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CONTINUED FRACTIONS, INTERMEDIATE FRACTIONS AND THEIR RELATION TO THE BEST APPROXIMATIONS

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Abstract. *The subject of this paper is the current state of art in theory of continued fractions, intermediate fractions and their relation to the best rational approximations of the first and second kind. The paper provides an overview of the some well known and even some new properties of continued fractions, and the various terms associated with them. In addition to intermediate fractions, paper considers the fine intermediate fractions and gave some statements to position these fractions in the continued fraction representation of numbers.*

Keywords: *continued fraction; convergent; intermediate fraction; irrational number; rational approximation.*

1. INTRODUCTION

Theory of continued fractions is one of important topics of mathematics, and it has been mostly explored in the last two centuries. Of most importance to this theory is the book “Continued fractions” by A. Ya. Khinchin [1], in which continued fractions and intermediate fractions are defined and discussed. Many authors in their books and papers analyse not only the continued fractions, but also analyse the other types of fractions. One of the problems in this theory is nomenclature, as there is no consensus about terminology between different sources [2-9]. Some of the most used terms are *best (rational) approximation of the first kind*, *best (rational) approximation of the second kind*, *intermediate fraction*, *semiconvergent*, *best-convergent*, *principal convergent*, *nonprincipal convergent* and *intermediate convergent*. Our intention is to explore this topic more thoroughly, compare the terms encountered from different sources and analyse the specific properties of these important and applicable fractions.

2. CONTINUED FRACTIONS

Continued fraction is an expression given in the following form:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{1}$$

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where $\alpha \in \mathbb{R}$, $a_0 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $a_i \in \mathbb{N} (i \geq 1)$. Values a_i we call *continued fraction digits* $(a_0, a_1, a_2, a_3, \dots)$. Continued fractions can be written in shorter form as

$$\alpha = [a_0; a_1, a_2, a_3, \dots]. \quad (2)$$

Throughout the paper we will consider only cases where α is positive real value, while whole analysis can be done analogously in case where α is negative value.

For given real number α, continued fraction digits are calculated by following steps:

$$\begin{aligned} x_0 &= \alpha \\ a_0 &= \lfloor x_0 \rfloor \\ x_{i+1} &= \frac{1}{x_i - a_i} \\ a_{i+1} &= \lfloor x_{i+1} \rfloor \end{aligned} \quad (3)$$

where $i = 0,1,2, \dots$. If α is a rational number then there is an index m, the smallest positive integer number such that the previous procedure ends with $x_m = a_m$. In that case

$$\alpha = [a_0; a_1, a_2, \dots, a_m]. \quad (4)$$

Otherwise, if α is an irrational number then there is an infinite sequence of continued fraction digits.

2.1. CONVERGENTS

For an arbitrary real number α, each fraction

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_{n-1,} a_n] \quad (5)$$

we call *convergent*. Convergents occur for both rational and irrational numbers and the condition $n < m$ must be satisfied for rational numbers.

For the sequence of convergents $\left(\frac{p_i}{q_i}\right)_{i=0..n}$, as sequence of fractions, the following recurrent relations hold:

$$\begin{aligned} p_i &= a_i \cdot p_{i-1} + p_{i-2}, \\ q_i &= a_i \cdot q_{i-1} + q_{i-2}, \end{aligned} \quad (6)$$

where $i = 2..n$, $p_0 = a_0$, $q_0 = 1$, $p_1 = a_0 a_1 + 1$ and $q_1 = a_1$.

2.2. SOME THEOREMS OF CONTINUED FRACTIONS

Further in this section are presented some theorems for continued fractions according to [1]. It should be noted that all the indices in the following theorems are from the set $\mathbb{N}_0$.

Theorem 1. *The numerators p_i and denominators q_i, defined by (6), are coprime integers.*

Theorem 2. *The sequence of denominators $q_1, q_2, \ldots, q_i, \ldots$ is a strictly increasing sequence of natural numbers. Absolute differences between convergents $\frac{p_{i+1}}{q_{i+1}}$ and $\frac{p_i}{q_i}$ is determined by*

$$\left|\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i}\right| = \frac{1}{q_i \cdot q_{i+1}} \tag{7}$$

monotonically converges to 0.

