ORIGINAL PAPER

SOME NOTES ON THE LAPLACIAN ENERGY OF EXTENDED ADJACENCY MATRIX

GULISTAN KAYA GOK[1], SERIFE BUYUKKOSE[2]

Manuscript received: 12.03.2020; Accepted paper: 29.06.2020;
Published online: 30.09.2020.

Abstract. *The extended adjacency matrix $A_e(G)$ of a graph G is determined by graph degrees. Some inequalities are found for the extended adjacency matrix including its vertices, its edges and its degrees in this paper. Also, some bounds are established for this matrix involving its energy and its laplacian energy.*

Keywords: *extended adjacency matrix; Laplacian energy; bounds.*

1. INTRODUCTION

Let *G* be a simple, connected graph on the vertex set *V(G)* and the edge set *E(G).* For $v_i \in V(G)$, the degree of the vertex v_i denoted by d_i, the maximum degree is denoted by Δ and the minimum degree is denoted by δ. If v_i and v_j are adjacent, then it is represented by $v_i \sim v_j$.

The *adjacency matrix* is a symmetric square matrix that determines the corner pairs in a graph. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ denote the eigenvalues of adjacency matrix. The greatest eigenvalue λ_1 is said to as the *spectral radius* of the graph *G*. The *energy* of graph *G* is defined as $E(\mathrm{G}) = \sum_{i=1}^{n} |\lambda_i|$. The *Laplacian matrix* of a graph G is represented by $L(G) = D(G) - A(G)$ where $D(G)$ is the degree matrix. The degree matrix is the diagonal matrix formed by the degree of each point belonging to G. The Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n$ are real. The graph *laplacian energy* $LE(\mathrm{G})$ is described by $LE = LE(\mathrm{G}) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$ with m edges and n vertices.

The *extended adjacency matrix* is defined as a new function in the study of Graph Theory. This symmetric matrix is represented by $A_e(G)$ and $A_e(G) = (a_{ij})$ is described as the $n \times n$ matrix such that,

$$a_{ij} = \begin{cases} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right); if\ i \sim j \\ 0 \qquad ; otherwise. \end{cases}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the eigenvalues of $A_e(G)$ and also, let α_1 be the largest eigenvalue of $A_e(G)$. Since the eigenvalues of $A_e(G)$ are real then $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. The *extended graph energy* $E_e(\mathrm{G})$ is introduced by $E_e(\mathrm{G}) = \sum_{i=1}^{n} |\alpha_i|$. For details of the mathematical theory of this subject see [1-6].

[1] Hakkari University, Department of Mathematics Education, 30000 Hakkari, Turkey.
E-mail: gulistankayagok@hakkari.edu.tr.
[2] Gazi University, Department of Mathematics, 06010 Ankara, Turkey. E-mail: sbuyukkose@gazi.edu.tr.

The narrative of this paper is as the following: In the second section, known and related studies are focused. In Section 3.1, some bounds are obtained for the extended adjacency matrix including its degrees and its edges. Also, some conclusions are given for the complement of $A_e(G)$ and subgraph of $A_e(G)$. In Section 3.2, the energy of extended adjacency matrix is stated with some relations and lemmas. In addition, some inequalities are pointed out for the laplacian energy of extended adjacency matrix using its vertices, its edges and its eigenvalues. Different equalities are expanded for the norm of extended adjacency matrix of some special graphs.

2. PRELIMINARIES

In this section, *extended graph laplacian matrix* is defined and some known results are stated that are needed in the next section.

The motivation of *extended graph laplacian matrix* comes from *extended graph adjacency matrix*. The *extended graph laplacian matrix* is qualified by $L_e(\mathrm{G}) = D_e(G) - A_e(G)$ *where* $D_e(G)$ *is the extended degree matrix. Let* $\rho_1, \rho_2, \dots, \rho_{n-1}, \rho_n$ be eigenvalues of *extended graph laplacian matrix*. These eigenvalues are real. The elementary properties of *eigenvalues of extended laplacian matrix* provide: $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \rho_i = 2m$, $\sum_{i=1}^{n} \rho_i^2 \geq 2m + \sum_{i=1}^{n} d_i^2 =$ $i{=}1n\mu i2.$ Also, $i{=}1n\mu i2{=}2m{+}Z1(\mathrm{G})$ such that $Z1(\mathrm{G})$ is the first Zagreb index.

