ORIGINAL PAPER

N-BISHOP DARBOUX VECTOR OF A TIMELIKE CURVE IN MINKOWSKI 3-SPACE

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Abstract. *It is known that a Bishop frame of a curve is one of the effective alternative approach in the differential geometry. Recently, several important works have been done about the Bishop frames. The aim of our paper is to investigate the N-Bishop frame for timelike curves in Minkowski space. We define the N-Bishop frame for the timelike curve in Minkowski space. Then, we consider some properties of the frame. Moreover, we describe the N-Bishop Darboux frame for the first time. Additionally, we compute some geometrical characterizations for the N-Bishop Darboux axis and momentum rotation vector.*
Keywords: *N-Bishop; timelike curve; Minkowski 3-space.*

1. INTRODUCTION

Bishop frame was developed by L. Bishop in 1975 in order to built a more practical alternative parallel frame without using the second derivative on the curve [1]. Recently, many studies have begun to be done in order to investigate the invariants of the curve via Bishop frame. The Bishop frame has been investigated by many researchers for various special curves such as involute evolute curves, Bertrand curves, helix, slant helix until now. Particularly, Bükcü and his colleagues examined the Bishop Darboux axis of a special Bishop movement and slant helix due to the Bishop frame [2]. Yücesan et al. studied on the Darboux rotation axis of Lorentz space curves [3]. Afterwards, Karacan and Bükcü computed the Bishop frame and the Bishop Darboux rotation axes for the spacelike curves in Minkowski 3-space [4-7]. Moreover, Yılmaz and his colleagues created a new Bishop alternative frame included the binormial vector instead of tangent vector [8]. S. Yılmaz and et. al. investigated new spherical indicators and some characterizations according to the new type Bishop frame [9]. In 2015, Yilmaz and Ünlütürk examined the characterizations of spacelike curves around Minkowski with the new Bishop frame [10]. Much work has been done on the Bishop frame as defined by Bishop and Yılmaz. Furthermore, a new frame called N-C-W frame was created by Scofield provided some different approach for slant helis by Uzunoğlu [11,12]. Then a new Bishop frame called N-Bishop frame was introduced by Keskin. et al. with using the frame methods in [13].

In our paper, timelike curves were investigated due to N-Bishop frame in Minkowski 3-space. Firstly, the N-Bishop frame for the timelike curves were introduced and then the transition matrix of the N-Bishop frame were obtained. Later, the properties of these curves related to normal expansions were examined. Finally, the N-Bishop Darboux frame was firstly defined and some theoretical properties were presented in the last part of the paper.

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2. MATERIALS AND METHODS

Minkowski space-time model was developed by H. Minkowski in 1907. He realized that there could be a 4-component space-time that combined time with three spatial dimensions. Minkowski space is introduced by the metric $g(\eta,\mu)=\eta_1\mu_1-\eta_2\mu_2+\eta_3\mu_3$ and denoted by $\mathbb{R}_1^3=\left(\mathbb{R}^3, g(,)\right)$ for $\eta=(\eta_1,\eta_2,\eta_3)$, $\mu=(\mu_1,\mu_2,\mu_3)$. The metric $g(,)$ is called the Lorentzian metric, which is a non-degenerate metric of index 1. There are three different vector type in Minkowski space. A vector $\eta\in\mathbb{R}_1^3$ is called spacelike, if $g(\eta,\eta)>0$ or $\eta=0$. It is called timelike, if $g(\eta,\eta)<0$; it is called a lightlike vector, if $g(\eta,\eta)=0$ and $\eta\neq 0$. The vectoral product in Minkowski space is denoted by $\times_L$. Moreover, the norm in Minkowski space is defined by $\|\eta\|_L=\sqrt{|g(\eta,\eta)|}$. The vectoral product of timelike curves is described by

$$\eta\times_L\mu=\left(\eta_2\mu_3-\eta_3\mu_2,\eta_1\mu_3-\eta_3\mu_1,\eta_1\mu_2-\eta_2\mu_1\right) \tag{1}$$

for the vectors $\eta,\mu\in\mathbb{R}_1^3$. If the unit vectors e_1,e_2,e_3 are spacelike, timelike, spacelike, respectively, then the vectoral product provides the equations

$$e_1\times_L e_2=e_3,\quad e_2\times_L e_3=e_1,\quad e_3\times_L e_1=-e_2\,. \tag{2}$$

More basic features about Minkowski-3 space should be found in [14, 15]. It is known that the derivative formula of Serret-Frenet frame in the matrix form can be written by

$$\begin{bmatrix} T'(s)\\ N'(s)\\ B'(s)\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\-\kappa&0&\tau\\0&-\tau&0\end{bmatrix}\begin{bmatrix}T(s)\\N(s)\\B(s)\end{bmatrix}.$$

where κ and τ are curvatures of the Serret-Frenet frame in Euclidean 3-space [15].

