ORIGINAL PAPER

THE NUMERICAL SOLUTION OF THREE POINT THIRD ORDER BOUNDARY VALUE PROBLEMS IN ODES

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Abstract. *In the present article, third order differential equation and corresponding boundary value problems considered for the numerical solution. An improved finite difference method developed for the approximate numerical solution. The proposed method converges and O(h²), the order of convergence under appropriate condition established. In the numerical experiment, model problems were considered for the purpose of testing the performance of the proposed method. The numerical results generated in numerical experiment are in good agreement with the theoretical discussion and development of the proposed method.*

Keywords: *boundary value problems; convergence analysis; finite difference method; three point boundary conditions; third order differential equation.*
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1. INTRODUCTION

Let us consider third order differential equation and corresponding boundary value problem in the following form

$$\frac{d^3u}{dx^3} = f(x,u), \qquad a < x < b \tag{1}$$

and boundary conditions are

$$\left(\frac{du(x)}{dx}\right)_{x=a} = \beta_0\,, \ \ u\left(\frac{a+b}{2}\right) = \alpha_0 \ \text{ and } \ \left(\frac{du(x)}{dx}\right)_{x=b} = \beta_1 \tag{2}$$

and α_0, β_0 , β_1 are either real finite constants or real continuous function of x i.e. $\alpha_0(x)$, $\beta_0(x)$, $\beta_1(x)$. Let us assume that $f(x,u)$ is continuous function in $[a,b]$.

The study and explaining problems in social and natural sciences one of the best methods is mathematical modeling and application of different orders of differential equations. Third order differential reported on the mathematical theory of channel flow in fluid dynamics by Jayaraman [1], the thyroid-pituitary homeostatic mechanism in biology by Danziger [2], physical oceanography by Dunbar [3], an active pulse transmission line simulating nerve axon by Nagumo [4], sandwich beam by Krajcinovic [5], Nagumo equation by McKean [6] and the references therein. To solve these problems analytically in realistic conditions is some time either not possible or very difficult. However, some work has been reported on theoretical development on different aspects such as positive, oscillations, asymptotic behavior of the solutions of third-order differential equations in the literature by

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Gregus [7], Padhi [8]. Sometimes these third order differential equations and corresponding boundary value problems have either no unique solution or no qualitative analytical solution exists. Thus an approximate numerical solution to these problems becomes very necessary. There is a finite difference Al-Said [9] and Pandey [10], Adomian decomposition Biazar [11], a method of superposition Na [12] and references in it to a different class of approximate numerical solution technique for the solution to these problems.

In this article we have modified and developed an efficient numerical technique using a finite difference method for the approximate numerical solution of problem (1). The existence and uniqueness of the solution for the problem (1) are assumed Agarwal [13]. We have presented our work in this article as follows. In the next section we will discuss the finite difference method and in Section 3 we will estimate error in proposed method. In Section 4, we have analyzed the convergence of the proposed method and the applications of the proposed method to the model problems and empirical analyses were made to demonstrate the efficiency in Section 5. Discussion and conclusion on the performance of the method are outlined in Section 6.

2. DEVELOPMENT OF THE FINITE DIFFERENCE METHOD

We substitute domain $R = [a, b]$ by a set of nodes and we wish to determine the numerical solution of the problem (1) at these nodes. So we partitioned region of interest $[a, b]$ for the purpose of solution of problem (1) and we generate $N - 1$, an odd number of equidistance nodes $a = x_0 < x_1 <..< x_N = b$, such that $b = a + Nh$, where h is the distance between two successive nodes. We also generate another set of nodes in which the nodes are separated by length $\frac{h}{2}$ and denote these nodes as $x_{i-\frac{1}{2}}, i = 1,2,\cdots, N$. Thus the region $[a, b]$ covered by nodes as shown below in the Fig. 1.

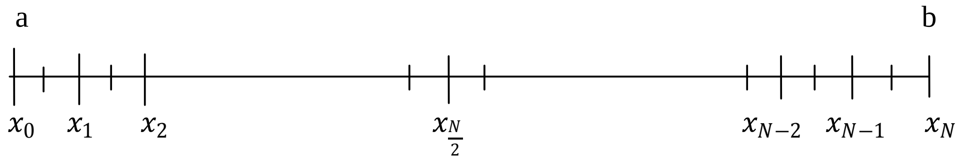

Figure 1. Arrangement of nodes.

