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A MUNTZ-LEGENDRE APPROACH TO OBTAIN SOLUTIONS OF SINGULAR PERTURBED PROBLEMS

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Abstract. *Singularly perturbed differential equations are encountered in mathematical modelling of processes in physics and engineering. Aim of this study is to give a collocation approach for solutions of singularly perturbed two-point boundary value problems. The method provides obtaining the approximate solutions in the form of Müntz-Legendre polynomials by using collocation points and matrix relations. Singularly perturbed problem is transformed into a system of linear algebraic equations. By solving this system, the approximate solution is computed. Also, an error estimation is done using the residual function and the approximate solutions are improved by means of the estimated error function. Two numerical examples are given to show the applicability of the method.*

Keywords: *singular perturbed differential equations; Müntz-Legendre polynomials; collocation method.*

1. INTRODUCTION

Singularly perturbed differential equations including a small parameter ε are encountered in mathematical modeling of processes in many fields such as fluid mechanics, fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto-hydrodynamics, rarefied gas dynamics, oceanography, and other domains of the great world of fluid motion. As a few examples of the aforementioned equations, we can give the following examples of major problems: boundary layer problems, Wentzel, Kramers and Brillouin (WKB) problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers and magneto-hydrodynamics duct problems at high Hartman numbers. Since, these problems depend on a small positive parameter ε, the solution of the equation varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. Also, the problems possesses boundary layers show rapid change in the solution near one of the boundary points. Although there is a wide class of asymptotic expansion methods available for solving the mentioned problems, difficulties are available in applying these asymptotic expansion methods, such as finding the appropriate asymptotic expansions in the inner and outer regions. So, more efficient and simpler numerical techniques are needeed to obtain the solutions of singularly perturbed boundary value problems.

In the recent years, for solving the mentioned problems, some authors have published the papers including various methods such as the finite difference methods [1], the B-spline

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collocation method [2], the B-splines with artificial viscosity [3], the Chebysev method [4], the second order spline finite difference method [5], the Bessel collocation method [6], the exponential polyomial approach [7] and an algorithm of finite difference method [8]. On the otther hand, Müntz-Legendre polynomails are used for solutions of systems of differential equations with functional arguments [8].

In this study, we will focus on the problems of the form,

$$\varepsilon y''(x) + p(x)y'(x) + r(x)y(x) = s(x), \qquad 0 \le x \le 1 \tag{1}$$

$$y(0) = \alpha \text{ and } y(1) = \beta . \tag{2}$$

Hence, we will obtain the approximate solutions of singular perturbed differential equations by using the method in [8]. By using this method, the approximate solutions are computed in form

$$y(x) = \sum_{n=0}^{N} a_n L_n(x), \quad n = 0,1,2,\ldots,N \tag{3}$$

where $L_n(x)$ are the Müntz-Legendre polynomials defined by the formula

$$L_n(x) = \sum_{j=n}^{N} (-1)^{N-j} \binom{N+1+j}{N-n} \binom{N-n}{N-j} x^j, \; 0 \le x \le 1. \tag{4}$$

In the problem $y^{(0)}(x) = y\,(x)$ are the unknown function $p(x), r(x)$ and $s(x)$ functions defined in the interval $0 \le x \le 1$. On the other hand, α, β are real constants, a_n, $n = 0,1,2,...,N$ are unknown Müntz-Legendre coefficients.

2. METHOD OF SOLUTIONS

At the beginning let us consider the Eq.(1) and try to construct the matrix form of each term in the equation. The approximate solution $y(x)$ given by the relation (3) and its derivatives can be written in the matrix form as,

$$\left[y(x)\right] = \mathbf{L}(x)\mathbf{A}\ , \ \left[y^{(k)}(x)\right] = \mathbf{L}^{(\mathrm{k})}(x)\mathbf{A} \tag{5}$$

where $\mathbf{L}(x) = \left[L_0(x)\ L_1(x)\ ...\ L_N(x)\right]$ and $\mathbf{A} = \left[a_0\ a_1\ ...\ a_N\right]^T$. On the other hand, $\mathbf{L}(x)$ matrix and its derivative can be represented as,