Furthermore, in 1975 Bishop is described a new alternative frame which is relatively parallel the unit tangent field T. The differentiation matrix of the Bishop frame is given by

$$\begin{bmatrix} T'(s)\\ N_1'(s)\\ N_2'(s)\end{bmatrix}=\begin{bmatrix}0&k_1&k_2\\-k_1&0&0\\-k_2&0&0\end{bmatrix}\begin{bmatrix}T(s)\\N_1(s)\\N_2(s)\end{bmatrix}$$

where k_1 and k_2 are curvatures of the Bishop frame in Euclidean 3-space [1]. Bukcu et al. defined the Bishop frames of timelike and spacelike curves [2, 4-7]. Moreover, in 2010, Yilmaz et al. developed a new alternative Bishop frame called type-2 Bishop Frame which is parallel to binormial vector field, and its derivativation matrix is described by

$$\begin{bmatrix} N_1'(s)\\ N_2'(s)\\ B'(s)\end{bmatrix}=\begin{bmatrix}0&0&-k_1\\0&0&-k_2\\k_1&k_2&0\end{bmatrix}\begin{bmatrix}N_1(s)\\N_2(s)\\B(s)\end{bmatrix}$$

where k_1 and k_2 are curvatures, [8]. Additionally, another new frame $\{N, C, N\times C = W\}$ was defined by P.D. Scofield et. al. in 2016 where $N, C = \dfrac{N'}{\|N'\|}$ and $W = \dfrac{\tau T + \kappa B}{\|\kappa^2 + \tau^2\|}$. The derivative matrix of the alternative moving frame is given with

$$\begin{bmatrix} N'(s) \\ C'(s) \\ W'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ C(s) \\ W(s) \end{bmatrix}$$

where $f = \sqrt{\kappa^2 + \tau^2}$ and $g = \dfrac{\kappa^2 \left(\dfrac{\tau}{\kappa}\right)'}{\kappa^2 + \tau^2} = \sigma f$ are the differentiable functions [11, 12].

In 2017, Keskin and Yaylı defined N-Bishop frame for a normal direction curve. The differentiation matrix of the N-Bishop frame [13] is computed by

$$\begin{bmatrix} N'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(s) \\ N_1(s) \\ N_2(s) \end{bmatrix},$$

3. RESULTS AND DISCUSSION

Now we will introduce the N-Bishop frame for a timelike curve and give a new definition called N-Bishop Darboux vector in Minkowski 3-space. Furhermore we will present some geometric properties of these new constructions.

If the curve χ is a timelike curve, then the tangent vector T is timelike and the principal normal and binormal vectors N, B are spacelike, i.e., $\langle T, T\rangle = -1,\ \langle N, N\rangle = 1,\ \langle B, B\rangle = 1.$ For the timelike curve, the orthonormal vectors of the N-Bishop frame N, N_1, N_2 are taken spacelike, timelike, spacelike, respectively. In this case, because of the inner and outer products, the properties

$$\begin{gathered} \langle N, N\rangle = 1,\ \langle N_1, N_1\rangle = -1,\ \langle N_2, N_2\rangle = 1, \\ N \times_L N_1 = N_2,\ N_1 \times_L N_2 = N,\ N_2 \times_L N = -N_1 \end{gathered}$$

are satisfied in Minkowski 3-space.

Theorem 3.1. Let the curve χ be a timelike curve with unit speed in Minkowski 3-space. If the frame $\{N, N_1, N_2\}$ is adapted N-Bishop frame, then we get

$$\begin{bmatrix} N' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & p_{01} & -p_{02} \\ p_{01} & 0 & -p_{12} \\ p_{02} & -p_{12} & 0 \end{bmatrix} \begin{bmatrix} N \\ N_1 \\ N_2 \end{bmatrix},$$

for some functions $p_{ij},\, i, j = 0,1,2.$

Proof: When the curve χ is a timelike curve, N_1 is timelike, N and N_2 are spacelike. The inner products of the vectors can be denoted by