The *extended graph laplacian energy* $LE_e(\mathrm{G})$ is described by $LE_e = LE_e(\mathrm{G}) = \sum_{i=1}^{n} \left| \rho_i - \frac{2m}{n} \right|$.

Lemma 2.1. [7] Let $M = (m_{ij})$ be an $\mathrm{n} \times \mathrm{n}$ irreducible nonnegative matrix and $R_1(\mathrm{M}) = \sum_{j=1}^{m} m_{ij}$. Then,

$$(minR_1(\mathrm{M}): 1 \leq \mathrm{i} \leq \mathrm{n}) \leq \lambda_1(\mathrm{M}) \leq (maxR_1(\mathrm{M}): 1 \leq \mathrm{i} \leq \mathrm{n}).$$

Lemma 2.2. [2] If *G* is a simple, connected graph and $\lambda_1(\mathrm{G})$ is the spectral radius then

$$\lambda_1(\mathrm{G}) \leq \left(\sqrt{m_i m_j}: 1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}, v_i, v_j \in E\right).$$

Lemma 2.3. [8] Let *M* be $\mathrm{s} \times \mathrm{s}$ symmetric matrix and let M_l be its leading $l \times l$ submatrix. Then,

$$\sigma_{s-l+1}(\mathrm{B}) \leq \sigma_{l-i+1}(B_k) \leq \sigma_{l-i+1}(\mathrm{B}).$$

where $\sigma_i(\mathrm{B})$ is the $i-th$ largest eigenvalue of *B* for $i = 1, 2, \dots, l$.

Lemma 2.4. [9] Let *G* has a diameter D. Then,

$$\lambda_1 \geq (n-1)^{\frac{1}{D}}.$$

3. MAIN RESULTS

3.1. ON THE EIGENVALUES

In this subsection, we deal with some bounds on $\alpha_1(\mathrm{G})$ for the extended adjacencey matrix in terms of the vertices, the edges and the degrees. Indeed, we examine some conclusions for a complement of graph and a subgraph.

Theorem 3.1.1. If G is a simple, connected graph and α_1 is the largest eigenvalue of $A_e(G)$, then

$$\alpha_1 \leq \frac{1}{2}\sqrt{\left(\frac{4e^2}{d_1^2}+\Delta\right)\left(\frac{4e^2}{d_1^2}+\delta\right)}.$$

Proof: Let $D(\mathrm{G})^{-1}A_e(G)D(\mathrm{G}) = M(\mathrm{G})$ and let $X = (x_1, x_2, \dots, x_n)^T$ be an eigenvector of $M(\mathrm{G})$. Also, $x_i = 1$ and $0 < x_i \leq 1$ for every k, $x_j = max_k(x_k: v_i, v_k \in E, i \sim k)$. Then,

$$M(\mathrm{G}) = \begin{cases} \frac{1}{2}\left(\frac{{d_i}^2+{d_j}^2}{d_i d_j}\right); if i \sim j \\ 0 \qquad ; otherwise. \end{cases}$$