Theorem 3. *Each convergent $\frac{p_i}{q_i}$ with odd index i is larger than adjacent convergents $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_{i+1}}{q_{i+1}}$. Each convergent $\frac{p_i}{q_i}$ with an even index i is smaller than the adjacent convergents $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_{i+1}}{q_{i+1}}$.*

Let as emphasise the following form [1].

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^{i+1} a_{i+1}}{q_{i+1} \cdot q_{i-1}}. \tag{8}$$

Theorem 4. *The subsequence of convergents with even indices is a sequence of monotonically increasing numbers, and the subsequence of convergents with odd indices is a sequence of monotonically decreasing numbers:*

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \alpha < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}. \tag{9}$$

Theorem 5. *Sequence of segments $\left[\frac{p_0}{q_0}, \frac{p_1}{q_1}\right], \left[\frac{p_1}{q_1}, \frac{p_2}{q_2}\right], \ldots, \left[\frac{p_{i-1}}{q_{i-1}}, \frac{p_i}{q_i}\right], \ldots$ is a sequence of segments arranged by order of inclusion. Each member of the sequence contains all subsequent members of the sequence. The length of the segments in sequence monotonically converges to 0.*

The following theorem gives one estimate of the error.

Theorem 6.

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2} \tag{10}$$

2.3. *CONTINUED FRACTIONS AND BEST APPROXIMATIONS*

As mentioned, we came across many different terms for continued fractions and fractions that can be derived from them. It happened that different terms sometimes correspond to the same set of fractions. Further in this section, we state two basic definitions of continued fractions and we give one overview of the used terminology.

Within Definitions 1 and 2 was considered the quality of the approximation of a real number from [1], and also [2-9].

Definition 1. For $\alpha \in \mathbb{R}$, the fraction $\frac{p}{q}$ is *best (rational) approximation of the first kind* if for every fraction $\frac{r}{s}$ stands:

$$\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{r}{s}\right| \tag{11}$$

where $\frac{r}{s} \neq \frac{p}{q}$, $0 < s \leq q$.

Term *best (rational) approximation of the first kind* for the same set of fractions was used by A. Ya. Khinchin [1], M. Thill [2], A. Cortzen [3] and J. Douthett and R. Krantz [4], as shown in *Table 1*. On the other hand, some of the terms, which were used for the same set of fractions are:

- *best Huygens approximation* by S. Khrushchev [5],
- *continued fraction* by M. Schechter [6].

In this paper, we will use the term ***best approximation of the first kind*** introduced by Definition 1.

Definition 2. For $\alpha \in \mathbb{R}$, the fraction $\frac{p}{q}$ is *best (rational) approximation of the second kind* if for every fraction $\frac{r}{s}$ stands:

$$|q\alpha - p| < |s\alpha - r| \tag{12}$$

where $\frac{r}{s} \neq \frac{p}{q}$, $0 < s \leq q$.

Term *best (rational) approximation of the second kind* for the same set of fractions was used by A. Ya. Khinchin [1], M. Thill [2], A. Cortzen [3], J. Douthett and R. Krantz [4] and M. Schechter [6], as shown in *Table 1*. On the other hand, terms, which were used for the same set of fractions are:

- *best Lagrange approximation* by S. Khrushchev [5],
- *principal convergent* by S. Khrushchev [5] and J. Douthett and R. Krantz [4],
- *convergent* by A. Honingh [7] and N. Carey and D. Clampitt [8].

In this paper, we will use the term ***best approximation of the second kind*** introduced by Definition 2.

The following continued fractions theorems refer to kind of approximations defined in Definitions 1 and 2 [1].

Theorem 7. *Every best approximation of the second kind is a convergent.*

Theorem 8. *Every best approximation of the second kind must necessarily be a best approximation of the first kind. The converse is not true.*

Theorem 8 can be illustrated with Fig. 1.