The numerical approximation of $u(x)$ and $f(x, u)$ at nodal point x_i will be respectively denoted by u_i and f_i. Applying these notations at node x_i, the problem $(1 - 2)$ reduced to the following discrete problem,

$$\left(\frac{d^3u(x)}{dx^3}\right)_{x=x_i} = f\big(x_i, u(x_i)\big)$$
$$u_i''' = f_i \tag{3}$$

and boundary conditions are

$$u_0' = \beta_0\,,\quad u_{\frac{N}{2}} = \alpha_0 \ \text{ and } \quad u_N' = \beta_1 \tag{4}$$

The discretization idea for the problem $(1 - 2)$ with forcing function $f(x, u, u')$ and same boundary conditions were proposed in Pandey [14]. Following those thought we propose difference method for the numerical solution of the problem (3), N even no. of discrete points in the region of interest $[a,b]$,

$$
\begin{aligned}
2u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} &= -hu'_{i-1} + \frac{h^3}{48}\left(21f_{i-\frac{1}{2}} + 25\,f_{i+\frac{1}{2}}\right), \qquad i = 1 \\
-u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} + u_{i+\frac{3}{2}} &= \frac{h^3}{2}\left(f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}\right), \qquad 1 < i < \frac{N}{2} \\
-u_{i-\frac{3}{2}} + 6u_{i-\frac{1}{2}} + 3u_{i+\frac{1}{2}} &= 8u_i - \frac{h^3}{16}\left(f_{i-\frac{3}{2}} - 9\,f_{i-\frac{1}{2}}\right), \qquad i = \frac{N}{2} \\
u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= -\frac{h^3}{2}\left(f_{i-\frac{3}{2}} + f_{i-\frac{1}{2}}\right), \qquad \frac{N}{2} < i < N \\
u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} &= hu'_i - \frac{h^3}{48}\left(25f_{i-\frac{3}{2}} + 21\,f_{i-\frac{1}{2}}\right), \qquad i = N
\end{aligned}
\tag{5}
$$

We have obtained difference method (5), a system of equation in which variables are $u_{i-\frac{1}{2}},\ i = 1,\cdots,N$. We solved the system of equations (5) for these variables and to compute $u_j,\ j = 0,\cdots,N,$ the approximate numerical value of the solution of the problem (1) used following second order difference approximations,

$$
u_{j-1} = \begin{cases}
u_{j-\frac{1}{2}} - h\,u'_{j-1}, j = 1 \\
\frac{1}{2}\left(u_{j-\frac{3}{2}} + u_{j-\frac{1}{2}}\right), j = 2,\cdots,\frac{N}{2}-1,\frac{N}{2}+3,\cdots,N \\
2u_{j-\frac{1}{2}} - u_j, j = \frac{N}{2} \\
2u_{j-\frac{3}{2}} - u_{j-2}, j = \frac{N}{2}+2 \\
u_{j-\frac{3}{2}} + \frac{h}{2}u'_{j-1}, j = N+1
\end{cases}
\tag{6}
$$

3. THE ERROR ESTIMATION OF THE METHOD

In this section we discuss the local truncation error estimation of the proposed finite difference method (5). Let us consider the following linear combination of the solution and derivatives of the solution of the problem (1), $u_{i-\frac{1}{2}}, u_{i+\frac{1}{2}}, u_{i+\frac{3}{2}}$, u'_{i-1}, $u'''_{i-\frac{1}{2}}$ and $u'''_{i+\frac{1}{2}}$ i.e.

$$
h^3\left(a_0u'''_{i-\frac{1}{2}} + a_1u'''_{i+\frac{1}{2}}\right) + \mathrm{h}b_0\,u'_{i-1} + c_0u_{i-\frac{1}{2}} + c_1u_{i+\frac{1}{2}} + c_2u_{i+\frac{3}{2}} = 0 \tag{7}
$$

The parameters $a_0, a_1, b_0, c_0, c_1, c_2$ in linear combination (7) are constant. To determine these parameters, expand each term in (7) in a Taylor series about the nodal point $x_i - \frac{h}{2}$ i.e $u_{i-\frac{1}{2}}$. In Taylor series using (3) and compare the coefficients of $\mathrm{h}^{\mathrm{p}}, \mathrm{p} = 01,2,\cdots,4$. Thus we obtained following system of linear equations,

$$
a_1 - \frac{1}{48}b_0 + \frac{1}{24}c_1 + \frac{2}{3}c_2 = 0 \tag{8}
$$

$$
\begin{gathered}
a_0 + a_1 + \frac{1}{8}b_0 + \frac{1}{6}c_1 + \frac{4}{3}c_2 = 0 \\
-b_0 + c_1 + 4c_2 = 0 \\
b_0 + c_1 + 2c_2 = 0 \\
c_0 + c_1 + c_2 = 0
\end{gathered}
$$

