ORIGINAL PAPER

A SHORT NOTE ON AVI CIRCLE

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Abstract. *In this note we study about a circle (refered as avi circle) which passes through the six notable touch points and having center at centroid (G) of triangle ABC and radius as* $\sqrt{\frac{S_A+S_B+S_C}{9}}$.

Keywords: *centroid, power of point, avi circle, Stewart's theorem.*

1. INTRODUCTION

Let ΔABC be a reference triangle, Suppose B_A, C_A are the points where the tangents drawn from vertex A touches the semicircle which is constructed on BC taking BC as diameter, similarly define the points C_B, A_B, A_C and B_C. All the six points B_A, C_A, C_B, A_B, A_C and B_C are concyclic (Fig. 1). For recognisitaion sake let us call the circle which passes through all the six points B_A, C_A, C_B, A_B, A_C and B_C as ***avi circle.*** In this short note we study about this circle.

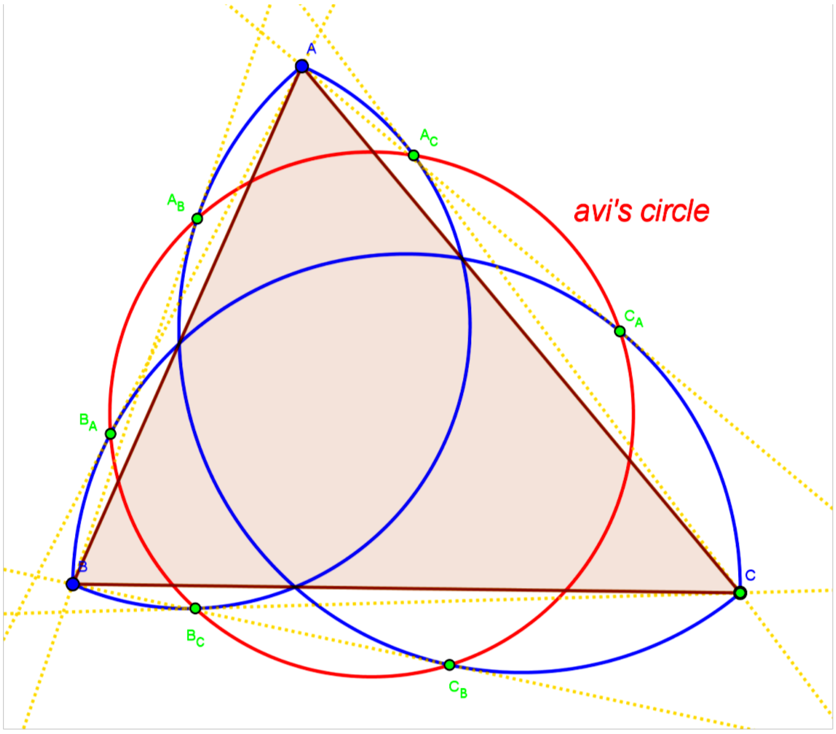

Figure 1. Avi's circle.

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Before proving our main theorem, let us prove some related lemmas.

Lemma 1. Let the points P and D are the foot of perpendicular and median drawn from the vertex A to the side BC respectively then the five points A, B_A, P, D and C_A are concyclic (refer Fig. 2).

Proof: It is clear that D is the mid point of BC as well as center of semicircle constructed on BC taking BC as diameter, and since the lines AB_A, AC_A are tangents from A to the semicircle constructed on BC taking BC as diameter.

So $AB_A \perp B_AD$ and $AC_A \perp C_AD$

Hence the four points A, B_A, D, C_A are concyclic.

Now since $AP \perp BC$ it implies $\angle AB_AD = \angle APD = 90^0$

It proves that the point P is concylic with the circle which passes through the points A, B_A, D, C_A.

Hence all the five points A, B_A, P, D and C_A are concyclic.

