ORIGINAL PAPER

INVESTIGATION OF SOME RESULTS ARISING FROM POST QUANTUM CALCULUS

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Abstract. *This article is pedestal for the* (*p*,*q*)*-calculus connecting two concepts of* (*p*,*q*)*-derivatives and* (*p*,*q*)*-integrals. The purpose of this paper is to establish different type of identities for* (*p*,*q*)*-calculus. Some special cases of the* (*p*,*q*)*-midpoint, Simpson, Averaged midpoint trapezoid, and trapezoid type integral identities are also derived.*

Keywords: (*p*,*q*)*-derivative;* (*p*,*q*)*-integrals;* (*p*,*q*)*-Hermit–Hadamard's inequality;* (*p*,*q*)*-midpoint type inequality.*

1. INTRODUCTION

In mathematics, the investigation of calculus with no limits is known in the literature as quantum calculus (also called q-calculus). The famous mathematician Euler instituted the study of q-calculus in the eighteenth century by proposing the parameter q in Newton's work of infinite series. In early twentieth century, Jackson [1] has started a symmetric study of q-calculus and developed q-definite integrals. Quantum calculus has huge applications in many mathematical areas. These new quantum assessments for Hermite Hadamard type inequalities have potential applications in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory, number theory, orthogonal theory of relativity and numerical analysis.

Quantum calculus has received exceptional interest by many researchers and hence it appears as a connection between mathematics and physics. Interested readers are referred to [2-4] for some current advances in the theory of quantum calculus and theory of inequalities. Recently, Tunç and Göv [5-7] studied the concept of the (*p*,*q*)-calculus over the intervals of $[a, b] \subset \mathbb{R}$. The the (*p*,*q*) derivative and the (*p*,*q*) integral were explain and some basic properties are given. Furthermore, they obtained some new result for the the (*p*,*q*)-calculus of several important integral inequalities. Currently, the (*p*,*q*)-calculus is being investigated extensively by many researchers, and a variety of new results can be found in the literature [8-13] and the references cited therein.

In 1893, Hadamard [14] explored one of the fundamental inequalities as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

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In 2014, Tariboon and Ntouyas [15] investigated the extension to q-calculus on the finite interval of (1), which is called the q-Hermite–Hadamard inequality, and some important inequalities:

$$f(\frac{qa+b}{1+q}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{qf(a)+f(b)}{1+q}$$

In 2018, Alp et al [16] proved the (p, q) Hermite –Hadamard inequality,

$$f\left(\frac{qa+pb}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)dx \leq \frac{qf(a)+pf(b)}{p+q}$$

2. PRELIMINARIES AND AUXILIARY RESULTS

In this section, we recall some previously known concepts and basic results. Throughout this section, we soppose f = [a, b] ⊂ ℝ be an interval and q be a constant with, $0 < q < p \leq 1$. The definations for (p, q)- derivative and (p, q)- integral were given in [5-6].

Definition 2.1. Let $f : [a;\ b] \to \mathbb{R}$ be a continuous function, and let $x \in [a,b]$. Then, the (p, q)-derivative of f on $[a,b]$ at x is is characterized by the expression

$$\mathrm{aDp,q}\, f\ (\mathrm{t}) = \frac{f(pt+(1-p)a)-f(qt+(1-q)a)}{(p-q)(t-a)} \quad \mathrm{t} \neq \mathrm{a}. \tag{2.1}$$

$$\mathrm{aDp,q}\, f(\mathrm{a}) = \lim_{t\to a} \mathrm{aD_{p,q}} f\ (\mathrm{t})\ .$$

Obviously, a function f is said to be $(\mathrm{p,q})$--differentiable on $[\mathrm{a,b}]$, if $\mathrm{aDp,q}f(\mathrm{t})$ exists for all $\mathrm{t} \in [\mathrm{a,b}]$.

If $\mathrm{a} = 0$ in (2.1), then $0\mathrm{Dp,q}f(\mathrm{t}) = \mathrm{Dp,q}f(\mathrm{t})$ is familiar $(\mathrm{p,q})$ derivative of f at t ∈[a,b] defined by the expression

$$\mathrm{Dp,q}\, f(\mathrm{t}) = \frac{f(pt)-f(qt)}{(p-q)t} \quad \mathrm{t} \neq 0 \tag{2.2}$$

Furthermore, if $\mathrm{p} = 1$ in (2.2), then it reduces to Dq f (x), which is the q -derivative of the function

$$\mathrm{D_q} f(\mathrm{t}) = \frac{f(t)-f(\mathrm{qt})}{(1-\mathrm{q})\mathrm{t}}\ , \qquad \mathrm{t}\neq 0 \tag{2.3}$$

