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PATHWAY FRACTIONAL FORMULA AND INTEGRAL INVOLVING INCOMPLETE H-FUNCTION

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Abstract. *In the present paper we derive a pathway fractional integral formula for the incomplete H-function and presumably new integral involving product of incomplete H-function and a general class of polynomials having general arguments. A large number of integrals involving various simpler functions follow as special cases of these integrals.*

Keywords: *incomplete H-function; pathway fractional integral, general class of polynomials; incomplete hypergeometric function.*

1. INTRODUCTION

The incomplete H-functions by Srivastava et al. [1] $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ will be represented and defined in the following manner

$$\gamma_{p,q}^{m,n}(z) = \gamma_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right]$$

$$= \gamma_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1, A_1, x), (a_2, A_2) \dots, (a_p, A_p) \\ (b_1, B_1), (b_2, B_2) \dots, (b_q, B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_{\mathcal{L}} g(s,x) z^{-s}\, ds, \quad (1)$$

where

$$g(s,x) = \frac{\gamma(1-a_1-A_1 s, x)\prod_{j=1}^{m}\Gamma(b_j+B_j s)\prod_{j=2}^{n}\Gamma(1-a_j-A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j s)\prod_{j=n+1}^{p}\Gamma(a_j+A_j s)} \quad (2)$$

and

$$\Gamma_{p,q}^{m,n}(z) = \Gamma_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_1, A_1, x), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right]$$

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$$= \Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1,x),(a_2,A_2)\dots,(a_p,A_p)\\(b_1,B_1),(b_2,B_2)\dots,(b_q,B_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} G(s,x)z^{-s}\,ds, \tag{3}$$

where

$$G(s,x) = \frac{\Gamma(1-a_1-A_1 s,x)\prod_{j=1}^{m}\Gamma(b_j+B_j s)\prod_{j=2}^{n}\Gamma(1-a_j-A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_j s)\prod_{j=n+1}^{p}\Gamma(a_j+A_j s)} \tag{4}$$

The incomplete H-function $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ in (1) and (3) exist for all $x \geqq 0$ under the same contour and the same sets of conditions as stated for H-function in [2]. The incomplete H-functions in (1) and (3) are symmetric in the set of parameters pairs $(a_2,A_2)\dots,(a_n,A_n)$ in the set of parameters pairs $(a_{n+1},A_{n+1})\dots,(a_p,A_p)$ in the set of parameter pairs $(b_1,B_1)\dots,(b_m,B_m)$, and in the set of parameter pairs $(b_{n+1},B_{m+1})\dots,(b_q,B_q)$. The definitions (1) and (3) readily yield the following decomposition formula

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = \mathrm{H}_{p,q}^{m,n}(z) \tag{5}$$

2. PATHWAY INTEGRAL OPERATOR OF AN INCOMPLETE H-FUNCTIONS

Let $f(x) \in L(a,b), \eta \in \mathbb{C}, \Re(\eta) > 0, a > 0$ and let us take a "pathway parameter" $\alpha < 1$. The pathway fractional integration operator studied in the paper is defined and represented as follows [3]:

$$\left(P_{0^+}^{(\eta,\alpha)} f\right)(x) = x^{\eta}\int_0^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt \tag{6}$$

The pathway model related to above operator was introduced by Mathai [4] and studied further by Mathai and Haubold [5,6]. For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p. d. f.):

$$f\,(x) = c|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{(1-\alpha)}} \tag{7}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \left[1-a(1-\alpha)|x|^{\delta}\right] > 0, \gamma > 0$, where c is the normalizing constant and α is called the pathway parameter. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2}\frac{\delta[a(1-\alpha]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{(1-\alpha)}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha}+1\right)}, \quad \text{for } \alpha < 1 \tag{8}$$

$$c = \frac{1}{2}\frac{\delta[a(1-\alpha]^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta}\right)}, \text{ for } \frac{1}{\alpha-1}-\frac{\gamma}{\delta} > 0, \alpha > 1 \tag{9}$$

$$c = \frac{1}{2}\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)},\ \alpha \to 1 \tag{10}$$

We observe that for $\alpha < 1$ it is a finite range density with $\left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and (7) remains in the extended generalized type-1 beta family. The pathway density in (7), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p. d. f.

For $\alpha > 1$, we have

$$f\,(x) = c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{(\alpha-1)}} \tag{11}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \alpha > 1$, which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the Student-t density, the Cauchy density and many more.

Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (7) and (11) take the exponential from, since

$$\lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\eta}{(1-\alpha)}} = \lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\eta}{(\alpha-1)}}$$

$$= c|x|^{\gamma-1} exp\left(-a\eta|x|^{\delta}\right) \tag{12}$$

This include the generalized Gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities.