$$\mathbf{L}(x) = \mathbf{X}(x)\mathbf{F}^T, \qquad \mathbf{L}^{(\mathrm{k})}(x) = \mathbf{X}^{(\mathrm{k})}(x)\mathbf{F}^T \tag{6}$$

where $\mathbf{X}(x) = \left[1\ x\ ...\ x^N\right]$ is the bases matrix and the matrix $\mathbf{F}$ is defined by

$$\mathbf{F}=\begin{bmatrix} (-1)^N\binom{N+1}{N} & (-1)^{N-1}\binom{N+2}{N}\binom{N}{N-1} & (-1)^{N-2}\binom{N+3}{N}\binom{N}{N-2} & \cdots & (-1)^1\binom{2N}{N}\binom{N}{1} & (-1)^0\binom{2N+1}{N} \\ 0 & (-1)^{N-1}\binom{N+2}{N-1} & (-1)^{N-2}\binom{N+3}{N-1}\binom{N-1}{N-2} & \cdots & (-1)^1\binom{2N}{N-1}\binom{N-1}{1} & (-1)^0\binom{2N+1}{N-1} \\ 0 & 0 & (-1)^{N-2}\binom{N+3}{N-2} & \cdots & (-1)^1\binom{2N}{N-2}\binom{N-2}{1} & (-1)^0\binom{2N+1}{N-2} \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & (-1)^1\binom{2N}{1} & (-1)^0\binom{2N+1}{1} \\ 0 & 0 & 0 & \cdots & 0 & (-1)^0\binom{2N+1}{0} \end{bmatrix}.$$

Using the relations (5) and (6), $y(x)$ and $y^{(k)}(x)$ can be written in the form

$$\left[y(x)\right]=\mathbf{X}(x)\mathbf{F}^T\mathbf{A}\ \ ,\ \left[y^{(k)}(x)\right]=\mathbf{X}^{(k)}(x)\mathbf{F}^T\mathbf{A} \qquad (7)$$

Besides the relation $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(k)}(x)$ can be written as $\mathbf{X}^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k$ where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Finally the matrix $\left[y^{(k)}(x)\right]$ can be represented as

$$\left[y^{(k)}(x)\right]=\mathbf{X}(x)\mathbf{B}^k\mathbf{F}^T\mathbf{A}. \qquad (8)$$

On the other hand, the matrix forms of the conditions given by (2) can be written as

$$y(0)=\mathbf{X}(0)\mathbf{F}^T\mathbf{A}=[\alpha] \text{ and } y(1)=\mathbf{X}(1)\mathbf{F}^T\mathbf{A}=[\beta]. \qquad (9)$$

We are now ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, we substitute the matrix relations into Eq. (1) and thus we have

$$\varepsilon\mathbf{X}(x)\mathbf{B}^2\mathbf{F}^T\mathbf{A}+p(x)\mathbf{X}(x)\mathbf{B}\mathbf{F}^T\mathbf{A}+r(x)\mathbf{X}(x)\mathbf{F}^T\mathbf{A}=s(x). \qquad (10)$$

For calculating the Müntz-Legendre coefficient matrix $\mathbf{A}$ numerically, let us replace the collocation points, defined by the formula $x_i=(1/N)i$, $i=0,1,\ldots,N$, into the equation (10). By this way, we have a system of linear algebraic equations as

$$\varepsilon\mathbf{X}(x_i)\mathbf{B}^2\mathbf{F}^T\mathbf{A}+p(x_i)\mathbf{X}(x_i)\mathbf{B}\mathbf{F}^T\mathbf{A}+r(x_i)\mathbf{X}(x_i)\mathbf{F}^T\mathbf{A}=s(x_i),\quad i=0,1,\ldots,N.$$