Definition 2.2. Let $f : [\mathrm{a};\ \mathrm{b}] \to \mathbb{R}$ be a continuous function. The definite (p, q) integral on $[\mathrm{a,b}]$ is defined as:

$$\int_{\mathrm{a}}^{\mathrm{t}} f(\mathrm{x})_{\mathrm{a}}\, d_{(p,q)}\mathrm{x} = (p-q)(t-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left\{\frac{q^n}{p^{n+1}}t + \left(1-\frac{q^n}{p^{n+1}}\right)a\right\} \tag{2.4}$$

for $t \in [a, b]$. If $c \in (a, t)$, then the $(\mathrm{p,q})$- definite integral on $[c, t]$ is expressed as:

$$\int_c^t f(x)\, \mathrm{ad(p,q)x} = \int_a^t f(x)\, \mathrm{ad(p,q)x} - \int_a^c f(x)\, \mathrm{ad(p,q)x} \tag{2.5}$$

If $p = 1$ in (2.4), then one can get the classical q-definite integral on $[a, b]$ defined by

$$\int_{\mathrm{a}}^{\mathrm{t}} f(\mathrm{x})_{\mathrm{a}}\, d_{(1,q)}x = (1-q)(t-a)\sum_{n=0}^{\infty} q^n f\{q^n t + (1-q^n)a\} \tag{2.6.}$$

If $a = 0$ in (2.4), then one can get the classical (p, q)-definite integral defined by

$$\int_0^{\mathrm{t}} f(\mathrm{x})_0\, d_{(p,q)}\mathrm{x} = (p-q)t\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\, f\left(\frac{q^n}{p^{n+1}}\mathrm{t}\right).$$

The proofs of the following lemmas were given in [17].

Lemma 2.1 Let $f\colon \mathrm{I} \to \mathbb{R}$ be a continous and q-differentiable function on I^0 with $0 < q < 1$. Thus the identity

$$\lambda[\mu f(\mathrm{b}) + (1-\mu)f(\mathrm{a})] + (1-\lambda)f(\mu\mathrm{b} + (1-\mu)\mathrm{a}) - \left(\frac{1}{b-a}\right)\int_{\mathrm{a}}^{\mathrm{b}} f(\mathrm{x})_{\mathrm{a}}\, d_q\mathrm{x}$$
$$= (\mathrm{b}-\mathrm{a})\left[\left\{\int_0^{\mu} (\mathrm{qt} + \lambda\mu - \lambda)\mathrm{aD_q}\, f(\mathrm{tb} + (1-\mathrm{t})\mathrm{a})_0 \mathrm{d_q t}\right\} + \left\{\int_{\mu}^{1} (\mathrm{qt} + \lambda\mu - 1)\mathrm{aD_q}\, f(\mathrm{tb} + (1-\mathrm{t})\mathrm{a})_0 \mathrm{d_q t}\right\}\right]$$

holds for all $\lambda, \mu \in [0,1]$, if $\mathrm{aDq}\, f$ is integrable on I .

Lemma 2.2. Let $\mu \in [0, 1]$ and $\daleth \in [0,\infty)$

$$\int_0^{\mu} t^{\daleth} \mathrm{d_q}\, \mathrm{t} = (1-\mathrm{q})\sum_{\mathrm{n}=0}^{\infty} \mu^{\daleth+1}\, \mathrm{q}^{(\daleth+1)\mathrm{n}} = \frac{\mu^{\daleth+1}(1-\mathrm{q})}{1-\mathrm{q}^{\daleth+1}}$$

Lemma 2.3. Let $\lambda, \mu \in [0,1]$ *and* $\tau \in [0, \infty)$. Then we have

$$\int_0^{\mu} \mathrm{t}^{\tau}\, |\mathrm{qt} - (\lambda - \lambda\mu)|_0 \mathrm{d_q t} =$$
$$\begin{cases} \dfrac{\mu^{\daleth+1}(1-\mathrm{q})(\lambda-\lambda\mu)}{1-\mathrm{q}^{\tau+1}} - \dfrac{\mu^{\daleth+2} q(1-\mathrm{q})}{1-\mathrm{q}^{\tau+2}} & \lambda\mu + q \le \lambda \\ \dfrac{2(1-q)^2(\lambda-\lambda\mu)^{\jmath+2}}{(1-q^{\jmath+1})(1-q^{\jmath+2})} + \dfrac{\mu^{\daleth+2} q(1-q)}{1-q^{\jmath+2}} - \dfrac{\mu^{\daleth+1}(1-q)(\lambda-\lambda\mu)}{1-q^{\jmath+1}} & \lambda\mu + q > \lambda \end{cases}$$