It is known that $(M(\mathrm{G}))X = \alpha_1(\mathrm{G})\mathrm{X}$ from the above definition. It is obtained by necessary calculations that $(M(\mathrm{G}))X = \frac{1}{2}\sum_k\left(\left(\frac{d_k}{d_1}\right)^2+1\right)x_k$. By Lemma 2.1, $\alpha_1 x_i = \frac{1}{2}\sum_k\left(\left(\frac{d_k}{d_1}\right)^2+1\right)x_k$. It is easy to see that $\alpha_1 \leq \frac{1}{2}\left(\frac{4e^2}{d_1^2}+d_i\right)$. From the jth equation of the same equation, $\alpha_1 \leq \frac{1}{2}\left(\frac{4e^2}{d_1^2}+d_j\right)$. Multiplying and by taking the square root the $i-th$ and the *j-th* inequality, $\alpha_1 \leq \frac{1}{2}\sqrt{\left(\frac{4e^2}{d_1^2}+d_i\right)\left(\frac{4e^2}{d_1^2}+d_j\right)}$. (By the help of Lemma 2.2) Since $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, then $\alpha_1 \leq \frac{1}{2}\sqrt{\left(\frac{4e^2}{d_1^2}+\Delta\right)\left(\frac{4e^2}{d_1^2}+\delta\right)}$.

Theorem 3.1.2. If G is a simple, connected graph with the degree d, then

$$\alpha_1 \leq \frac{1}{2}\sqrt{\left(\frac{n}{d_n}d_i+m_i\right)\left(\frac{n}{d_n}d_j+m_j\right)}$$

where m_i is the average degree of *G*.

Proof: Similarly Theorem 3.1.1, $(M(\mathrm{G}))X = \alpha_1\mathrm{X}$ is obtained.

$$\begin{aligned}\alpha_1 x_i &= \frac{1}{2}\sum_k\left(\frac{d_i}{d_k}+\frac{d_k}{d_i}\right)x_k \\ &= \left(\frac{1}{2}d_i\sum_k\frac{1}{d_k}+\frac{1}{2}\sum_k\frac{d_k}{d_i}\right)x_k.\end{aligned}$$

Each of d_k's can be taken as d_n in the above equation. Also, $\sum_k \frac{d_k}{d_i} = m_i$. It follows that, $\alpha_1 x_i = \frac{1}{2}\left(\frac{nd_i}{d_n} + m_i\right) x_k$. Considering the $j-th$ equation of the above equation, it gets $\alpha_1 \leq \frac{1}{2}\sqrt{\left(\frac{n}{d_n} d_i + m_i\right)\left(\frac{n}{d_n} d_j + m_j\right)}$.

Corollary 3.1.3. Let $A_e(G')$ be an extended adjacency matrix of connected subgraph of graph G with the minimum degree δ. Then,

$$\frac{2(m-\delta)}{n-1} \leq \alpha_1(G') \leq \frac{1}{2}\sqrt{\left(\frac{(n-1)d_i}{d_{n-1}} + m_i - \delta\right)\left(\frac{(n-1)d_j}{d_{n-1}} + m_j - \delta\right)}.$$

Proof: It is known that $\alpha_1(G) \geq \alpha_1(G')$. Thus, $\frac{2m'}{n'} \geq \frac{2(m-\delta)}{n-1}$ where $\alpha_1(G') = \alpha_1(A_{(n-1)}), m', n' \in G'$. By the Theorem 3.1.2, it has

$$\alpha_1(G') \leq \frac{1}{2}\sqrt{\left(\frac{(n-1)d_i}{d_{n-1}} + m_i - \delta\right)\left(\frac{(n-1)d_j}{d_{n-1}} + m_j - \delta\right)}.$$

Hence,

$$\frac{2(m-\delta)}{n-1} \leq \alpha_1(G') \leq \frac{1}{2}\sqrt{\left(\frac{(n-1)d_i}{d_{n-1}} + m_i - \delta\right)\left(\frac{(n-1)d_j}{d_{n-1}} + m_j - \delta\right)}.$$

Theorem 3.1.4. Let G be a graph with *n* nodes, then $\alpha_1 \leq \frac{1}{2}\left(\frac{n}{d_n} + 1\right)$.