Solving (8), we have

$$(a_0, a_1, b_0, c_0, c_1, c_2) = \frac{1}{48}(-21, -25, 48, 96, -144, 48) \tag{9}$$

To estimate error, use (9) in (7), we have

$$-\frac{31}{1920}h^5 u^{(5)}_{i-\frac{1}{2}} + o(h^6) = 0$$

first truncated term in Taylor series is,

$$T_i = -\frac{31}{1920}h^5 u^{(5)}_{i-\frac{1}{2}}$$

Thus, we estimated the error term in system of equations (5) for $i = 1$. Following similar manner we estimate error terms for other relation in (5),

$$T_i = \begin{cases} -\frac{31}{1920}h^5 u^{(5)}_{i-\frac{1}{2}} & i = 1 \\ o(h^6) \qquad 1 < i < N, & i \neq \frac{N}{2} \\ \frac{1}{16}h^5 u^{(5)}_{i-\frac{1}{2}} & i = \frac{N}{2} \\ \frac{31}{1920}h^5 u^{(5)}_{i-\frac{1}{2}} & i = N \end{cases} \tag{10}$$

4. THE STUDY OF CONVERGENCE OF THE PROPOSED METHOD

In this section we analyze method (5) for the purpose of its convergence. For the purpose of convergence analysis of the proposed method (5) we will consider following linear boundary value problem,

$$\frac{d^3u}{dx^3} = f(x), \qquad a < x < b \tag{11}$$

subject to boundary conditions

$$\left(\frac{du(x)}{dx}\right)_{x=a} = \beta_0, \;\; u\left(\frac{a+b}{2}\right) = \alpha_0 \;\text{ and }\; \left(\frac{du(x)}{dx}\right)_{x=b} = \beta_1 \tag{12}$$

We will solve the problem (11) using proposed method (5) and we write in the matrix equation form. Let $U_{\frac{1}{2}}, U_{\frac{3}{2}}, \cdots, U_{N-\frac{1}{2}}$ be the approximate solution of the problem (11) and we write

$$\boldsymbol{A} \cdot \boldsymbol{U} = \boldsymbol{Rhs} \tag{13}$$

Thus, (13) is the matrix equation of the our proposed finite difference equation (5), where **A** , **U** and **Rhs** are respectively the coefficient matrix, column vector of the solution and the column vector of known values of boundary, forcing function in (11). These matrices, vectors are given as,

$$\boldsymbol{A} = \begin{pmatrix} 2 & -3 & 1 & & & & & & 0 \\ -1 & 3 & -3 & 1 & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & & \\ & & -1 & 3 & -3 & 1 & & & \\ & & & -1 & 6 & 3 & & & \\ & & & & 1 & -3 & 3 & -1 & \\ & & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & 1 & -3 & 3 & -1 \\ 0 & & & & & & 2 & -5 & 3 \end{pmatrix}_{N\times N}, \quad \boldsymbol{U} = \left(U_{\frac{1}{2}}, U_{\frac{3}{2}}, \cdots, U_{N-\frac{1}{2}}\right)^T$$

and

$$\boldsymbol{Rhs} = \begin{pmatrix} -h\beta_0 + \dfrac{h^3}{48}(21f_{\frac{1}{2}} + 25f_{\frac{3}{2}}) \\ \dfrac{h^3}{2}\left(f_{\frac{3}{2}} + f_{\frac{5}{2}}\right) \\ \vdots \\ 8\alpha_0 - \dfrac{h^3}{16}\left(f_{\frac{N}{2}-\frac{3}{2}} - 9f_{\frac{N}{2}-\frac{1}{2}}\right) \\ -\dfrac{h^3}{2}\left(f_{\frac{N}{2}-\frac{1}{2}} + f_{\frac{N}{2}+\frac{1}{2}}\right) \\ \vdots \\ h\beta_1 - \dfrac{h^3}{48}(25f_{N-\frac{3}{2}} + 21f_{N-\frac{1}{2}}) \end{pmatrix}$$