$$\begin{cases} (1-\mathrm{q})\sum_{n=0}^{\infty} q^n\,(\lambda-\lambda\mu-q^{n+1})(1-q^n)^{\tau}, & \lambda\mu+q\leq\lambda \\ \begin{pmatrix} 2(1-q)(\lambda-\lambda\mu)^2\sum_{n=0}^{\infty} q^n(1-q^n)[1-q^{n-1}(\lambda-\lambda\mu)]^{\tau} \\ -(1-\mathrm{q})\sum_{n=0}^{\infty} q^n\,(\lambda-\lambda\mu-q^{n+1})(1-q^n)^{\tau}, \ \lambda\mu+q>\lambda \end{pmatrix} \end{cases}$$

Lemma 2.4. Let $\lambda,\in[0,1]and\ \tau\ \in[0,\infty)$. Then we have

$$\int_0^1 \mathrm{t}^{\tau}\,|\mathrm{qt}-(1-\lambda\mu)|_0 \mathrm{d_q t} =$$

$$\begin{cases} \frac{(1-\mathrm{q})(1-\lambda\mu)}{1-\mathrm{q}^{\tau+1}}-\frac{q(1-\mathrm{q})}{1-\mathrm{q}^{\tau+2}} & \lambda\mu+q\leq 1, \\ \frac{2(1-q)^2(1-\lambda\mu)^{\jmath+2}}{(1-q^{\jmath+1})(1-q^{\jmath+2})}+\frac{q(1-q)}{1-q^{\jmath+2}}-\frac{(1-q)(1-\lambda\mu)}{1-q^{\jmath+1}} & \lambda\mu+q>1 \end{cases}$$

$$\begin{cases} (1-\mathrm{q})\sum_{n=0}^{\infty} q^n\,(1-\lambda\mu-q^{n+1})(1-q^n)^{\tau}, \ \lambda\mu+q\leq 1 \\ \begin{pmatrix} 2(1-q)(1-\lambda\mu)^2\sum_{n=0}^{\infty} q^n(1-q^n)[1-q^{n-1}(1-\lambda\mu)]^{\tau} \\ -(1-\mathrm{q})\sum_{n=0}^{\infty} q^n\left(1-\lambda\mu-q^{n+1}\right)(1-q^n)^{\tau}, \quad \lambda\mu+q\leq 1 \end{pmatrix} \end{cases}$$

Lemma 2.5. Let $\lambda,\,\mu\in[0,1]and\ \ \theta\in[0,\infty)$. Then we have

$$\int_0^1 |\mathrm{qt}-(1-\lambda\mu)|^{\theta}_{\ 0}\ \mathrm{d_q t} =$$

$$\begin{cases} (1-\mathrm{q})\sum_{n=0}^{\infty} q^n\,(1-\lambda\mu-q^{n+1}\mu)^{\theta}, \quad 0\leq\lambda\mu\leq 1-q, \\ (1-q)\begin{pmatrix} (1-\lambda\mu)^{\theta+1}\sum_{n=0}^{\infty} q^{n-1}(1-q^n)^{\theta}+\sum_{n=0}^{\infty} q^n(q^{n+1}-1+\lambda\mu)^{\theta} \\ -(1-\lambda\mu)^{\theta+1}\sum_{n=0}^{\infty} q^{n-1}\,(q^n-1)^{\theta} \quad, \quad (1-q)<\lambda\mu\leq 1 \end{pmatrix} \end{cases}$$

3. RESULTS AND DISCUSSION

In this section, we introduce some new post quantum -integral identites and post quantum estimates for midpoint, Simpson, averaged midpoint-trapezoid, and trapezoid -type integral identites.

Lemma 3.1. Let f : I→ ℝ be a continuous and differentiable function on I with $0<q<p\leq 1$. Then the identity

$$\lambda\left[\int_0^{\mu} \mu f(b)+(1-\mu)f(a)\right]+(1-\lambda)f(\mu b+(1-\mu)a)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)\ a d_{p,q}\,x$$

$$= (\mathrm{b}-\mathrm{a}) \int_0^{\mu} (qt+\lambda\mu-\lambda)_a\, D_{p,q} f(tb+(1-t)a)_0 d_{p,q}t + \int_{\mu}^{1} (qt+\lambda\mu-1)_a\, D_{p,q} f(tb + (1-t)a)_0 d_{p,q}t$$

Holds for all $\lambda, \mu \in [0,1], if$ aDp, qf $is\ intergrable\ on\ I$.