Proof: Similarly Theorem 3.1.1, $\alpha_1 x_i = \sum_k \frac{1}{2}\left(\frac{d_i^2 + d_k^2}{d_i d_k}\right) x_k$. By the Cauchy Schwarz inequality,

$$\alpha_1 x_i \leq \frac{1}{2}\sum_k \frac{(d_i + d_k)^2}{d_i d_k} x_k$$

$$\leq \frac{1}{2}\sum_k \frac{(d_i + d_k)}{d_i d_k} x_k$$

$$\leq \frac{1}{2}\left(\sum_k \frac{1}{d_k}\right) + \frac{1}{d_i}\left(\sum_k 1\right) x_k.$$

Each of d_k's can be taken as d_n repeatly in in the above inequality. It is seen that

$$\alpha_1 x_i = \frac{1}{2}\left(\frac{n}{d_n} + \frac{1}{d_i} d_i\right) = \frac{1}{2}\left(\frac{n}{d_n} + 1\right).$$

By the Lemma 2.2, $\alpha_1 \leq \frac{1}{2}\left(\frac{n}{d_n} + 1\right)$.

Corollary 3.1.5. If G and $\overline{\mathrm{G}}$ are connected non-singular graphs of $A(\mathrm{G})$, then

$$n-1 \leq \alpha_1(\mathrm{G}) + \alpha_1(\overline{\mathrm{G}}) \leq \frac{n}{2}\left(\frac{n-1}{d_n(n-1-d_n)}\right) + 1.$$

where $\overline{\mathrm{G}}$ be a complement of graph G.

Proof: Applying the Theorem 3.1.4, it is seen that

$$\alpha_1(\mathrm{G}) + \alpha_1(\overline{\mathrm{G}}) \leq \frac{1}{2}\left(\frac{n}{d_n} + 1\right) + \frac{1}{2}\left(\frac{n}{n-1-d_n} + 1\right).$$

Thus, the inequality has

$$\alpha_1(\mathrm{G}) + \alpha_1(\overline{\mathrm{G}}) \leq \frac{1}{2}\left(\frac{n}{d_n} + \frac{n}{n-1-d_n} + 2\right).$$

On the other hand,

$$\alpha_1(\mathrm{G}) + \alpha_1(\overline{\mathrm{G}}) \geq \frac{2m}{n} + \frac{2\overline{\mathrm{m}}}{n}.$$

Since $m + \overline{\mathrm{m}} = \frac{1}{2}\big(n(n-1)\big)$, then $\alpha_1(\mathrm{G}) + \alpha_1(\overline{\mathrm{G}}) \geq \ n - 1$. Hence,

$$n - 1 \leq \alpha_1(\mathrm{G}) + \alpha_1(\overline{\mathrm{G}}) \leq \frac{n}{2}\left(\frac{n-1}{d_n(n-1-d_n)}\right) + 1.$$

3.2. THE ENERGY AND LAPLACIAN ENERGY FOR EXTENDED ADJACENCY MATRIX

In this subsection, some inequalities are given for the energy $E_e(G)$ and some relations are obtained on laplacian energy $L_e(G)$. In addition, some expressions are represented for the norm of extended adjacency matrix of some special graphs by the direct computations.

Theorem 3.2.1. Let G be a connected graph and $trLE_e(\mathrm{G})$ be a trace of extended Laplacian matrix $L_e(G)$. Then,

$$E_e(G) \leq \sqrt{trLE_e(G) - l} + \Delta.$$

where $l = (n-1)^{\frac{1}{D}}$ and D is a diameter.

Proof: It is known that $E(G) = \sum_{i=1}^{n} \lambda_i^2 = 2m$. Thus, $\sum_{i=1}^{n} \alpha_i^2 = 2m$. If necessary, calculations are done, $traceLE_e(\mathrm{G}) = 2m$ is obtained. Hence,

$$trLE_e(\mathrm{G}) = \alpha_1^2 + \sum_{i=2}^{n} \alpha_i^2.$$

$$trLE_e(\mathrm{G}) - \alpha_1^2 = \sum_{i=2}^{n} \alpha_i^2.$$

It follows that $\sqrt{trLE_e(\mathrm{G}) - \alpha_1^2} \geq \sum_{i=2}^{n} |\alpha_i|$. Hence, $\sqrt{trLE_e(\mathrm{G}) - \alpha_1^2} + |\alpha_1| \geq E_e(G)$. Applying Lemma 2.4, the energy gets $E_e(G) \leq \sqrt{trLE_e(\mathrm{G}) - l} + |\alpha_1|$. Since $|\alpha_1| < \Delta$ then,