$$\begin{array}{lll} p_{00} = \langle N', N \rangle = 0, & p_{10} = \langle N_1', N \rangle, & p_{20} = \langle N_2', N \rangle, \\ -p_{01} = \langle N', N_1 \rangle, & -p_{11} = \langle N_1', N_1 \rangle = 0, & -p_{21} = \langle N_2', N_1 \rangle, \\ p_{02} = \langle N', N_2 \rangle, & p_{12} = \langle N_1', N_2 \rangle, & p_{22} = \langle N_2', N_2 \rangle = 0. \end{array}$$

for some functions $p_{ij},\, i, j = 0,1,2.$ Then we get

$$\begin{array}{lll} -p_{00} = \langle N', N \rangle = 0, & -p_{10} = \langle N_1', N \rangle, & -p_{20} = \langle N_2', N \rangle, \\ p_{01} = \langle N', N_1 \rangle, & p_{11} = \langle N_1', N_1 \rangle = 0, & p_{21} = \langle N_2', N_1 \rangle, \\ p_{02} = \langle N', N_2 \rangle, & p_{12} = \langle N_1', N_2 \rangle, & p_{22} = \langle N_2', N_2 \rangle = 0. \end{array}$$

The derivative of the inner product $\langle N, N_1 \rangle = 0$, then we obtain

$$\begin{aligned} &\langle N', N_1 \rangle + \langle N, N_1' \rangle = 0 \\ &-p_{01} + p_{10} = 0 \\ &p_{01} = p_{10}. \end{aligned}$$

Also taking the derivative of the inner product $\langle N, N_2 \rangle = 0$, we compute

$$\begin{aligned} &\langle N', N_2 \rangle + \langle N, N_2' \rangle = 0 \\ &p_{02} + p_{20} = 0 \\ &p_{02} = -p_{20}. \end{aligned}$$

The derivation of $\langle N_1, N_2 \rangle = 0$ is

$$\begin{aligned} &\langle N_1', N_2 \rangle + \langle N_1, N_2' \rangle = 0 \\ &p_{12} + (-p_{21}) = 0 \\ &p_{12} = p_{21}. \end{aligned}$$

Consequently the derivation matrix of the adapted N-Bishop frame $\{N, N_1, N_2\}$ is calculated by

$$\begin{bmatrix} N' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & p_{01} & -p_{02} \\ p_{01} & 0 & -p_{12} \\ p_{02} & -p_{12} & 0 \end{bmatrix} \begin{bmatrix} N \\ N_1 \\ N_2 \end{bmatrix}$$

Theorem 3.2. Let χ be a time-like curve with unit speed. Let $\{N, C, W\}$ and $\{N, N_1, N_2\}$ be alternative frame and Bishop frame. The relation between them is

$$\begin{bmatrix} N \\ C \\ W \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N \end{bmatrix}.$$

Proof: The tangent vector according to frame $\{N, N_1, N_2\}$ is defined by

$$W = \sinh\theta N_1 + \cosh\theta N_2 .$$

Differentiating the above equation with respect to s yields

$$W' = -g.C = (\theta' - p_{12})(\cosh\theta N_1 + \sinh\theta N_2) + (\sinh\theta p_{01} + \cosh\theta p_{02})N . \tag{3}$$

The Eq. (3) indicates that M is relatively parallel necessary and sufficient condition is $\theta' - p_{12} = 0$. In the Eq.(3) , let us take $\theta' = -g$ and $C = \cosh\theta N_1 + \sinh\theta N_2$. In addition, we can take the coefficients as $p_{01} = k_1 = f\cosh\theta$ and $p_{02} = k_2 = -f\sinh\theta$. Also the equations $f = \|N'\|_L = \sqrt{\kappa^2 + \tau^2} = \sqrt{\left|k_2^2 - k_1^2\right|}$ and $\theta = -\int_{s_0}^{s} g(t) = -\tanh\left(\dfrac{k_2}{k_1}\right)$ are satisfied.

Theorem 3.3. Let the curve χ be a C^k timelike curve in Minkowski 3-space which is regular; that is the velocity never vanishes $k \geq 2$. Then for any vector X_0 at $\alpha(t_0)$ there is a unique C^{k-1} relatively parallel field X along α such that $X(t_0) = X_0$ and the scalar product of two relatively is constant.