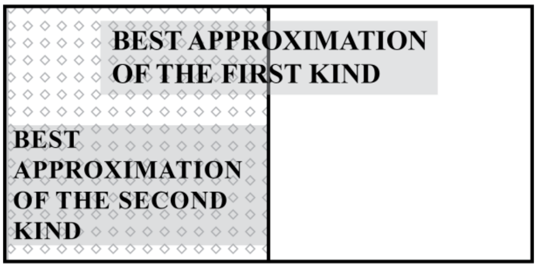

Figure 1. Theorem 8 illustration.

3. INTERMEDIATE FRACTIONS

The quality of the approximation of a real number by a fraction can also be observed based on the following definition according to [1], and also [2-9].

Definition 3. *Intermediate fractions* of $\alpha \in \mathbb{R}$, which are denoted with $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$, may appear between two successive convergents $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$, and are defined by:

$$p_{i+1}^{(j)} = j \cdot p_i + p_{i-1}$$
$$q_{i+1}^{(j)} = j \cdot q_i + q_{i-1}, \tag{13}$$

where $j \in \{1,2,\dots,a_{i+1}-1\}$.

Term *intermediate fraction* for the same set of fractions was used by A. Ya. Khinchin [1], M. Thill [2] and M. Schechter [6], as shown in *Table 1*. On the other hand, terms, which are used for the same set of fractions are:

- *semiconvergent* by A. Cortzen u [3],
- *intermediate convergent* by J. Douthett and R. Krantz [4],
- *nonprincipal convergent* by S. Khrushchev [5],
- *semi-convergent* by A. Honingh u [7] and N. Carey and David Clampitt [8].

In this paper, we will use the term ***intermediate fraction*** introduced by Definition 3.

We can present two consecutive convergents using continued fraction digits:

$$\frac{p_i}{q_i} = [a_0; a_1, \dots, a_i] \tag{14}$$

and

$$\frac{p_{i+1}}{q_{i+1}} = [a_0; a_1, \dots, a_i, a_{i+1}]. \tag{15}$$

From there, if there are intermediate fractions between these two successive convergents, can be present as following

$$\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} = [a_0; a_1, \dots, a_i, j], \tag{16}$$

where $j \in \{1,2,\dots,a_{i+1}-1\}$.

Table 1. Terms used for fractions from Definitions 1, 2 and 3 in the sources.

	References	**A.Ya. Khinchin [1]**	**M. Thill [2]**	**A. Cortzen [3]**	**J. Douthett, R. Krantz [4]**	**S. Khrushchev [5]**	**A. Honingh [6]**	**N. Carey, D. Clampitt [7]**	**M. Schechter [8]**
Definition 1	**Best approx. of the first kind**	+	+	+	+				
	Best Huygens approx.					+			
	Continued fraction								+
Definition 2	**Best approx. of the second kind**	+	+	+	+				+
	Best Lagrange approx.					+			
	Principal convergent				+	+			
	Convergent						+	+	
Definition 3	**Intermediate fraction**	+	+						+
	Semiconvergent			+					
	Intermediate convergent				+				
	Nonprincipal convergent					+			
	Semi-convergent						+	+	

Note that intermediate fractions are between converges, and therefore cannot be the best approximation of the second kind.

Example 1. By the example of the number π, we will show that several intermediate fractions can be found between two observed adjacent convergents. These intermediate fractions can approximate the number π better or worse with respect to the observed convergents. The continued fraction record of the number π is

$$\pi = 3.14159265359\dots = [3; 7,15,1,292,1,1,1,2,1,1,3\dots].$$

The first two convergents of the number π are:

$$\frac{p_0}{q_0} = \frac{a_0}{1} = \frac{3}{1} = [3]$$

$$\frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{22}{7} = [3; 7].$$

The observed convergents are the best approximations of the first kind and the best approximations of the second kind, because the inequalities in Definitions 1 and 2 are satisfied, for every fraction $\frac{r}{s}$, where $\frac{r}{s} \neq \frac{p}{q}$, $0 < s \leq q$. See Theorems 7 and 8.

