PRELIMINARIES

In this section, some elementary definitions and theorems are given in Euclidean space and dual space. Then, definitions of geodesic spray and integral curve in Riemannian manifold are presented.

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Definition 1: Let α be a regular curve with $\alpha''(s) \neq 0$. For every point of $\alpha(s)$, the set $\{T(s), N(s), B(s)\}$ is called the Frenet frame along (s) , where

$$T(s) = \alpha'(s), N(s) = \frac{\alpha''(s)}{||\alpha''(s)||}, B(s) = T(s) \times N(s)$$

are the unit tangent, principal normal and binormal vectorsof the curve at the point $\alpha(s)$, respectively [3].

Definition 2: Let M be a hypersurface in $\mathbb{R}^3$ and let $\alpha: I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{\boldsymbol{d}}{\boldsymbol{ds}}\big(\alpha(s)\big) = X(\alpha(s)) \text{ (for all } s \in I)$$

where X is a smooth tangent vector field on M [5]. We have

$$\mathbf{TM} = \cup_{\mathbf{p}\in\mathbf{M}} \mathrm{T_pM} = \boldsymbol{\chi}(\mathbf{M}),$$

where T_pM is the tangent space of M at p and $\boldsymbol{\chi}(\mathbf{M})$is the space of vector fields on M.

Definition 3: For any parametrized curve $\alpha: I \to M$, $\bar{\alpha}: I \to TM$ given by

$$\bar{\alpha}(s) = \big(\alpha(s), \alpha'(s)\big) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift α on TM [5]. Thus, we can write

$$\frac{\boldsymbol{d\bar{\alpha}}}{\boldsymbol{ds}} = \frac{\boldsymbol{d}}{\boldsymbol{ds}}\big(\alpha'(s)|_{\alpha(s)}\big) = D_{\alpha'(s)}\alpha'(s)$$

where D is the Levi-Civita connection on $\mathbb{R}^3$.

Definition 4: The set $D = \{(a, a^*) | a, a^* \in \mathbb{R}\}$ is called dual numbers set with product, addition operations as follows:

For the dual numbers $A = (a, a^*)$ and $B = (b, b^*)$,

$$\oplus: D \times D \to D\ ,$$

$$(A, B) \to A \oplus B = (a, a^*) \oplus (b, b^*) = (a + b, a^* + b^*)$$

$$\odot: D \times D \to D$$

$$(A, B) \to A \odot B = (a, a^*) \odot (b, b^*) = (ab, ab^* + a^*b)$$

Definition 5: The set $D^3 = \left\{\vec{A} \mid \vec{A} = \vec{a} + \varepsilon\overrightarrow{a^*}, \vec{a}, \overrightarrow{a^*} \in \mathbb{R}^3\right\}$ is called D-Modul [2]. Addition and scalar product on D^3 are given as follows:

$$\oplus: D^3 \times D^3 \to D^3, \vec{A} \oplus \vec{B} = \big(\vec{a} + \vec{b}\big) + \varepsilon(\overrightarrow{a^*} + \overrightarrow{b^*})$$

$$\odot: D \times D^3 \to D^3\ , \vec{\lambda} \odot \vec{A} = \lambda\vec{a} + \varepsilon(\lambda\overrightarrow{a^*} + \lambda^*\vec{a})$$

Definition 6: Let M be a surface on E^n and β be a curve on the M. T is the unit tangent vector of β and if

$$D_T T = 0\ ,$$

β called a geodesic curve on E^n. If

$$\bar{D}_T T = 0\ ,$$

β called a geodesic curve on M [6].

Definition 7: $X \in \chi(M)$ differentiable vector field is a called geodesic spray on TM if

$$\mathbf{X}(\mathbf{V}) = -< V, S(V) > \mathbf{N_p}$$

for $V \in TM$. N unit normal vector field for M [6].

Definition 8: For an $\bar{M}$- vector field $= Z_t + Z_n$, a curve βon M is called an $\bar{M}$- integral curve of Z if

$$Z_t(\beta(t)) = (\frac{d\beta}{dt})\Big|_{\beta(t)}.$$

Definition 9: An $\bar{M}$- vector field is called as an $\bar{M}$- geodesic spray if for $V \in TM$

$$Z_t(V) = \left(\frac{d\lambda}{dt} - < V, S(V) >\right) N.$$

3. $\bar{M}$-GEODESIC SPRAY AND $\bar{M}$-INTEGRAL CURVE IN DUAL SPACE

This section presents the definitions of $\bar{M}$-geodesic spray and $\bar{M}$ – integral curve in dual space.