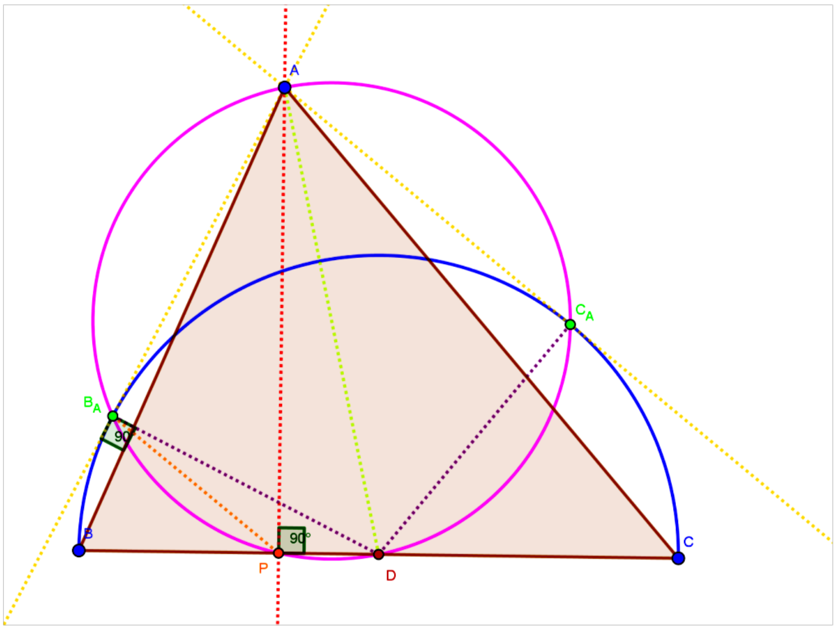

Figure 2. The five points A, B_A, P, D and C_A are concyclic

Lemma 2. The lines B_AC_A and AP intersects at orthocenter (H) of the triangleABC and also

$$AH.HP = B_AH.HC_A = \frac{S_AS_BS_C}{4\Delta^2}$$

where

$$2S_A = b^2 + c^2 - a^2 = 2bc\cos A,\ 2S_B = c^2 + a^2 - b^2 = 2ca\cos B\ \ 2S_c = a^2 + b^2 - c^2 = 2ab\cos C$$

using conway notation[1] (refer Fig. 3).

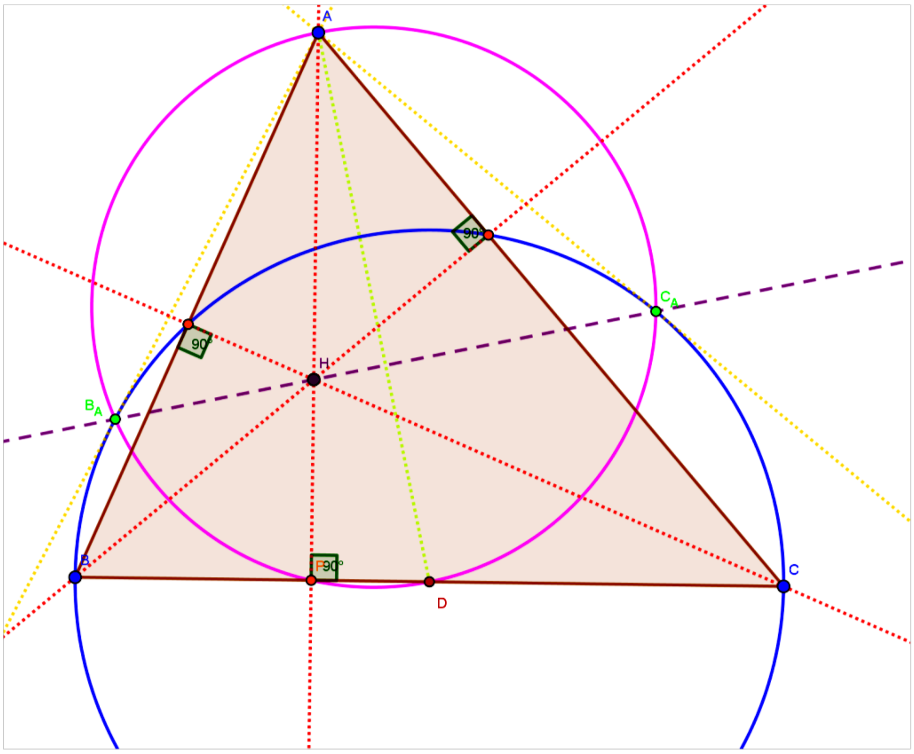

Figure 3. The lines B_AC_A and AP intersects at orthocenter (H)

Proof: For proving Lemma 2, we will make use of Homogeneous barycentric coordinates.