Indeed, by Definition 1:

$$\left|\pi - \frac{3}{1}\right| < \left|\pi - \frac{r}{s}\right|,$$
$$\left|\pi - \frac{22}{7}\right| < \left|\pi - \frac{r}{s}\right|, \tag{17}$$

and by Definition 2:

$$|1 \cdot \pi - 3| < |s\pi - r|,$$
$$|7 \cdot \pi - 22| < |s\pi - r|. \tag{18}$$

There are intermediate fractions between convergents $\frac{3}{1}$ and $\frac{22}{7}$:

$$\frac{p_1^{(1)}}{q_1^{(1)}} = \frac{4}{1} = [3; 1],$$
$$\frac{p_1^{(2)}}{q_1^{(2)}} = \frac{7}{2} = [3; 2],$$
$$\frac{p_1^{(3)}}{q_1^{(3)}} = \frac{10}{3} = [3; 3],$$
$$\frac{p_1^{(4)}}{q_1^{(4)}} = \frac{13}{4} = [3; 4], \tag{19}$$
$$\frac{p_1^{(5)}}{q_1^{(5)}} = \frac{16}{5} = [3; 5],$$
$$\frac{p_1^{(6)}}{q_1^{(6)}} = \frac{19}{6} = [3; 6].$$

Based on the inequality in Definition 1, $\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{r}{s}\right|$, we can determine which intermediate fractions satisfy this inequality, in such a way that they are a better approximation of the number π with respect to the convergent $\frac{3}{1}$. For intermediate fractions $\frac{p_1^{(1)}}{q_1^{(1)}}$, $\frac{p_1^{(2)}}{q_1^{(2)}}$ and $\frac{p_1^{(3)}}{q_1^{(3)}}$ the following inequalities are not satisfied

$$\left|\pi - \frac{4}{1}\right| < \left|\pi - \frac{3}{1}\right|,$$
$$\left|\pi - \frac{7}{2}\right| < \left|\pi - \frac{3}{1}\right|, \tag{20}$$
$$\left|\pi - \frac{10}{3}\right| < \left|\pi - \frac{3}{1}\right|.$$

On the other hand, for intermediate fractions $\frac{p_1^{(4)}}{q_1^{(4)}}$, $\frac{p_1^{(5)}}{q_1^{(5)}}$ and $\frac{p_1^{(6)}}{q_1^{(6)}}$ the following inequalities are satisfied:

$$\left|\pi - \frac{13}{4}\right| < \left|\pi - \frac{3}{1}\right|,$$
$$\left|\pi - \frac{16}{5}\right| < \left|\pi - \frac{3}{1}\right|, \tag{21}$$
$$\left|\pi - \frac{19}{6}\right| < \left|\pi - \frac{3}{1}\right|.$$

Table 2 shows the approximations of the number π using the first two convergents (denoted by *) and the intermediate fractions between them. The first column shows a rational representation; the second column shows continued fraction record; and the third column shows the deviation of these approximations from the real value of the number π. The deviation is determined on the basis of formulas $\pi - \frac{p_i}{q_i}$ and $\pi - \frac{p_i^{(j)}}{q_i^{(j)}}$. All the fractions that are the best approximation of the first kind are bold in Table 2.

Table 2. The first two convergents of the number π and the intermediate fractions between them.

Convergents and intermediate fractions	Continued fraction digits	Deviation
3/1	**= [3]** *	**0.1415926535**
4/1	= [3; 1]	- 0.8584073464
7/2	= [3; 2]	- 0.3584073464
10/3	= [3; 3]	- 0.1917406797
13/4	**= [3; 4]**	**- 0.1084073464**
16/5	**= [3; 5]**	**- 0.0584073464**
19/6	**= [3; 6]**	**- 0.0250740131**
22/7	**= [3; 7]** *	**- 0.0012644892**

As can be seen from Table 2, the convergent $\frac{22}{7}$ better approximates the number π compared to its predecessor convergent $\frac{3}{1}$, as convergent $\frac{22}{7}$ has a smaller deviation than the number π. Also, based on deviations in relation to convergent $\frac{3}{1}$:

- intermediate fractions $\frac{4}{1}$, $\frac{7}{2}$ and $\frac{10}{3}$ are worse approximations of the number π,
- intermediate fractions $\frac{13}{4}$, $\frac{16}{5}$ and $\frac{19}{6}$ are better approximations of the number π. ■

Some intermediate fractions are also the best approximations of the first kind. This is confirmed by Definitions 1 and 3 and Example 1.