Proof: To prove that $\langle X, Y\rangle_L$ is constant, we see that it is trivial for the tangential and normal parts. In this way, we suppose X and Y are normal, with derivatives $\lambda_1 N$ and $\lambda_2 N$. Then the derivative of $\langle X, Y\rangle_L$ is

$$\begin{aligned} \frac{d}{dt}\langle X, Y\rangle_L &= \langle X', Y\rangle_L + \langle X, Y'\rangle_L \\ &= \langle \lambda_1 N, Y\rangle_L + \langle X, \lambda_2 N\rangle_L \\ &= \lambda_1 \langle N, Y\rangle_L + \lambda_2 \langle X, N\rangle_L \\ &= 0 \end{aligned}$$

where $\langle X, Y\rangle_L$ is constant. As a result, the derivative matrix of the N-Bishop frame for the timelike curve in Minkowski 3-space

$$\begin{bmatrix} N' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} N \\ N_1 \\ N_2 \end{bmatrix}$$

where $f = \|N'\|_L = \sqrt{\kappa^2+\tau^2} = \sqrt{\left|k_2^2-k_1^2\right|}$ and $\theta = -\int_{s_0}^{s} g(t) = -\tanh\left(\dfrac{k_2}{k_1}\right)$.

Theorem 3.4. Let $\chi = \chi(s)$ be a regular timelike curve and k_1, k_2 be N-Bishop curvatures of this curve in Minkowski-3 space. The normal direction curve $\int Nds$ is spherical curve if and only if $1-\lambda_1 k_1 - \lambda_2 k_2 = 0$ where λ_1 and λ_2 are constant.

Proof: $(\Rightarrow)$ Taking the curve γ as spherical curve $\gamma(s) = \int N(s)ds$, then we have $\langle \gamma - P, \gamma - P\rangle = r^2$ where P is a point on the sphere S^2. If we take the derivation of the both sides, we get

$$\langle \gamma - P, \gamma - P\rangle = r^2$$
$$\left\langle (\gamma - P)', \gamma - P\right\rangle + \left\langle \gamma - P, (\gamma - P)'\right\rangle = 0$$
$$\langle N, \gamma - P\rangle = 0$$

where λ_1 and λ_2 are constant. Then it is easy to see that $\gamma - P = \lambda_1 N_1 + \lambda_2 N_2$. Thus N, N_2 is spacelike vectors and N_1 is timelike $\lambda_1 = \langle \gamma - P, N_1\rangle$ and $\lambda_2 = \langle \gamma - P, N_2\rangle$ are obtained. By derivating λ_1 and λ_2, we have

$$\begin{aligned} \lambda_1' &= \langle \gamma - P, N_1\rangle' \\ &= \langle N, N_1\rangle + k_1\langle \gamma - P, N\rangle \\ &= k_1 \langle \gamma - P, N\rangle \end{aligned} \quad \text{and} \quad \begin{aligned} \lambda_2' &= \langle \gamma - P, N_2\rangle' \\ &= \langle N, N_2\rangle + k_2\langle \gamma - P, N\rangle \\ &= k_2 \langle \gamma - P, N\rangle. \end{aligned}$$

Furthermore, derivating $\langle \gamma - P, N\rangle$, we get

$$\begin{aligned} \langle \gamma - P, N\rangle' &= 1 + \langle \lambda_1 N_1 + \lambda_2 N_2, k_1 N_1 - k_2 N_2\rangle \\ &= 1 - \lambda_1 k_1 - \lambda_2 k_2. \end{aligned}$$

In that case, we obtain $1-\lambda_1 k_1 - \lambda_2 k_2 = 0.$

$(\Leftarrow)$ If (k_1, k_2) is on the line, the equation $1-\lambda_1 k_1 - \lambda_2 k_2 = 0$ holds where λ_1 and λ_2 are constant. The vector can be written as

$$\overrightarrow{p\gamma} = \lambda_1 N_1 + \lambda_2 N_2$$
$$\begin{aligned} -p' &= -\gamma' + \left(\lambda_1 N_1 + \lambda_2 N_2\right)' \\ &= \left(-1 + \lambda_1 k_1 + \lambda_2 k_2\right) N. \end{aligned}$$

Then we have $1-\lambda_1 k_1-\lambda_2 k_2=0$. Moreover, the equations

$$\|p\gamma\|^2=\langle\gamma-p,\gamma-p\rangle,$$
$$\frac{d}{ds}\langle\gamma-p,\gamma-p\rangle=\langle N,\gamma-p\rangle=0$$

are satisfy. Thus $\langle\gamma-p,\gamma-p\rangle=r^2$ is constant and timelike curve lies on a Lorentzian sphere of radius r and center p.