Proof: Using of identity transformation, we attain

$$(\mathrm{b}-\mathrm{a}) \int_0^{\mu} (qt+\lambda\mu-\lambda)_a\, D_{p,q} f(tb+(1-t)a)_0 d_{p,q}t + \int_{\mu}^{1} (qt+\lambda\mu-1)_a\, D_{p,q} f(tb + (1-t)a)_0 d_{p,q}t$$

$$= (\mathrm{b}-\mathrm{a}) \left\{ \int_0^1 (qt+\lambda\mu-1) \mathrm{aDp,q} f(tb+(1-t)a) d_{p,q}t + \int_0^{\mu} (1-\lambda) \mathrm{aDp,q} f(tb + 1-tadp,qt \right. \tag{3.1}$$

From definition 2.1, we get

$$\mathrm{aDp,q} f(\mathrm{tb}+(1-\mathrm{t})\mathrm{a}) = \frac{f(ptb+(1-pt)a) - f[q(tb+(1-t)a)] + (1-q)a}{(p-q)(tb+(1-t)a-a)}$$
$$= \frac{f(ptb+(1-pt)a) - f[qtb+(1-qt)a]}{t(p-q)(b-a)}$$

Make use of the above calculation and Definiation 2.2, we have

$$\int_0^1 tD_{p,q} f(tb+(1-t)a)_a\, d_{p,q}t = \int_0^1 \frac{f(ptb+(1-pt)a) - f(qtb+(1-qt)a)}{(p-q)(b-a)} a d_{p,q}t.$$

$$= \frac{1}{b-a}\left[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f(\frac{q^n}{p^n} + \left(1-\frac{q^n}{p^n}\right)a\right) - \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^{n+1}b}{p^{n+1}} + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)a\right) \right]$$

$$= \frac{1}{b-a}\left\{ \frac{1}{q} f(b) - (\frac{1}{q}-\frac{1}{p}) \sum_{n=1}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n b}{p^n} + \left(1-\frac{q^n}{p^n}\right)a\right) \right\}$$

$$= \frac{1}{b-a}\left\{ \frac{1}{q} f(b) - (\frac{p-q}{qp}) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n b}{p^n} + \left(1-\frac{q^n}{p^n}\right)a\right) \right\}$$

$$= \frac{f(b)}{q(b-a)} - \frac{1}{qp(b-a)^2} \int_a^{pb+(1-p)a} f(x)_a d_{p,q}x \tag{3.2}$$

$$= \int_0^1 D_{p,q} f(tb+(1-t)a)_a\, d_{p,q}t$$

$$= \int_0^1 \frac{f(ptb+(1-pt)a) - f[qtb+(1-qt)a+(1-q)a)]}{t(p-q)(b-a)} d_{p,q}t,$$

$$= \frac{1}{b-a}\left\{ \sum_{n=0}^{\infty} f\left(\frac{q^n}{p^n} b + \left(1-\frac{q^n}{p^n}\right)a\right) - \sum_{n=0}^{\infty} f\left(\frac{q^{n+1}}{p^{n+1}} b + \left(1-\frac{q^{n+1}}{p^{n+1}}\right)a\right) \right\}$$

$$= \frac{f(b)-f(a)}{b-a}. \tag{3.3}$$

and

$$\int_0^{\mu} D_{p,q} f(tb+(1-t)a)_a \, d_{p,q}t = \int_0^{\mu} \frac{f(ptb+(1-pt)a) - f(qtb+(1-qt)a)}{t(p-q)(b-a)} d_{p,q}t$$

$$= \frac{1}{b-a}\left\{\sum_{n=0}^{\infty} f(\frac{q^n p}{p^{n+1}}\mu b + \left(1-\frac{q^n p}{p^{n+1}}\mu)a\right) - \sum_{n=0}^{\infty} f(\frac{q^{n+1}}{p^{n+1}}\mu b + \left(1-\frac{q^{n+1}}{p^{n+1}}\mu)a\right)\right\}.$$

$$= \frac{f(\mu b+(1-\mu)a)-f(a)}{b-a}. \tag{3.4}$$

Substituting (3.2), (3.3) and (3.4) into (3.1), we get the desired result.

Remark 3.2: In Lemma 3.1, if we take take p $=1$, then we recapture Lemma 2.1.