Definition 4. Each intermediate fraction of α, which is also the best approximation of the first kind, is called *the fine intermediate fraction*. The remaining intermediate fractions are called *the non-fine intermediate fractions*.

Example 2. According to this definition, intermedia fractions $\frac{13}{4}$, $\frac{16}{5}$ and $\frac{19}{6}$ are examples of the fine intermediate fractions for the number π. On the other hand, intermediate fractions $\frac{4}{1}$, $\frac{7}{2}$ and $\frac{10}{3}$ are examples of the non-fine intermediate fractions for the number π. ■

Thus, the set of the intermediate fractions of the observed number consists of fine and non-fine intermediate fractions as illustrated in Fig. 2.

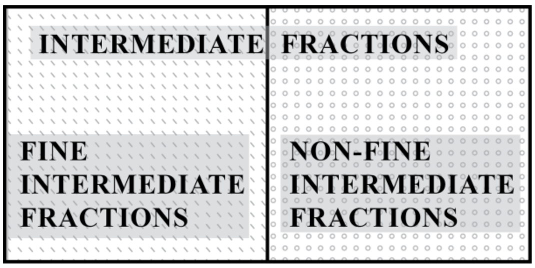

Figure 2. Intermediate fractions.

Definition 4 can be illustrated with Fig. 3.

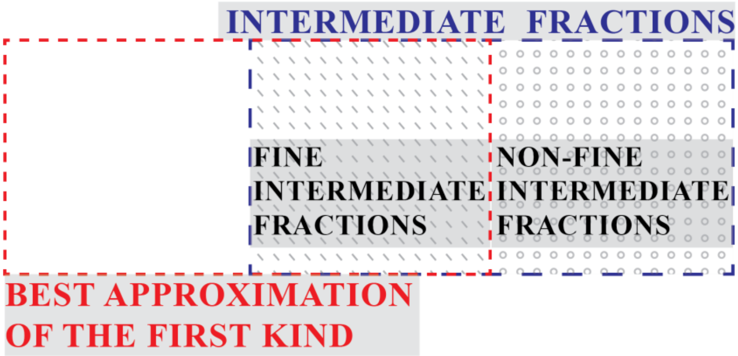

Figure 3. Definition 4.

The following theorem discusses the connection between the best approximations of the first, the best approximations of the second kind and intermediate fractions.

Theorem 9. *Every best approximation of the first kind of a number α is either the best approximation of the second kind or an intermediate fraction of the continued fraction representing that number.*

Theorem 9 can be illustrated with Fig. 4.

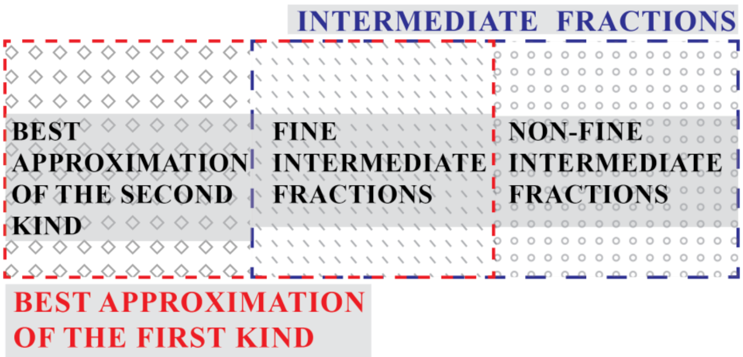

Figure 4. Theorem 9.

4. SOME STATEMENTS ABOUT INTERMEDIATE FRACTIONS AND FINE INTERMEDIATE FRACTIONS

One of the main results of this paper are the following statements, which discuss nature of intermediate fractions.