Hence, the above system can be expressed as

$$\left\{\boldsymbol{\varepsilon}\mathbf{X}\mathbf{B}^2\mathbf{F}^T+\mathbf{P}\mathbf{X}\mathbf{B}\mathbf{F}^T+\mathbf{R}\mathbf{X}\mathbf{F}^T\right\}\mathbf{A}=\mathbf{S}, \qquad (11)$$

where

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon & 0 & 0 & 0\\ 0 & \varepsilon & 0 & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \varepsilon\end{bmatrix},\ \mathbf{X}=\begin{bmatrix}\mathbf{X}(x_0)\\ \mathbf{X}(x_1)\\ \vdots\\ \mathbf{X}(x_N)\end{bmatrix}=\begin{bmatrix}1 & x_0 & x_0^{\ 2} & \cdots & x_0^{\ N}\\ 1 & x_1 & x_1^{\ 2} & \cdots & x_1^{\ N}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & x_N & x_N^{\ 2} & \cdots & x_N^{\ N}\end{bmatrix},\ \mathbf{P}=\begin{bmatrix}p(x_0) & 0 & 0 & 0\\ 0 & p(x_1) & 0 & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & p(x_N)\end{bmatrix},$$

$$\mathbf{R}=\begin{bmatrix}r(x_0) & 0 & 0 & 0\\ 0 & r(x_1) & 0 & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & r(x_N)\end{bmatrix} \text{ and } \mathbf{S}=\begin{bmatrix}s(x_0)\\ s(x_1)\\ \vdots\\ s(x_N)\end{bmatrix}.$$

Eq. (11) is called as the fundamental matrix equation. Briefly we can write the Eq.(11) as

$$\mathbf{W}\mathbf{A}=\mathbf{S} \text{ or } [\mathbf{W};\mathbf{S}] \qquad (12)$$

where $\mathbf{W}=\boldsymbol{\varepsilon}\mathbf{X}\mathbf{B}^2\mathbf{F}^T+\mathbf{P}\mathbf{X}\mathbf{B}\mathbf{F}^T+\mathbf{R}\mathbf{X}\mathbf{F}^T$. Here, Eq. (11) corresponds to a system of $(N+1)$ linear algebraic equations with unknown Müntz-Legendre coefficients $a_0,a_1,\ldots,a_N$. Briefly, the matrix forms for conditions (2) are

$$\mathbf{U}_1\mathbf{A}=[\alpha] \quad \text{or} \quad [\mathbf{U}_1;\alpha] \qquad (13)$$

$$\mathbf{U}_2\mathbf{A}=[\beta] \quad \text{or} \quad [\mathbf{U}_2;\beta] \qquad (14)$$

where $\mathbf{U}_1=\mathbf{X}(0)\mathbf{F}^T=\begin{bmatrix}u_{1\,0} & u_{1\,1} & u_{1\,2} & \cdots & u_{1\,N}\end{bmatrix}$ and $\mathbf{U}_2=\mathbf{X}(1)\mathbf{F}^T=\begin{bmatrix}u_{2\,0} & u_{2\,1} & u_{2\,2} & \cdots & u_{2\,N}\end{bmatrix}$.

Consequently, to obtain the solution of Eq. (1) under conditions (2), by replacing the row matrices (13) and (14) by two rows of the matrix (12), we have the required augmented matrix

$$\tilde{\mathbf{W}}\mathbf{A}=\tilde{\mathbf{S}}. \qquad (15)$$

For simplicity, if last two rows of the matrix (11) are replaced, the augmented matrix of the system (15) is as follows:

$$[\tilde{\mathbf{W}};\tilde{\mathbf{S}}]=\begin{bmatrix}w_{0\,0} & w_{0\,1} & w_{0\,2} & \cdots & w_{0\,N} & ; & s(x_0)\\ w_{1\,0} & w_{1\,1} & w_{1\,2} & \cdots & w_{1\,N} & ; & s(x_1)\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ w_{N-2\,0} & w_{N-2\,1} & w_{N-2\,2} & \cdots & w_{N-2\,N} & \vdots & s(x_{N-2})\\ u_{1\,0} & u_{1\,1} & u_{1\,2} & \cdots & u_{1\,N} & ; & \alpha\\ u_{2\,0} & u_{2\,1} & u_{2\,2} & \cdots & u_{2\,N} & ; & \beta\end{bmatrix}. \qquad (16)$$