Definition 3.5. Let the curve $\chi=\chi(\mathrm{s})$ be a regular timelike curve and k_1,k_2 be N-Bishop curvatures of this curve in Minkowski-3 space. The N-Bishop Darboux vector is defined by

$$\omega=N\times_L N'=k_2N_1-k_1N_2.$$

Also the unit vector of the ω is described by $E=\dfrac{\omega}{\|\omega\|}=\dfrac{k_2N_1-k_1N_2}{\sqrt{\left|k_1^2-k_2^2\right|}}$.

Theorem 3.6. If $\chi=\chi(\mathrm{s})$ is a regular timelike curve in Minkowski-3, then the properties hold

a) $N'\times_L N''=\left(k_1'k_2-k_1k_2'\right)N+f^2\omega$

b) $\det\left(N,N',N''\right)=k_1'k_2-k_1k_2'$

c) $\dfrac{\det\left(N,N',N''\right)}{\|N\times N'\|^2}=\dfrac{k_1'k_2-k_2'k_1}{k_1^2+k_2^2}$

where ω is the N-Bishop Darboux vector of a timelike curve, k_1,k_2 are N-Bishop curvatures and also $f=\sqrt{\left|k_2^2-k_1^2\right|}$.

Proof:

a) Using the second derivative formula of $N'=k_1N_1-k_2N_2$, from the derivative matrix of N-Bishop frame, we compute

b)

$$\begin{aligned}N''&=\left(k_1N_1-k_2N_2\right)'\\&=\left(k_1^2-k_2^2\right)N+k_1'N_1-k_2'N_2\\&=f^2N+k_1'N_1-k_2'N_2.\end{aligned}$$

Moreover, the following properties hold $\langle N'',N\rangle=f^2$, $\langle N'',N_1\rangle=-k_1'$, $\langle N'',N_2\rangle=-k_2'$. Substituting the Darboux vector $N'=\omega\times N$ in the following outer product, we get

$$\begin{aligned} N' \times N'' &= (\omega \times N) \times N'' \\ &= \langle N, N'' \rangle \omega - \langle \omega, N'' \rangle N \\ &= (k_1' k_2 - k_1 k_2') N + f^2 \omega. \end{aligned}$$

c) Using the property of the inner product we found the equation

$$\begin{aligned} \langle N, N' \times N'' \rangle &= \langle N, (k_1' k_2 - k_1 k_2') N + f^2 \omega \rangle \\ &= (k_1' k_2 - k_1 k_2') \langle N, N \rangle + f^2 \langle N, \omega \rangle \\ &= k_1' k_2 - k_1 k_2'. \end{aligned}$$

d) From the equality $N' = \omega \times N$, we can write $\omega = N \times N'$ and the norm of ω is obtained by

$$\|N \times N'\|_L = \|\omega\|_L = \sqrt{|k_1^2 - k_2^2|}.$$

Furthermore, the angle of $\theta = -\tanh^{-1}\left(\frac{k_2}{k_1}\right)$ can be also computed by

$$\begin{aligned} \theta' &= -\frac{\left(\frac{k_2}{k_1}\right)'}{1 - \left(\frac{k_2}{k_1}\right)^2} = \frac{k_1' k_2 - k_2' k_1}{k_1^2 - k_2^2} \\ &= -\frac{\langle N, N' \times N'' \rangle}{\|N \times N'\|^2} = \pm \frac{\det(N, N', N'')}{\|N \times N'\|^2} \\ &= \pm g(t). \end{aligned}$$

N-Bishop Darboux rotation can be splited into two rotation motions. The vector N_1 rotates around the vector N_2 with a k_1 angular speed that is

$$\begin{aligned} N_1' &= \omega \times N_1 \\ &= (k_2 N_1 - k_1 N_2) \times N_1 \\ &= k_2 N_1 \times N_1 - k_1 N_2 \times N_1 \\ &= k_1 N \end{aligned} \qquad \text{and} \qquad \begin{aligned} N_2' &= \omega \times N_2 \\ &= (k_2 N_1 - k_1 N_2) \times N_2 \\ &= k_2 N_1 \times N_2 - k_1 N_2 \times N_2 \\ &= k_2 N. \end{aligned}$$