Statement 10. *Intermediate fractions, if there are any between two consecutive convergents of the number α, $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$, are all located on the same side of the number α as the convergent $\frac{p_{i+1}}{q_{i+1}}$.*

$$\frac{p_{i+1}}{q_{i+1}} > \alpha \Rightarrow \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} > \alpha, \quad j \in \{1,2, \dots, a_{i+1} - 1\} \tag{22}$$

$$\frac{p_{i+1}}{q_{i+1}} < \alpha \Rightarrow \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} < \alpha, \quad j \in \{1,2,\dots,a_{i+1}-1\} \tag{23}$$

Proof: If we assume that $\frac{p_{i+1}}{q_{i+1}} > \alpha$, we need to prove that $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} > \alpha$. We can use recurrent formulas (6) to determine convergents:

$$p_{i+1} = a_{i+1} \cdot p_i + p_{i-1},$$

$$q_{i+1} = a_{i+1} \cdot q_i + q_{i-1}.$$

According to Definition 3, there is a natural number j such that the intermediate fractions are determined by recurrent formulas (13):

$$p_{i+1}^{(j)} = j \cdot p_i + p_{i-1},$$

$$q_{i+1}^{(j)} = j \cdot q_i + q_{i-1}.$$

Now, we establish a dependency between $p_{i+1}^{(j)}$ and p_{i+1}:

$$\begin{gathered} p_{i+1}^{(j)} = j \cdot p_i + p_{i-1} \\ p_{i+1}^{(j)} = j \cdot p_i + p_{i-1} + a_{i+1} \cdot p_i - a_{i+1} \cdot p_i \\ p_{i+1}^{(j)} = a_{i+1} \cdot p_i + p_{i-1} + (j - a_{i+1}) \cdot p_i \end{gathered} \tag{24}$$

and from there we obtain

$$p_{i+1}^{(j)} = p_{i+1} + (j - a_{i+1}) \cdot p_i. \tag{25}$$

We establish a dependency between $q_{i+1}^{(j)}$ and q_{i+1}:

$$q_{i+1}^{(j)} = q_{i+1} + (j - a_{i+1}) \cdot q_i. \tag{26}$$

The inequality $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} > \alpha$ is equivalent to the following inequalities:

$$\begin{gathered} p_{i+1}^{(j)} > \alpha \cdot q_{i+1}^{(j)} \\ p_{i+1} + (j - a_{i+1}) \cdot p_i > \alpha \cdot (q_{i+1} + (j - a_{i+1}) \cdot q_i) \\ p_{i+1} + (j - a_{i+1}) \cdot p_i > \alpha \cdot q_{i+1} + \alpha \cdot (j - a_{i+1}) \cdot q_i \\ p_{i+1} - \alpha \cdot q_{i+1} > \alpha \cdot (j - a_{i+1}) \cdot q_i - (j - a_{i+1}) \cdot p_i \end{gathered} \tag{27}$$

$$p_{i+1} - \alpha \cdot q_{i+1} > (j - a_{i+1}) \cdot (\alpha \cdot q_i - p_i)$$

and from there we obtain

$$\frac{p_{i+1}}{q_{i+1}} - \alpha > (a_{i+1} - j) \cdot \left(\frac{p_i}{q_i} - \alpha\right) \cdot \frac{q_i}{q_{i+1}}. \quad (28)$$

Let us analyse each member of the inequality (28):

- $\frac{p_{i+1}}{q_{i+1}} - \alpha > 0$ based on the assumption from the beginning of this proof, $\frac{p_{i+1}}{q_{i+1}} > \alpha$,
- $(a_{i+1} - j) > 0$,
- $\left(\frac{p_i}{q_i} - \alpha\right) < 0$ based on Theorem 4, $\frac{p_{i+1}}{q_{i+1}} > \alpha \Rightarrow \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} > \alpha$,
- $\frac{q_i}{q_{i+1}} > 0$.