Let $\bar{M}$ be Riemannian manifold and M be a hypersurface of $\bar{M}$ in dual space. $\bar{D}$ being the Riemannian connection on $\bar{M}$, S Weingarten map of M, $N = n + \varepsilon n^*$ being unit normal vector field of M.

$$\forall\ X = x + \varepsilon x^*, Y = y + \varepsilon y^* \in \ \chi(M)$$

we have the Gauss Equation given by

$$\bar{D}_X Y = D_X Y - < S(X), Y > N$$

$$\bar{D}_x y + \varepsilon[\bar{D}_{x^*} y + \bar{D}_x y^*] = D_x y - < S(x), y > n + \varepsilon[D_{x^*} y + D_x y^* - < S(x), y > n^* - < S(x^*), y > n - < S(x), y^* > n]$$

$$\bar{D}_x y = D_x y - < S(x), y > n$$
$$\bar{D}_{x^*} y + \bar{D}_x y^* = D_{x^*} y + D_x y^* - < S(x), y > n^* - \quad < S(x^*), y > n - < S(x), y^* > n$$

where D is the Riemannian connection on M.

Definition 10: Let Z be a vector field on $\bar{M}$. $Z = z + \varepsilon z^*$ is called an $\bar{M}$- vector field on M if Z is a mapping which atteches to each point p in M, a vector Z_p in $Tp\bar{M}$, that is,

$$Z: M \to T_p\bar{M},$$

$$p \to Z_p$$

Any $\bar{M}$- vector field Z can be decomposed into its tangential and normal components given by

$$Z = Z_t + Z_n$$

where Z_t is a tanget vector field on M and Z_n is a vektor field of $\bar{M}$ defined on M which is normal to M at every point. We have

$$Z = Z_t + \lambda N$$

$$z + \varepsilon z^* = (z_1 + \varepsilon z_1{}^*) + \lambda(n + \varepsilon n^*)$$

where $\lambda \epsilon C^\infty(M, \mathcal{R})$.

Let $\beta = \beta_1 + \varepsilon\beta_1{}^*$be a curve passing through a point p on M and T denote the tangent vector field of a on M. Covariant differentiation of Z in the direction $T = t + \varepsilon t_1 *$ gives

$$\bar{D}_T Z = \bar{D}_T Z_t + \bar{D}_T \lambda N$$
$$= D_T Z_t + \lambda S(T) + \left(\frac{d\lambda}{dt} - < S(T), Z_t >\right) N$$

After some calculations, we obtain

$$\bar{D}_T Z = tan\bar{D}_T Z + nor\bar{D}_T Z,$$

where

$$tan\bar{D}_{t_1} z = D_{t_1} z_t + \lambda S(t_1)$$
$$tan\left(\bar{D}_{t_1} \cdot z + \bar{D}_{t_1} z *\right) = D_{t_1} \cdot z + D_{t_1} z * + \lambda S(t_1{}^*)$$

and

$$nor\bar{D}_{t_1} z = (\frac{d\lambda}{dt} - < S(t_1) z_t >) n$$

$$nor\left(\bar{D}_{t_1} \cdot z + \bar{D}_{t_1} z *\right) = \left(\frac{d\lambda}{dt} - < S(t_1), z_t >\right) n^* - < S(t_1{}^*), z_t > n - < S(t_1), z_t{}^* > n.$$

Definition 11: [8] For an $\bar{M}$- vector field $Z = Z_t + Z_n$, a curve β c M is called an $\bar{M}$- integral curve of Z if

$$Z_t(\beta(t)) = (\frac{d\beta}{dt})\Big|_{\beta(t)} ,$$

$$z_t(\beta_1) = \frac{d\beta_1}{dt}$$

$$z_t^*(\beta_1) + z_t(\beta_1^*) = \frac{d\beta_1^*}{dt} \ .$$

Definition 12: [8] β c M being a differentiable curve, the curve $\bar{\beta}: I \to TM$ given by

$$\bar{\beta}(t) = \dot{\beta}(t)\big|_{\beta(t)},$$

$$\bar{\beta}_1(t) = \dot{\beta}_1(t)\big|_{\beta(t)}$$

$$\bar{\beta}_1^*(t) = \dot{\beta}_1^*(t)\big|_{\beta(t)}$$

is called as the naturel lift of α on the manifold TM.