Matrix **A** is invertible Jain [15] and Varga [16]. Let $\boldsymbol{B} = (b_{l,m})_{N\times N}$ be the inverse of matrix $\boldsymbol{A}$, where

$$b_{l,m} = \frac{1}{8(N+1)} \begin{cases} (3N-4m+1)(N+1)m - 4(N-m+1)(l-1)l, & l \le m < \frac{N}{2} \\ N+1, & l = m = \frac{N}{2} \\ -(N-2l+1)(N+2l-1)(N-m+2), & \frac{N}{2}+1 \le m < N, l \le m \\ -(N-2l+1)(N+2l-1), & l \le m = N \\ -(N-2l+1)(3N-2l+5)m, & m < \frac{N}{2}, \quad m \le l \\ (N-2l+1)(3N-2l+5)(1-m) - (N+1)(N-2m+1)(N-2m+3), & \\ & \frac{N}{2}+1 \le m \le l, \quad m \le l \end{cases} \tag{14}$$

Let R_l be row sum of matrix $\boldsymbol{B}$ and

$$R_l = \begin{cases} \dfrac{N(N^2-9N+14)}{96} + \dfrac{l(l^2+3l+2)}{6} - \dfrac{l}{8(N+1)}(4l(N+2)+N^2(l-1)+4N), & l < \dfrac{N}{2} \\ \dfrac{5N^3-9N^2+10N-12}{96} + \dfrac{l}{16(N+1)}\big((5l+3)N^2+8(2l+1)N-5l+21\big) & \\ \quad -\dfrac{l}{24}\big(3(N+1)(N+3)+2(l+1)(2l+1)\big) & \\ -\dfrac{1}{16(N+1)}\Big(lN\big(2+(N-4)(2N-l+3)\big)+2(N^2-1)-l(l-3)\Big), & l \geq \dfrac{N}{2} \end{cases} \tag{15}$$

R_l will be maximum if $l = N$ and

$$\begin{aligned} \max(R_l) &= \frac{N(N^3+16N^2+20N+62)}{96(N+1)} \\ \max(R_l) &\leq \frac{687}{12000}N^3 = \frac{687(b-a)^3}{12000h^3} \end{aligned} \tag{16}$$

Let $u_{\frac{1}{2}}, u_{\frac{3}{2}}, \cdots, u_{N-\frac{1}{2}}$ be the exact problem (11) and we write

$$\boldsymbol{A} \cdot \boldsymbol{u} = \boldsymbol{Rhs} + \boldsymbol{T} \tag{17}$$

where $\boldsymbol{T}$ is the truncation error in (5). The vectors $\boldsymbol{u}$ and $\boldsymbol{T}$ are as follows,

$$\boldsymbol{u} = \left(u_{\frac{1}{2}}, u_{\frac{3}{2}}, \cdots, u_{N-\frac{1}{2}}\right)^T$$

and

$$\boldsymbol{T} = (T_1, T_2, \cdots, T_N)^T$$

Let us define difference between exact and approximate solution of problem (11) i.e.

$$\varepsilon_{i-\frac{1}{2}} = u_{i-\frac{1}{2}} - U_{i-\frac{1}{2}}, \qquad i = 1,2,\cdots,N \tag{18}$$

Subtract (11) from (17) and apply (18), we have

$$\boldsymbol{A\varepsilon} = \boldsymbol{T} \tag{19}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_{\frac{1}{2}}, \varepsilon_{\frac{3}{2}}, \cdots, \varepsilon_{N-\frac{1}{2}})^T$.

So, we have

$$\parallel \boldsymbol{\varepsilon} \parallel = \parallel \boldsymbol{A}^{-1} \parallel \parallel \boldsymbol{T} \parallel \tag{20}$$

Let $M = \max_{x\in[a,b]} |u^{(5)}(x)|$ and using (10), (16) in (20), we have

$$\parallel \boldsymbol{\varepsilon} \parallel \leq \frac{687(b-a)^3h^2}{192000} M \tag{21}$$

From equation (21) it follows that $\parallel \boldsymbol{\varepsilon} \parallel \to \boldsymbol{0}$ as $h \to 0$. So it evident that method (5) converges and the order of the convergence of the proposed method (5) is at least quadratic i.e. $O(h^2)$.