$$E_e(G) \leq \sqrt{trLE_e(\mathrm{G}) - l} + \Delta.$$

Theorem 3.2.2. Let G be a graph of order n with m edges. Then,

$$E_e^1(\mathrm{G}) + E_e^2(\mathrm{G}) \leq 2\Delta + \sqrt{(n-1)\left(4m - \frac{8m^2}{n^2} + 4\sqrt{m - \frac{m}{n}}\right)}.$$

where E_e^1 and E_e^2 are the energies of two extended adjacency matrix of A_e^1 and A_e^2 in G, respectively.

Proof: Let α_i and α_j be eigenvalues of A_e^1 and A_e^2, respectively. Since $\mathrm{a}_1, \mathrm{b}_1 \geq \frac{2m}{n}$ it has

$$\begin{aligned}\sum_{i=2\, j=2}^{n} \left(\alpha_i + \alpha_j\right)^2 &\leq \sum_{i=2}^{n} \alpha_i^2 + \sum_{j=2}^{n} \alpha_j^2 + 2\sqrt{\sum_{i=2}^{n} \alpha_i^2 \sum_{j=2}^{n} \alpha_j^2} \\ &= 2m - \alpha_1^2 + 2m - \alpha_1^2 + \sqrt{(2m - \alpha_1^2)(2m - \alpha_1^2)} \\ &= 4m - \tfrac{8m^2}{n^2} + 4\sqrt{m - \tfrac{m}{n}}.\end{aligned}$$

Since $\alpha_1 \leq \Delta$, then

$$\begin{aligned}E_e^1(G) + E_e^2(G) &= |\alpha_1| + |\alpha_1| + \sqrt{(n-1)\sum_{i=2,j=2}^{n} \left(|\alpha_i| + |\alpha_j|\right)^2}. \\ &\leq 2\Delta + \sqrt{(n-1)\textstyle\sum_{i=2,j=2}^{n}\left(|\alpha_i| + |\alpha_j|\right)^2}.\end{aligned}$$

It follows that

$$E_e^1(G) + E_e^2(G) \leq 2\Delta + \sqrt{(n-1)\left(4m - \frac{8m^2}{n^2} + 4\sqrt{m - \frac{m}{n}}\right)}.$$

Theorem 3.2.3. Let $LE_e(\mathrm{G})$ be a Laplacian energy of $A_e(\mathrm{G})$. Then, for $1 \leq k \leq n$

$$LE_e(G) \geq 2max \sum_{i=1}^{k} \left(\rho_i - \frac{2m}{n}\right).$$

Proof: It is known that $LE_e(G) = \sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right|$. Hence,

$$\begin{aligned}LE_e(G) &= \sum_{i=1}^{s} \left(\rho_i - \frac{2m}{n}\right) + \sum_{i=s+1}^{n} -\left(\rho_i - \frac{2m}{n}\right) \\ &= \sum_{i=1}^{s} \rho_i - \frac{2m}{n} s - \sum_{i=s+1}^{n} \rho_i + \frac{2m}{n}(n-s) \\ &= \textstyle\sum_{i=1}^{s} \rho_i - \sum_{i=s+1}^{n} \rho_i - \frac{4ms}{n} + 2m.\end{aligned}$$

Since $\sum_{i=s+1}^{n} \rho_i = 2m - \sum_{i=1}^{s} \rho_i$, then

$$LE_e(G) = \sum_{i=1}^{s} \rho_i - \left(2m - \sum_{i=1}^{s} \rho_i\right) - \frac{4ms}{n} + 2m$$
$$= 2\left(\sum_{i=1}^{s} \left(\rho_i - \frac{2m}{n}\right)\right).$$