Definition 13: [8] An $\bar{M}$- vector field Z is called as an $\bar{M}$- geodesic spray if for $V \in TM$

$$Z_t(V) = \left(\frac{d\lambda}{dt} - < S(V), V >\right) N$$

$$z_t(v) = \left(\frac{d\lambda}{dt} - < S(v), v >\right) n$$

$$z_t^*(v) + z_t(v^*) = \left(\frac{d\lambda}{dt} - < S(v), v >\right) n^* + < S(v^*), v > n + < S(v), v^* > \ .$$

Theorem 1: [8] The natural lift $\bar{\beta} = \bar{\beta}_1 + \varepsilon\bar{\beta}_1^*$ of the curve $\beta = \beta_1 + \varepsilon\beta_1^*$ is an $\bar{M}$- integral curve of the $\bar{M}$- geodesic spray $Z = z + \varepsilon z^*$ if β is an $\bar{M}$- geodesic on M.

Proof: (⇒)Let $\bar{\beta} = \bar{\beta}_1 + \varepsilon\bar{\beta}_1^*$ be an $\bar{M}$- integral curve of the $\bar{M}$- geodesic spray $Z = z + \varepsilon z^*$.

Thus, we can write

$$Z_t(\bar{\beta}) = \frac{d\bar{\beta}}{dt}$$

$$z_t(\bar{\beta}_1) + \varepsilon[z_t^*(\bar{\beta}_1) + z_t(\bar{\beta}_1^*)] = \frac{d\bar{\beta}_1}{dt} + \varepsilon\frac{d\bar{\beta}_1^*}{dt} \ . \tag{1}$$

Since $Z = z + \varepsilon z^*$ is a $\bar{M}$- geodesic spray we have

$$Z_t(\bar{\beta}) = \left(\frac{d\lambda}{dt} - < S(\bar{\beta}), \bar{\beta} >\right) N$$

$$z_t(\bar{\beta}_1) + \varepsilon[z_t^*(\bar{\beta}) + z_t(\bar{\beta}_1^*)] = \left(\frac{d\lambda}{dt} - < S(\bar{\beta}_1), \bar{\beta}_1 >\right) n + \varepsilon\Big[\left(\frac{d\lambda}{dt} - < S(\bar{\beta}_1), \bar{\beta}_1 >\right) n^* + < S(\bar{\beta}_1^*), \bar{\beta}_1 > n + < S(\bar{\beta}_1), \bar{\beta}_1^* > n\Big] \tag{2}$$

Joining (1) and (2) we obtain that

$$\frac{d\bar{\beta}_1}{dt}+\varepsilon\frac{d\bar{\beta}_1{}^*}{dt}=(\frac{d\lambda}{dt}-< S(\bar{\beta}_1),\bar{\beta}_1 >)n+\varepsilon[\left(\frac{d\lambda}{dt}-< S(\bar{\beta}_1),\bar{\beta}_1 >\right)n^*+< S(\bar{\beta}_1{}^*),\bar{\beta}_1 > n$$
$$+< S(\bar{\beta}_1),\bar{\beta}_1{}^* > \mathrm{n}]$$

$$\bar{D}_\beta\dot{\beta}=(\frac{d\lambda}{dt}-< S(\bar{\beta}_1),\bar{\beta}_1 >)n+\varepsilon[\left(\frac{d\lambda}{dt}-< S(\bar{\beta}_1),\bar{\beta}_1 >\right)n^*+< S(\bar{\beta}_1{}^*),\bar{\beta}_1 > n$$
$$+< S(\bar{\beta}_1),\bar{\beta}_1{}^* > \mathrm{n}]$$

$$tan\bar{D}_\beta\dot{\beta}=0$$

This shows that β is an $\bar{M}$- geodesic on M which is to be shown.

$(\Rightarrow)$ Let β is an $\bar{M}$- geodesic on M. Thus

$$tan\bar{D}_\beta\dot{\beta}=0.$$

We obtain that

$$\bar{D}_\beta\dot{\beta}=(\frac{d\lambda}{dt}-< S(\bar{\beta}_1),\bar{\beta}_1 >)n+\varepsilon[\left(\frac{d\lambda}{dt}-< S(\bar{\beta}_1),\bar{\beta}_1 >\right)n^*+< S(\bar{\beta}_1{}^*),\bar{\beta}_1 > n$$
$$+< S(\bar{\beta}_1),\bar{\beta}_1{}^* > \mathrm{n}]$$

$$\frac{d\bar{\beta}_1}{dt}+\varepsilon\frac{d\bar{\beta}_1{}^*}{dt}=z_t(\bar{\beta}_1)+\varepsilon[z_t{}^*(\bar{\beta}_1)+z_t(\bar{\beta}_1{}^*)].$$

This shows that $\beta$is an $\bar{M}$- integral curve of the $\bar{M}$- geodesic spray $Z=z+\varepsilon z^*$.

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