A unit vector $\Upsilon = \frac{\omega}{\|\omega\|}$ rotates with $\theta' = \frac{k_1' k_2 - k_2' k_1}{|k_1^2 - k_2^2|}$. Also the vector N rotates with a $\|\omega\|$ angular speed round $\frac{\omega}{\|\omega\|}$. Bishop Darboux axis, also $N' = \omega \times N$. From now on we shall do a further study of momentum Bishop Darboux axis. We compute the unit vector

$$\Upsilon = \frac{\omega}{\|\omega\|} = \frac{k_2 N_1 - k_1 N_2}{\sqrt{\left|k_1^2 - k_2^2\right|}}.$$

Because of $\omega' = k_2' N_1 - k_1' N_2$, the derivative of the norm of Bishop Darboux vector is found by $\|\omega\|' = \frac{k_1 k_1' - k_2 k_2'}{\sqrt{\left|k_1^2 - k_2^2\right|}}$. Differentiating of the unit Bishop Darboux vector is obtained by

$$\begin{aligned}
\Upsilon' &= \left(\frac{\omega}{\|\omega\|}\right)' = \left(\frac{k_2 N_1 - k_1 N_2}{\sqrt{\left|k_1^2 - k_2^2\right|}}\right)' \\
&= \frac{-\left(k_1' k_2 - k_2' k_1\right)}{\sqrt{\left|k_1^2 - k_2^2\right|}} \frac{\left(k_1 N_1 - k_2 N_2\right)}{\sqrt{\left|k_1^2 - k_2^2\right|}} \\
&= -\theta'\left(\Upsilon \times N\right) + 0.N + 0.\Upsilon.
\end{aligned}$$

Also, the differentiation of the principal normal vector can be calculated as

$$N' = \|\omega\|\left(\frac{\omega}{\|\omega\|} \times N\right) = \|\omega\|\left(\Upsilon \times N\right) + 0.N + 0.\Upsilon..$$

The derivativation of the vectoral product of $\left(\Upsilon \times N\right)$ is

$$\begin{aligned}
\left(\Upsilon \times N\right)' &= \Upsilon' \times N + \Upsilon \times N' \\
&= -\theta'\left(-\langle \Upsilon, N\rangle N + \langle N, N\rangle \Upsilon\right) - \|\omega\|\left(-\langle \Upsilon, \Upsilon\rangle N + \langle N, \Upsilon\rangle \Upsilon\right) \\
&= -\theta'.\Upsilon + \|\omega\|\langle \Upsilon, \Upsilon\rangle N \\
&= 0.\left(\Upsilon \times N\right) + \varepsilon_1 \|\omega\| N - \theta'.\Upsilon,
\end{aligned}$$

where $\varepsilon_1 = \pm 1$. As a result, the derivative formula of the Bishop frame can be written in the matrix form as

$$\begin{bmatrix} \left(\Upsilon \times N\right)' \\ N' \\ \Upsilon' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1 \|\omega\| & -\theta' \\ \|\omega\| & 0 & 0 \\ -\theta' & 0 & 0 \end{bmatrix} \begin{bmatrix} \Upsilon \times N \\ N \\ \Upsilon \end{bmatrix},$$

where the first coefficient $\|\omega\| = \sqrt{k_1^2 + k_2^2}$ is grater than zero. The other coefficient is

$$\theta' = \frac{k_1' k_2 - k_2' k_1}{k_1^2 + k_2^2}, k_2 \neq 0, \ \left|k_1\right| \neq \left|k_2\right|.$$

Moreover, the vectors $\{(\Upsilon\times N), N, \Upsilon\}$ give a rotation motion with the rotation vector

$$\omega_1=\varepsilon_1\{\theta' N+\|\omega\|\Upsilon\}=\varepsilon_1\{\theta' N+\omega\} \text{ or } \pm(\theta' N+\omega)$$

The momentum rotation vector of the N-Bishop Darboux vector is shown as follows

$$N'=\omega_1\times N$$
$$(\Upsilon\times N)'=\omega_1\times(\Upsilon\times N)$$
$$\Upsilon'=\omega_1\times\Upsilon.$$

4. CONCLUSION

N-Bishop frame was introduced by Keskin et al. [13]. In our paper, we worked on the N-Bishop frame in Minkowski 3-space for the a timelike curve. Then we defined the N-Bishop Darboux axis for the first time. Moreover, we investigated some properties of Darboux rotation axis. We think that this study will contribute to many researchers particularly working in Minkowski space in future research.

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