If $\operatorname{rank}\tilde{\mathbf{W}}=\operatorname{rank}[\tilde{\mathbf{W}};\tilde{\mathbf{S}}]=N+1$, then we can write

$$\mathbf{A}=(\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{S}}. \qquad (17)$$

The unknown Müntz-Legendre coefficients matrix **A** is determined solving this linear system. Hence, by substituting determined Müntz-Legendre coefficients $a_0, a_1, \dots, a_N$ into Eq.(3), we get the approximate solution

$$y_N(x) = \sum_{n=0}^{N} a_n L_n(x) \cdot \tag{18}$$

3. ERROR ESTIMATION AND IMPROVED APPROXIMATE SOLUTIONS

In this part, we apply an error estimation technique and residual improvement [12, 13] for the considered problem and method. Firstly let us the approximate solution is susbstituted in Eq.(1) and then the equation is satisfied approximately, that is:

$$\varepsilon y_N''(x) + p(x) y_N'(x) + r(x) y_N(x) = s(x) + R_N(x), \tag{19}$$

where $R_N(x)$ is called as the residual function.

Let us write the error function as $e_N = y(x) - y_N(x)$ to obtain the error problem. let us subtract the equation (19) from the equation (1) side by side. So, we could reach the error problem with the homogenous conditions,

$$\varepsilon e_N''(x) + p(x) e_N'(x) + r(x) e_N(x) = -R_N(x). \tag{20}$$

$$e_N(0) = 0, \; e_N(1) = 0. \tag{21}$$

Here, the conditions in Eq.(21) are homogenous because the exact solution $y(x)$ and the approximate solutions $y_N(x)$ provide the conditions (2). Hence, solving the error problem (20)-(21) by the same method defined in section 2, an approximation $e_{N,M}$ can be find to the error function e_N. When solving this error problem, it is better to choose $M \geq N$. Also, the approximate solution $y_N(x)$ can be improved as $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$. So a new error function can be defined as improved absolute error function by the relation

$$\left|E_{N,M}(x)\right| = \left|y(x) - y_{N,M}(x)\right|.$$

4. NUMERICAL EXAMPLES

In this section, we apply the method to some examples.

Example 1. [5] First let us consider singularly perturbed equation of the form**,**

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon) y(x) = 0, \quad 0 \leq x \leq 1 \tag{22}$$

under the boundary conditions $y(0)=1+e^{-(1+\varepsilon)/\varepsilon}$ and $y(1)=1+e^{-1}$. The exact solution of the problem is $y(x)=e^{-x}+e^{(1+\varepsilon)(x-1)/\varepsilon}$. In Table 1 and Table 2; the actual absolute errors, the estimated absolute errors and the improved absolute errors are compared for (N,M)=(12,13) and $\varepsilon=10^{-3}$, $\varepsilon=10^{-5}$ respectively. In Table 3, a comparison is given between the exact solution and the results of other methods such as seventh order numerical method (SNM) [10], the Chebyshev method (CM) [4], the Bessel collocation method[6] and the present method for $y(x)$.On the other hand in Figs. 1-2; the exact solutions and the improved approximate solutions are drawn for $y(x)$ for different values of truncation limits.

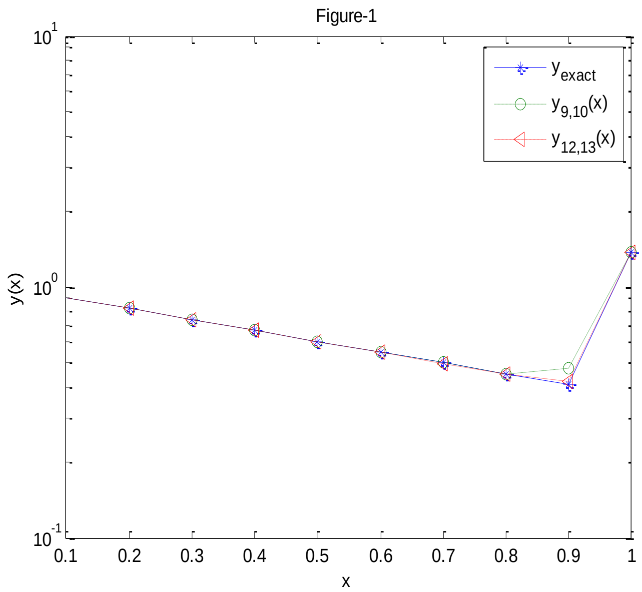

Figure 1. Comparison of exact solution and improved approximate solutions of $y(x)$ in Ex.1 for (N,M)=(9,10), (12,13) and $\varepsilon=10^{-3}$.