When $\alpha \to 1_{-}$, $\left[1 - \frac{a(1-\alpha)}{x}\right]^{\frac{\eta}{(1-\alpha)}} \to e^{-\frac{\alpha\,\eta}{x}t}$. Then, operator (6) reduces to the Laplace integral transform f f with parameter $\frac{\alpha\,\eta}{x}$:

$$\left(P_{0^+}^{(\eta,1)} f\right)(x) = x^{\eta} \int_0^{\infty} e^{-\frac{\alpha\,\eta}{x}} f(t)dt = x^{n} L_f\left(\frac{\alpha\,\eta}{x}\right) \tag{13}$$

When $\alpha = 0$, $a = 1$, then replacing η by $\eta - 1$ in (6) the integral operator reduces to the Riemann-Liouville fractional integral operator (For more details, we refer to [7-14].

Lemma 1. Let $\eta \in \mathbb{C}, \Re(\eta) > 0, \beta \in \mathbb{C}$ and $\alpha < 1$. If $\Re(\beta) > 0, \Re\left(\frac{\eta}{1-\alpha}\right) > -1$, then there holds the basic formula

$$P_{0^+}^{(\eta,\alpha)}\left[x^{\beta-1}\right] = \frac{x^{\eta+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\beta+1\right)} \tag{14}$$

Theorem 1. Let $\eta, \rho \in \mathbb{C}, \Re(\beta) > 0, \Re\left(1 + \frac{\eta}{1-\alpha}\right) > 0, \Re(\rho) > 0$ and $\alpha < 1, b \in \Re$. Then for the pathway fractional integral $P_{0_+}^{(\eta,\alpha)}$ the following formula holds for the image of an arbitrary incomplete H function

$$\left(P_{0_+}^{(\eta,\alpha)} t^{\rho-1} \Gamma_{p,q}^{m,n}\left[bt^{\beta}\left|\begin{matrix}(a_1,\alpha_{1,}x),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]\right) = \frac{x^{\eta+\rho}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}}$$

$$\times \Gamma_{p+1,q+1}^{m,n+1}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\left|\begin{matrix}(a_1,\alpha_1,x),(a_2,\alpha_2),\dots,(a_p,\alpha_p),(1-\rho,\beta)\\ (b_1,\beta_1),\dots,(b_q,\beta_q),(-\rho-\frac{\eta}{1-\alpha},\beta)\end{matrix}\right.\right] \tag{15}$$

Proof. Using the definitions (1.3) and (2.1), we have

$$\left(P_{0_+}^{(\eta,\alpha)}t^{\rho-1}\Gamma_{p,q}^{m,n}\left[bt^{\beta}\left|\begin{matrix}(a_1,\alpha_1,x),(a_2,\alpha_2),\dots,(a_p,\alpha_p)\\ (b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right]\right)$$

$$=x^{\eta}\int_0^{\frac{x}{a(1-\alpha)}}t^{\rho-1}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}\frac{1}{2\pi i}\int_L G(s,x)(bt^{\beta})^{-s}ds\,dt$$

Interchanging the order of integration and evaluating the integral by the beta function formula, it gives

$$=\frac{x^{\eta+\rho}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}}\frac{1}{2\pi i}\int_L G(s,x)\frac{\Gamma(\rho-\beta s)}{\Gamma\left(1+\rho+\frac{\eta}{1-\alpha}-\beta s\right)}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^{-s}ds$$

$$=\frac{x^{\eta+\rho}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}}$$

$$\times \Gamma_{p+1,q+1}^{m,n+1}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\left|\begin{matrix}(a_1,\alpha_1,x),(a_2,\alpha_2),\dots,(a_p,\alpha_p),(1-\rho,\beta)\\ (b_1,\beta_1),\dots,(b_q,\beta_q),(-\rho-\frac{\eta}{1-\alpha},\beta)\end{matrix}\right.\right]$$

The interchange of the order of integration is permissible under the conditions stated in the theorem due to convergence of the integrals involved in the process. This completes the proof of Theorem 1.

When $\alpha\to 1$, then (2.10) tends to

$$\frac{x^{\eta+\rho}\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{[a(1-\alpha)]^{\rho}}\frac{1}{2\pi i}\int_L G(s,x)\frac{\Gamma(\rho-\beta s)}{\Gamma\left(1+\rho+\frac{\eta}{1-\alpha}-\beta s\right)}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^{-s}ds\to$$

$$\frac{x^{\eta+\rho}}{(a\eta)^{\rho}}\frac{1}{2\pi i}\int_L G(s,x)\Gamma(\rho-\beta s)\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^{-s}ds$$

$$\to\frac{x^{\eta+\rho}}{(a\eta)^{\rho}}\Gamma_{p+1,q}^{m,n+1}\left[\frac{bx^{\beta}}{a^{\beta}\eta^{\beta}}\left|\begin{matrix}(a_1,\alpha_1,x),(a_2,\alpha_2),\dots,(a_p,\alpha_p),(1-\rho,\beta)\\ (b_1,\beta_1),(b_2,\beta_2),\dots,(b_q,\beta_q)\end{matrix}\right.\right] \tag{16}$$

as it gives the formula for Laplace transform of the incomplete H-function [3].