It will be shown that $\sum_{i=1}^{s}\left(\rho_i - \frac{2m}{n}\right) = max \sum_{i=1}^{k}\left(\rho_i - \frac{2m}{n}\right), 1 \le k \le n$ that is; $\sum_{i=1}^{k}\left(\rho_i - \frac{2m}{n}\right) \le \sum_{i=1}^{s}\left(\rho_i - \frac{2m}{n}\right)$ for any k, $k = 1,2,\dots,n$. If $k = s$ then the above equality holds. If $k \ne s$ then two cases are considered:

Case of $k < s$: Since $\sum_{i=1}^{s}\left(\rho_i - \frac{2m}{n}\right) \ge 0$ and $\sum_{i=1}^{s}\left(\rho_i - \frac{2m}{n}\right) = \sum_{i=1}^{k}\left(\rho_i - \frac{2m}{n}\right) + \sum_{i=k+1}^{s}\left(\rho_i - \frac{2m}{n}\right)$, then the equality holds.

Case of $k > s$: Since $\sum_{i=1}^{s}\left(\rho_i - \frac{2m}{n}\right) < 0$ and $\sum_{i=1}^{k}\left(\rho_i - \frac{2m}{n}\right) = \sum_{i=1}^{s}\left(\rho_i - \frac{2m}{n}\right) + \sum_{i=s+1}^{k}\left(\rho_i -\right.$ *2mn<i=1sρi−2mn*, then the equality holds. Hence,

$$LE_e(G) = \sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right| = 2\left(\sum_{i=1}^{s} \left(\rho_i - \frac{2m}{n}\right)\right)$$
$$= 2max \sum_{i=1}^{k} \left(\rho_i - \frac{2m}{n}\right) : 1 \le k \le n.$$

Theorem 3.2.4. Let G be a graph with defining relations. Then,

$$LE_e(G) + LE_e(\bar{G}) \le 4\left(\frac{n^2(n-1)^2}{4} - 2n^2m - nm - 2m^2\right)\left(\frac{1-n}{n^2}\right)$$

where $\bar{G}$ is a complement of G.

Proof: The Cauchy Schwarz inequality becomes that

$$\left(LE_e(G) + LE_e(\bar{G})\right)^2 \le \left(\sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right|\right)^2 + \left(\sum_{i=1}^{n} \left|\bar{\rho}_i - \frac{2\bar{m}}{n}\right|\right)^2$$
$$\le \sum_{i=1}^{n} \rho_i^2 - 2\sum_{i=1}^{n} \rho_i \frac{2m}{n} + \frac{4m^2}{n^2} + \sum_{i=1}^{n} \bar{\rho}_i^2 - 2\sum_{i=1}^{n} \bar{\rho}_i \frac{2\bar{m}}{n} + \frac{4\bar{m}^2}{n^2}.$$

Also, it represent that

$$\left(LE_e(G) + LE_e(\bar{G})\right)^2 \le \sqrt{4\left(\frac{n^2(n-1)^2}{4} - 2n^2m - nm - 2m^2\right)\left(\frac{1-n}{n^2}\right)}.$$

Hence,

$$LE_e(G) + LE_e(\bar{G}) \leq 4\left(\frac{n^2(n-1)^2}{4} - 2n^2m - nm - 2m^2\right)\left(\frac{1-n}{n^2}\right).$$

Corollary 3.2.5. Let K_n, P_n, $C_n$and $S_n$represent the complete graph, path, cycle and star graph, respectively. For $n > 2$, the following items provide

1) $\|A(P_n)\| = \frac{9}{4}$,
2) $\|A(K_n)\| = n - 1$,
3) $\|A(C_n)\| = 2$,
4) $\|A(S_n)\| = \frac{1}{2}\left(n^2 - n + 1 - \frac{1}{n}\right)$.

4. CONCLUSION

This paper focuses on the extended adjacency matrix of graphs. Firstly, different bounds are obtained for the eigenvalues of extended adjacency matrix. Then, the energy and laplacian energy are studied by the help of defining relations.

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