Based on this, the left side of the inequality (28) is positive, and on the right side one member is negative, which makes whole a right side product negative. From there we can conclude that the inequality (28) is true. Therefore, the inequality $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} > \alpha$ is also true, which is what we were supposed to prove. ■

Example 3. Let us observe the number π and its convergents $\frac{3}{1} < \pi$ and $\frac{22}{7} > \pi$. The intermediate fractions between them $\frac{4}{1}, \frac{7}{2}, \frac{10}{3}, \frac{13}{4}, \frac{16}{5}$ and $\frac{19}{6}$, are on the same side of the number π as $\frac{22}{7}$ (see Table 2). ■

Statement 11. *For intermediate fractions, if there are any between two consecutive convergents of the number* α, $\frac{p_i}{q_i}$ *and* $\frac{p_{i+1}}{q_{i+1}}$, *the following applies*

$$\left|\alpha - \frac{p_{i+1}^{(1)}}{q_{i+1}^{(1)}}\right| > \left|\alpha - \frac{p_{i+1}^{(2)}}{q_{i+1}^{(2)}}\right| > \cdots > \left|\alpha - \frac{p_{i+1}^{(a_{i+1}-1)}}{q_{i+1}^{(a_{i+1}-1)}}\right| > \left|\alpha - \frac{p_{i+1}}{q_{i+1}}\right|, \quad (29)$$

where $j \in \{1,2,\dots,a_{i+1}-1\}$.

Proof: Let us suppose that between two consecutive convergents of the numbers α, $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$, there are at least two intermediate fractions, $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$, where $j \in \{1,2,\dots,a_{i+1}-1\}$. As stated in Theorem 4, two consecutive convergents are on the different sides of the number α.

One is smaller and the other is larger than α. Suppose, for example, that $\frac{p_i}{q_i} < \alpha < \frac{p_{i+1}}{q_{i+1}}$. Based on Statement 10 and its proof, we know that all intermediate fractions $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$ are from the same side of the number α, as for the convergents $\frac{p_{i+1}}{q_{i+1}}$, (22) and (23). This means that

$$\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}} - \alpha > 0.$$

According to Definition 3, there is a natural number j such that the intermediate fractions are determined by recurrent formulas (13):

$$p_{i+1}^{(j+1)} = (j+1) \cdot p_i + p_{i-1} = p_{i+1}^{(j)} + p_i,$$
$$q_{i+1}^{(j+1)} = (j+1) \cdot q_i + q_{i-1} = q_{i+1}^{(j)} + q_i. \tag{30}$$

Now it is needed to be proved that $\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}} < \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$. Based on the previous consideration, we can state the following:

$$\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}} - \alpha = \frac{p_{i+1}^{(j)} + p_i}{q_{i+1}^{(j)} + q_i} - \alpha = \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} - \alpha + \frac{q_{i+1}^{(j)} \cdot p_i - q_i \cdot p_{i+1}^{(j)}}{{q_{i+1}^{(j)}}^2 + q_{i+1}^{(j)} \cdot q_i}. \tag{31}$$

In order for inequality $\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}} < \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$ to be correct, the inequality $q_{i+1}^{(j)} \cdot p_i - q_i \cdot p_{i+1}^{(j)} < 0$ needs to be true. Based on the initial assumption $\frac{p_i}{q_i} < \alpha < \frac{p_{i+1}}{q_{i+1}}$ and Statement 10, we can state that

$$\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} > \frac{p_i}{q_i}, \tag{32}$$

and from there

$$q_i \cdot p_{i+1}^{(j)} > q_{i+1}^{(j)} \cdot p_i, \tag{33}$$

and

$$q_{i+1}^{(j)} \cdot p_i - q_i \cdot p_{i+1}^{(j)} < 0. \tag{34}$$

From previous we obtain

$$\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}} - \alpha = \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} - \alpha + \frac{q_{i+1}^{(j)} \cdot p_i - q_i \cdot p_{i+1}^{(j)}}{{q_{i+1}^{(j)}}^2 + q_{i+1}^{(j)} \cdot q_i} < \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}} - \alpha, \tag{35}$$

and from there

$$\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}} < \frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}. \tag{36}$$

Thus, the fraction $\frac{p_{i+1}^{(j+1)}}{q_{i+1}^{(j+1)}}$, from the sequence of the intermediate fractions, is a better approximation than the fraction $\frac{p_{i+1}^{(j)}}{q_{i+1}^{(j)}}$. ■