5. NUMERICAL EXPERIMENTS

In this section, we have applied the proposed method (5) to solve linear and nonlinear model problems. We have used Gauss-Seidel and Newton Raphson method respectively to solve the system of linear and nonlinear equations arises from the equation (5). All computations were performed on a Windows 2007 home basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 GHz PC. Let u_i, the approximate value of the solution of the considered problem calculated by the method (5), and U_i function value of the solution $U(x)$ at the node point x_i. We have defined MAE, the maximum absolute error as $MAE(\mathrm{u}) = \max_{1\le i\le N, i\ne \frac{N}{2}} |\mathrm{U_i} - \mathrm{u_i}|$.

These *MAE* are shown for different value of N in Tables 1-2. The stopping criterion for iteration is either the maximum difference between two successive iterates is less than 10^{-7}or the number of iterations reached 2×10^5.

Example 1. The linear model problem with different boundary conditions in Jator [17] is given as:

$$\frac{d^3u(x)}{dx^3} = \mathrm{xu(x)} + (\mathrm{x}^3 - 2\mathrm{x}^2 - 5\mathrm{x} - 3)\mathrm{e}^\mathrm{x}, \quad 0 < x < 1,$$

subject to the boundary conditions $\mathrm{u}'(0) = 1$, $\mathrm{u}'(1) = -\mathrm{e}^1$ and $u\left(\frac{1}{2}\right) = \frac{1}{4}\mathrm{e}^{\frac{1}{2}}$. The analytical solution of the problem is $u(x) = \mathrm{x}(1.0 - \mathrm{x})\mathrm{e}^\mathrm{x}$. The MAE computed by the method (5) for different values of N are presented in Table 1.

Example 2. The nonlinear parametric model boundary value problem is given as,

$$\frac{d^3u(x)}{dx^3} = \mathrm{Ae}^{-\mathrm{x}-2\mathrm{u(x)}} + f(x), \quad 0 < x < 1,$$

subject to the boundary conditions $\mathrm{u}'(0) = -\mathrm{A}$, $\mathrm{u}'(1) = \frac{-\mathrm{A}}{\mathrm{A}+\mathrm{e}^1}$ and $u\left(\frac{1}{2}\right) = \ln\left(1 + \mathrm{Ae}^{-\frac{1}{2}}\right)$.

The source function $f(x)$ is computed so that the analytical solution of the problem is $u(x) = \ln(1 + \mathrm{Ae}^{-\mathrm{x}})$. The MAE computed by the method (5) for different values of N and parameter A are presented in Table 2.

Table 1. Maximum absolute error $|\mathbf{U_i} - \mathbf{u_i}|$ in example 1 .

N	MAE	N	MAE
8	.20181668(-1)	128	.82711260(-4)
16	.51804939(-2)	256	.20708463(-4)
32	.13114511(-2)	512	.51809149(-5)
64	.32985915(-3)	1024	.12957032(-5)

Table 2. Maximum absolute error $|\mathbf{U_i} - \mathbf{u_i}|$ in example 2 .

N	MAE			
	A=.04	A=.2	A=1.0	A=5.0
8	.70834563(-4)	.26756640(-3)	.48827914(-3)	.40303537(-3)
16	.17883383(-4)	.67350928(-4)	.12207014(-3)	.11106949(-3)
32	.44926771(-5)	.16895685(-4)	.30517563(-4)	.27808279(-4)
64	.11258921(-5)	.42312160(-5)	.76293925(-5)	.69573067(-5)
128	.28181317(-6)	.10587190(-5)	.19073474(-5)	.17399869(-5)
256	.70495665(-7)	.26479457(-6)	.47683708(-6)	.43508054(-6)
512	.17629212(-7)	.66213013(-7)	.11920929(-6)	.10878061(-6)
1024	.44079651(-8)	.16555050(-7)	.29802319(-7)	.27196463(-7)

We observed from tabulated numerical results for problem 1 that the error decreases as step length decreases. The order of accuracy is $O(h^2)$ and in good agreement with the theoretically established order of accuracy. Results for problem 2, also showed that the error decreases as step length decreases. The effect of a change of the parameter A is visible on MAE. Thus we conclude that our proposed method converges and results approve the estimated order of accuracy.

6. CONCLUSION

An accurate numerical technique developed for the numerical solution of three point boundary value problem in ordinary differential equation. The proposed finite difference method is convergent. The performance of the proposed method was tested on linear and nonlinear model problems. Numerical results for the considered problem are in good agreement with the theoretical development and discussion.

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