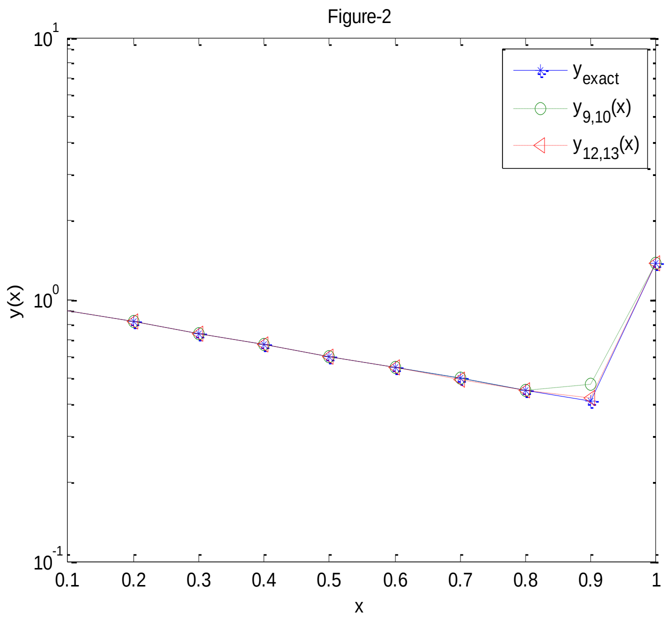

Figure 2. Comparison of exact solution and improved approximate solutions of $y(x)$ in Ex. 1 for (N,M)=(9,10), (12,13) and $\varepsilon=10^{-5}$.

Table 1. Comparison of actual absolute errors, estimated absolute errors and improved absolute errors of Example 1 for (N,M)=(12,13) and ε = 10^{-3}.

	Actual Absolute Errors for N=12	Estimated Absolute Errors for N=12, M=13	Improved Absolute Errors for N=12, M=13
x_i	$\lvert e_{12}(x_i)\rvert$	$\lvert e_{12,13}(x_i)\rvert$	$\lvert E_{12,13}(x_i)\rvert$
0	0	2.3639e-018	0
0.2	7.9850e-004	1.4854e-003	6.8694e-004
0.4	6.6896e-004	1.2331e-003	5.6409e-004
0.6	5.5279e-004	1.0055e-003	4.5273e-004
0.8	2.9933e-005	4.9068e-004	4.6074e-004
1	2.6186e-012	2.8553e-011	2.2886e-011

Table 2. Comparison of actual absolute errors, estimated absolute errors and improved absolute errors of Example 1 for (N,M)=(12,13) and ε = 10^{-5}.

	Actual Absolute Errors for N=12	Estimated Absolute Errors for N=12, M=13	Improved Absolute Errors for N=12, M=13
x_i	$\lvert e_{12}(x_i)\rvert$	$\lvert e_{12,13}(x_i)\rvert$	$\lvert E_{12,13}(x_i)\rvert$
0	0	8.6982e-019	0
0.2	8.2362e-004	1.5351e-003	7.1149e-004
0.4	6.9057e-004	1.2753e-003	5.8476e-004
0.6	5.7117e-004	1.0406e-003	4.6945e-004
0.8	5.5415e-005	5.1905e-004	4.6364e-004
1	3.3055e-012	1.3974e-011	4.5540e-013

Table 3. Comparison of numerical values for $y(x)$ in Example 1 with different methods.