If a triangle ABC has side lengths BC = a, CA = b, AB = c then A = (1 : 0 : 0), B = (0 : 1 : 0) , C = (0 : 0 : 1) , D = (0 : 1 : 1) in homogeneous barycentric coordinates with reference to ABC [1]. The coordinates of P = (0: bcosC : ccosB) = $\left(0:S_C:S_B\right)$ and H = (tanA : tanB : tanC) = $\left(S_BS_C:S_CS_A:S_AS_B\right)$ [using conway's notation].

Now the equation of the circle which contains the five points A, B_A, P, D and C_A in homogeneous barycentric coordinates is given by

$$2a^2yz+2b^2zx+2c^2xy-\left(x+y+z\right)\left(S_By+S_Cz\right)=0 \tag{1}$$

and the equation of the circle which is constructed on BC taking BC as diameter in homogeneous barycentric coordinates is given by

$$a^2yz+b^2zx+c^2xy-S_Ax\left(x+y+z\right)=0 \tag{2}$$

Now it is clear that the radical axis of (1) and (2) is the line B_AC_A

So the equation of the line B_AC_A is given by

$$2S_Ax-S_By-S_Cz=0 \tag{3}$$

Now it is easy to verify that the point H(orthocenter) lies on the line (3).

Hence the lines B_AC_A and AP intersects at H orthocenter of the triangleABC and since the points A, B_A, P, C_A are concyclic(using lemma-1), the lines B_AC_A and AP intersects at H. So using chords property $AH.HP=B_AH.HC_A$.

Since AH=2RcosA, HP= 2RcosBcosC,

$$B_AH.HC_A = AH.HP = 2R\cos A.2R\cos B\cos C = 4R^2\cos A\cos B\cos C = \frac{S_AS_BS_C}{4\Delta^2}$$

since abc=4RΔ

Hence proved.

Now let us prove our main theorem.

Theorem 1. The six points B_A, C_A, C_B, A_B, A_C and B_C are concyclic (for reconginisation sake let us call the circle as avi circle)(Fig. 4).