Example 4. For intermediate fractions $\frac{4}{1}$, $\frac{7}{2}$, $\frac{10}{3}$, $\frac{13}{4}$, $\frac{16}{5}$ and $\frac{19}{6}$, which are between consecutive convergents $\frac{3}{1}$ and $\frac{22}{7}$ of the number π, we can establish the following relation (see Table 2):

$$\left|\pi - \frac{4}{1}\right| > \left|\pi - \frac{7}{2}\right| > \left|\pi - \frac{10}{3}\right| > \left|\pi - \frac{13}{4}\right| > \left|\pi - \frac{16}{5}\right| > \left|\pi - \frac{19}{6}\right| > \left|\pi - \frac{22}{7}\right|. \tag{37}$$

Definition 5. There are intermediate fractions between the two consecutive convergents of a number α, $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$. Intermediate fraction

$$\frac{p_{i+1}^{(a'_{i+1})}}{q_{i+1}^{(a'_{i+1})}} = [a_0; a_1, \ldots, a_i, a'_{i+1}], \tag{38}$$

where $0 < a'_{i+1} < a_{i+1}$, which fulfils the condition

$$\left|\alpha - \frac{p_{i+1}^{(1)}}{q_{i+1}^{(1)}}\right| > \left|\alpha - \frac{p_{i+1}^{(2)}}{q_{i+1}^{(2)}}\right| > \cdots > \left|\alpha - \frac{p_{i+1}^{(a'_{i+1}-1)}}{q_{i+1}^{(a'_{i+1}-1)}}\right| > \left|\alpha - \frac{p_i}{q_i}\right| > \left|\alpha - \frac{p_{i+1}^{a'_{i+1}}}{q_{i+1}^{a'_{i+1}}}\right|, \tag{39}$$

is called *the first fine intermediate fraction*.

Example 5. Intermediate fraction $\frac{13}{4}$, which is in the sequence of intermediate fractions $\frac{4}{1}$, $\frac{7}{2}$, $\frac{10}{3}$, $\frac{13}{4}$, $\frac{16}{5}$ and $\frac{19}{6}$ between consecutive convergents $\frac{3}{1}$ and $\frac{22}{7}$ of the number π, is the first fine intermediate fraction (see Table 2). Inequalities presented by (39), then clearly state:

$$\left|\pi - \frac{4}{1}\right| > \left|\pi - \frac{7}{2}\right| > \left|\pi - \frac{10}{3}\right| > \left|\pi - \frac{3}{1}\right| > \left|\pi - \frac{13}{4}\right|. \tag{40}$$

The following statement is based on Statement 11 and Definition 5.

Statement 12. *For fine intermediate fractions, if there are any between two consecutive convergents of the number* α, $\frac{p_i}{q_i}$ *and* $\frac{p_{i+1}}{q_{i+1}}$, *the following applies*

$$\left|\alpha - \frac{p_i}{q_i}\right| > \left|\alpha - \frac{p_{i+1}^{(a'_{i+1})}}{q_{i+1}^{(a'_{i+1})}}\right| > \left|\alpha - \frac{p_{i+1}^{(a'_{i+1}+1)}}{q_{i+1}^{(a'_{i+1}+1)}}\right| > \cdots > \left|\alpha - \frac{p_{i+1}^{(a_{i+1}-1)}}{q_{i+1}^{(a_{i+1}-1)}}\right| > \left|\alpha - \frac{p_{i+1}}{q_{i+1}}\right|. \quad (41)$$

Example 6. For convergents $\frac{3}{1}$ and $\frac{22}{7}$ of the number π and fine intermediate fractions between them $\frac{13}{4}$, $\frac{16}{5}$ and $\frac{19}{6}$ we can establish the following relation (see Table 2):

$$\left|\pi - \frac{3}{1}\right| > \left|\pi - \frac{13}{4}\right| > \left|\pi - \frac{16}{5}\right| > \left|\pi - \frac{19}{6}\right| > \left|\pi - \frac{22}{7}\right|. \quad (42)$$

5. CONCLUSION

By introducing the notion of fine intermediate fraction and listing some new features of such continued fractions, we have come to the conclusion that they also have real application in many fields of science and techniques that use the theory of continued fractions, [6, 8, 10-18]. Some of these real-world applications will be the subject of future research and quality analysis of rational approximations.

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