x_i	Exact solution	SNM[10]	CM[4] N=20	Bessel Collocation method $\varepsilon=10^{-3}$ [6] $N=12$ $M=13$	Present Method $\varepsilon=10^{-3}$, $N=12$, $M=13$
0.0	1.000000	0.999000	0.999999	1.000000	1.000000
0.1	0.904837	0.903837	0.904813	0.904022	0.904022
0.2	0.818731	0.817731	0.818698	0.818044	0.818043
0.3	0.740818	0.717924	0.740863	0.740202	0.740201
0.4	0.670320	0.669320	0.670386	0.669756	0.669755
0.5	0.606531	0.605531	0.606373	0.606022	0.606021
0.6	0.548812	0.547812	0.548888	0.548359	0.548358
0.7	0.496585	0.495585	0.496934	0.496148	0.496148
0.8	0.449329	0.448329	0.449007	0.448868	0.448868
0.9	0.406570	0.405070	0.404981	0.420754	0.420753
1.0	1.367880	1.367879	1.367879	1.367880	1.367879

Example 2. [7] Now let us consider the singularly-perturbed differential equation

$$\varepsilon y''(x)+\left(1-x/2\right)y'(x)-1/2y(x)=0\,,\quad 0\le x\le 1 \tag{23}$$

with the boundary conditions $y(0)=0$ and $y(1)=1$. The exact solution of the problem is given by $y(x)=1/(2-x)-1/2e^{(-x+x^2/4)/\varepsilon}$. In Fig. 3, the actual error function $\lvert e_5(x)\rvert=\lvert y(x)-y_5(x)\rvert$, the improved error functions $\lvert E_{5,6}(x)\rvert=\lvert y(x)-y_{5,6}(x)\rvert$ and $\lvert E_{7,8}(x)\rvert=\lvert y(x)-y_{7,8}(x)\rvert$ are compared by their graphs. Also in Table 4, the numerical results for $y(x)$ are calculated at different collocations points and compared with other methods such as the B-splines method (BM)[2], Kevorkian and Cole's method (KCM) [11], the Chebyshev method (CM) [4], the Bessel collocation method [6].

As a result, it is seen from these examples that the present method gives effective results.

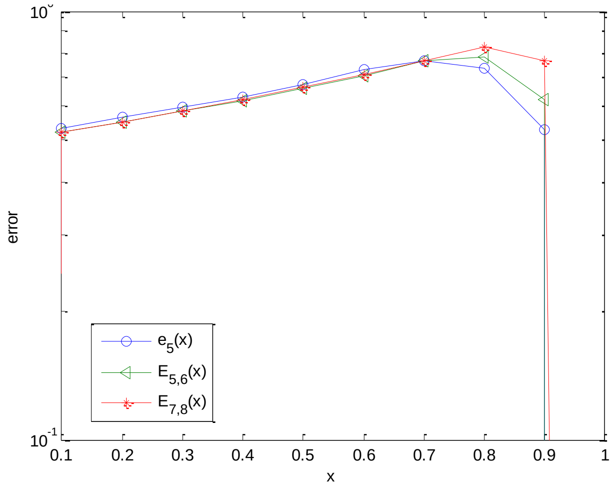

Figure 3. Comparison of actual absolute error and improved absolute error functions of Example 2 for (N,M)=(5,6),(7,8) and ε = 5^{-4}.

Table 4. Comparison of numerical values of $y(x)$ in Example 2 with different methods for ε = 2^{-4}.

x_i	Exact solution	BM [2] $N = 200$	KCM [11]	CM [4] $N = 100$	Bessel Collocation method [6] $N = 12$, $M = 13$	Present method $N = 12$, $M = 13$
0.01	0.0762	0.0848	0.0763	0.0816	0.0770	0.0770
0.02	0.1413	0.1568	0.1414	0.1513	0.1428	0.1427
0.03	0.1971	0.2180	0.1971	0.2110	0.2472	0.1990
0.05	0.2858	0.3143	0.2859	0.3059	0.2885	0.2885
0.1	0.4212	0.4566	0.4212	0.4502	0.4246	0.4245
0.2	0.5316	0.5641	0.5316	0.5666	0.5342	0.5342
0.5	0.6662	0.6896	0.6662	0.7000	0.6603	0.6603
0.9	0.9090	0.9174	0.9091	0.9224	0.8700	0.8700

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