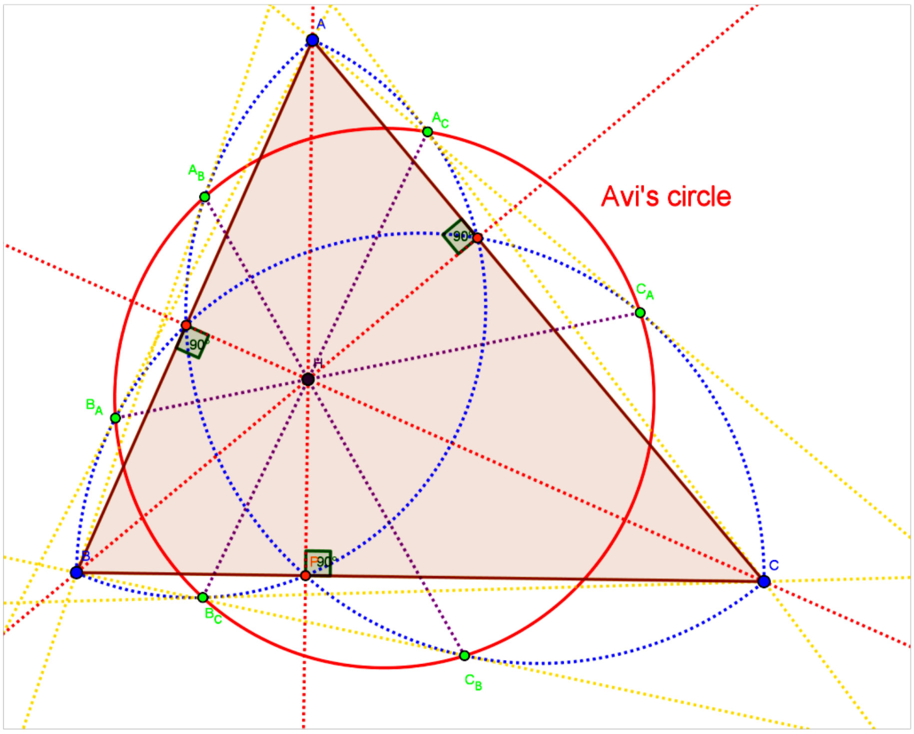

Figure 4. The six points B_A, C_A, C_B, A_B, A_C and B_C lie on Avi's Circle

Proof: Using lemma-3, we can prove that $B_AH.HC_A = C_BH.HA_B = A_CH.HB_C = \frac{S_AS_BS_C}{4\Delta^2}$.

That is the line segments B_AC_A, C_BA_B and A_CB_C are concurrent at H such that

$$B_AH.HC_A = C_BH.HA_B = A_CH.HB_C$$

Hence using chords property(power of point) we can conclude that the six points B_A, C_A, C_B, A_B, A_C and B_C are concyclic.

Hence proved

Theorem 2: Avi circle has center at centroid (G) of triangle ABC and radius as $\sqrt{\frac{S_A+S_B+S_C}{9}}$ (refer Fig. 5).

Proof: To prove that G is the center of avi circle, it is enough to prove that

$$GB_A = GC_A = GC_B = GA_B = GA_C = GB_C = \sqrt{\frac{S_A + S_B + S_C}{9}} .$$

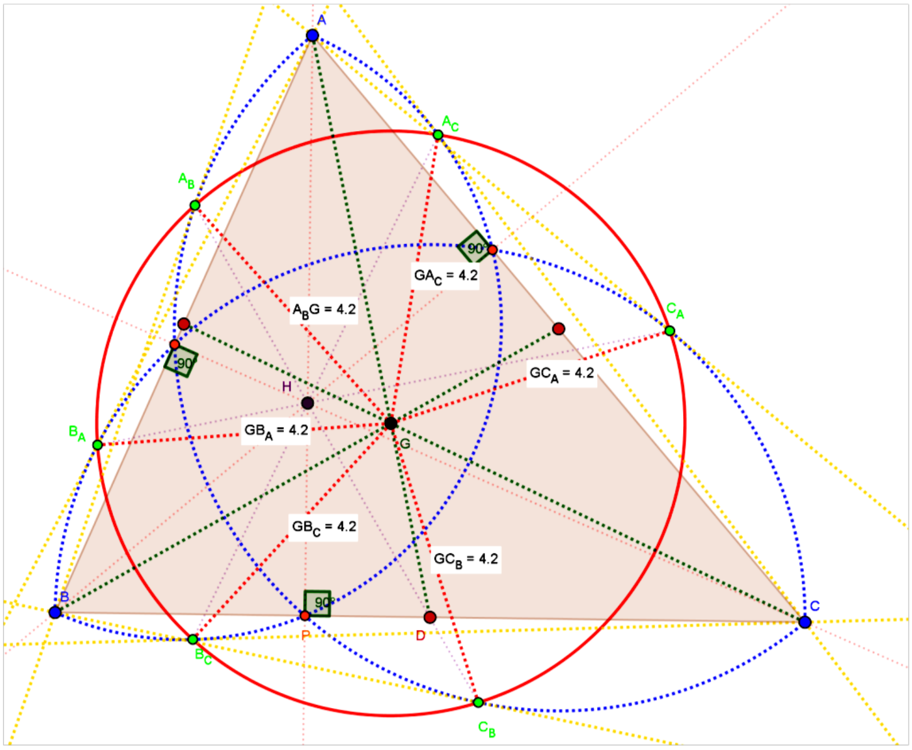

Figure 5. The centroid G is the center of Avi's Circle

Let us compute lengths of AB_A, GB_A (refer Fig. 6)