3. INTEGRAL INVOLVING INCOMPLETE H-FUNCTION AND POLYNOMIAL

The general class of polynomials introduce by Srivastava [15] defined in the following manner

$$S_V^U(y) = \sum_{K=0}^{[\frac{V}{U}]} \frac{(-V)_{UK} A(V,K)}{K!} y^K, \qquad V = 0,1,2,\dots, \tag{17}$$

where U is an arbitrary positive integer and coefficients $A(V,K), (V,K \geq 0)$ are arbitrary constants, real or complex.

Theorem 2. The following integral holds true:

$$\int_0^\infty z^{\lambda-1} \left[z + a + (z^2 + 2az)^{\frac{1}{2}}\right]^{-\nu} \gamma_{p,q}^{m,n} \left[\omega \left[z + a + (z^2 + 2az)^{\frac{1}{2}}\right]^{-\mu}\right]$$

$$\times\ S_V^U \left[\tau \left[z + a + (z^2 + 2az)^{\frac{1}{2}}\right]^{-\alpha}\right] dz$$

$$= 2a^{-\nu} \left(\frac{1}{2} a\right)^{\lambda} \Gamma(2\lambda) \sum_{K=0}^{[\frac{V}{U}]} (-V)_{UK} A(V,K) \frac{\left(\tau/_{a^\alpha}\right)^K}{K!}\ F_{p+2,q+2}^{m,n+2}$$

$$\times \left[\omega a^{-\mu} \left| \begin{matrix} (-\nu - \alpha K, \mu), (1 + \lambda - \nu - \alpha K, \mu), (a_1, \alpha_1, x)(a_2, \alpha_2), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q), (1 - \nu - \alpha K, \mu), (-\nu - \alpha K - \lambda, \mu) \end{matrix} \right. \right] \tag{18}$$

where
(i) $\mu > 0, Re(\lambda, \nu, \alpha) > 0\ \ x \geq 0$
(ii) $Re(\lambda) - Re(\nu) - \mu \min_{1 \leq j \leq m} Re\left(\frac{b_j}{\beta_j}\right) < 0$

Proof. To obtain the result (2) the first express Incomplete H-function involved in its left-hand side in terms of contour integral using equation (3) and the general class of polynomials $S_V^U(y)$ in series form given by equation (17). Interchanging the order of integration and summation which is permissible under the conditions stated with (18) and evaluating the *x*-integral with the help of the result given below [16]:

$$\int_0^\infty z^{\lambda-1} \left[z + a + (z^2 + 2az)^{\frac{1}{2}}\right]^{-\nu} dz$$

$$= 2\nu a^{-\nu} \left(\frac{1}{2} a\right)^{\lambda} [\Gamma(1 + \nu + \lambda]^{-1} \Gamma(2\lambda)\, \Gamma(\nu - \lambda), \quad 0 < Re(\lambda) < \nu,$$

We easily arrive at the desired result (3.2). If in the integral (18) we reduce $S_V^U(y)$ to unity and incomplete H-function to incomplete Gauss hypergeometric function [17], we arrive at the following result after a little simplification

$$\int_0^\infty z^{\lambda-1} \left[z + a + (z^2 + 2az)^{\frac{1}{2}}\right]^{-\nu}$$

$$\times \quad {}_2\Gamma_1[a, b; c; \tau(z + a + (z^2 + 2az)^{\frac{1}{2}})^{-1}]dz$$

$$= 2^{1-\lambda}\nu\Gamma(2\lambda)a^{\lambda-\nu}\frac{\Gamma(\nu-\lambda)}{\Gamma(\nu+\lambda+1)}$$

$$\times \quad {}_4\Gamma_3(a, b, \nu-\lambda, \nu+1; c, \nu, \nu+\lambda+1; {}^{\tau}/_{a}) \tag{19}$$

where

$$0 < Re(\lambda) < Re(\nu), |\tau| < |a|, x \geq 0.$$

The importance of the result given by (18) lies in the fact that it not only gives the value of the integral but also ‘augments’ the coefficients in the series in the integrand to give a ${}_4\Gamma_3$ series as the integrated series. A number of other integrals involving functions that are special cases of incomplete H-function [13] and/or the general class of polynomials can also be obtained from (18) but we do not record them here.

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