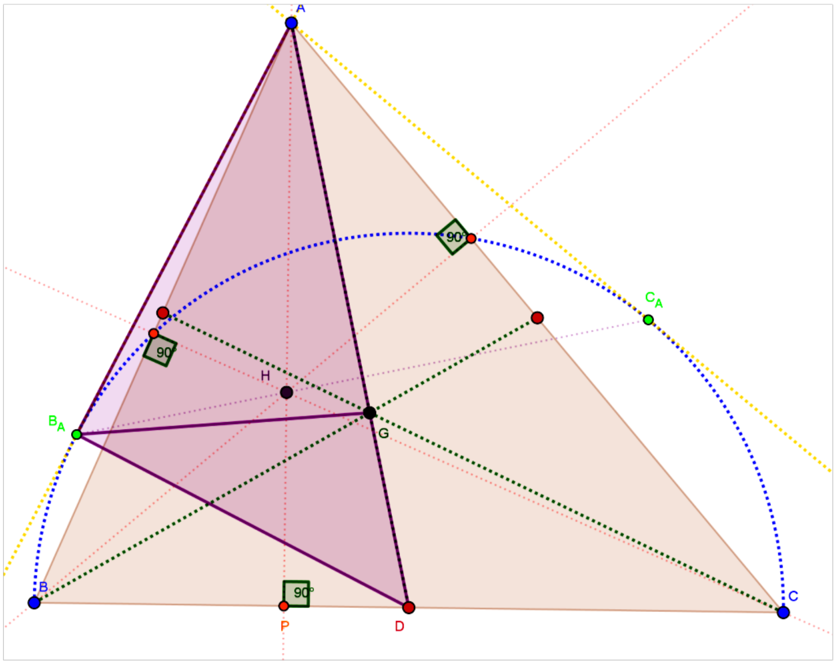

Figure 6. Computing the lengths of AB_A, GB_A

Consider triangle AB_AD,

Since $\angle AB_AD = 90^0 \Rightarrow AD^2 = AB_A{}^2 + B_AD^2$ by replacing

$$B_AD = \frac{a}{2},\ AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},$$

we get $AB_A{}^2 = \frac{b^2 + c^2 - a^2}{2} = S_A$ and from the triangle AB_AD, centroid(G) lies on AD such that AG:GD = 2:1.

So using stewarts theorem, $B_AG^2 = \frac{DG.AB_A{}^2}{AD} + \frac{AG.DB_A{}^2}{AD} - AG.GD$

by replacing

$$B_AD = \frac{a}{2},\ AG = \frac{2AD}{3}, DG = \frac{AD}{3}, AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},\ AB_A{}^2 = \frac{b^2 + c^2 - a^2}{2} = S_A$$

we get $B_AG^2 = \frac{b^2 + c^2 - a^2}{6} + \frac{a^2}{6} - \frac{2b^2 + 2c^2 - a^2}{18} = \frac{a^2 + b^2 + c^2}{18} = \frac{S_A + S_B + S_C}{9}$

In the similar manner we can prove

$$GB_A = GC_A = GC_B = GA_B = GA_C = GB_C = \sqrt{\frac{S_A + S_B + S_C}{9}}.$$

Hence proved

Notes:

1. The equation of the avi circle in homogeneous barycentric coordinates is given by

$$2(a^2yz + b^2zx + c^2xy) - (S_Ax^2 + S_By^2 + S_Cz^2) = 0$$

or

$$3(a^2yz + b^2zx + c^2xy) - (x + y + z)(S_Ax + S_By + S_Cz) = 0 .$$

2. The avi circle is also called as "Orthoptic Circle of the Steiner Inellipse"[2]. This short note gives a new way construction of Orthoptic Circle of the Steiner Inellipse.

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[1] Yiu, P., *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

[2] http://mathworld.wolfram.com/OrthopticCircleoftheSteinerInellipse.html.