Remark 3.3: Consider lemma 3.1.

(i) Putting $\mu = 0, we\ have$

$$f\text{ (a)} - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x)_a \, d_{p,q}x = (b-a)\int_0^1 (qt-1)_a D_{p,q} f(tb+(1-t)a)_0 d_{p,q}t.$$

(ii) Putting $\mu = 1, we\ have$

$$f\text{ (b)} - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x)_a \, d_{p,q}x = (b-a)\int_0^1 qt_a D_{p,q} f(tb+(1-t)a)_0 d_{p,q}t.$$

(iii) Putting $\mu = \frac{p}{p+q}$

$$\lambda \frac{Pf(b)+qf(a)}{p+q} + (1-\lambda) f\left(\frac{pb+qa}{p+q}\right) - \frac{1}{p(b-a)} \int_a^{pb+(1-p)a} f(x)_0 \, d_{p,q}x$$

$$= (\mathrm{b}-a)\left\{\int_0^{\frac{p}{p+q}} \left(qt - \frac{\lambda q}{p+q}\right) D_{p,q} f(tb+(1-t)a)_0 d_{p,q}t + \int_{\frac{p}{p+q}}^{1} \left(qt - \frac{\lambda q}{p+q} - 1\right) D_{p,q} f(tb \right.$$

$$\left. + (1-t)a)_0 d_{p,q}t \right\}$$

Remark 3.4. Consider Lemma 3.1.:

(i) Putting λ=0, we get

$$f(\mu b+(1-\mu)a)-\frac{1}{b-a}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x=(b-a)$$

$$\left\{\int_0^{\mu} qt_a\, D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t+\int_{\mu}^{1} qt_a\, D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t\right\}$$

Substituting $\mu=\frac{p}{p+q}$ (p,q)- midpoint –type intergral identity is obtained and previously proved in [16].

$$f\left(\frac{pb+qa}{p+q}\right)-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x$$

$$=\left\{\int_0^{\frac{p}{p+q}} qt_a\, D_{p,q}f(tb+(1-t)a)_0 d_q t+\int_{\frac{p}{p+q}}^{1}(qt-1)\, D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t\right\}$$

(ii) Putting $\lambda=\frac{1}{3}$, we get

$$\frac{1}{3}[\mu f(\mathrm{b})+(1-\mu)f(a))+2f(\mu\mathrm{b})+(1-\mu)a)]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x$$

$$=(b-a)\left\{\int_0^{\mu}\left(qt+\frac{1}{3}\mu-\frac{1}{3}\right)\ {}_aD_{p,q}f(tb+(1-t)a)_0 d_q t+\int_{\mu}^{1}\left(qt+\frac{1}{3}\mu-1\right)_a D_{p,q}f(tb\right.$$

$$\left.+(1-t)a)_0 d_{p,q}t\right\}$$

Specially taking $\mu=\frac{p}{p+q}$, we obtain the simpson-type integral identity.

$$\frac{1}{3}\left[\frac{qf(a)+pf(b)}{p+q}+2f\left(\frac{pb+qa}{p+q}\right)\right]-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x=$$

$$=(b-a)\left\{\int_0^{\frac{p}{p+q}}\left(qt+\frac{q}{3(p+q)}\right)_a D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t\right.$$

$$\left.+\int_{\frac{p}{p+q}}^{1}\left(qt+\frac{p}{3(p+q)}-1\right)_a D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t\right\}$$

(iii) Putting $\lambda = \frac{1}{2}$, we get

$$\frac{1}{2}[\mu f(\mathrm{b}) + (1-\mu)f(a)) + f(\mu\mathrm{b} + (1-\mu)a)] - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x$$

$$= (b-a)\left\{\int_0^{\mu}\left(qt + \frac{1}{2}\mu - \frac{1}{2}\right)_a D_{p,q}f(tb+(1-t)a)_0 d_q t + \int_{\mu}^{1}\left(qt + \frac{1}{2}\mu - 1\right)_0 D_{p,q}f(tb + (1-t)a)_0 d_{p,q}t\right\}$$

Specially taking $\mu = \frac{p}{p+q}$, we obtain the averaged midpoint-trapezoid-type integral identity.

$$\frac{1}{2}\left[\frac{qf(a)+pf(b)}{p+q} + f\left(\frac{pb+qa}{p+q}\right)\right] - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x$$

$$= (b-a)\left\{\int_0^{\frac{p}{p+q}}\left(qt + \frac{q}{2(p+q)}\right)_a D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t + \int_{\frac{p}{p+q}}^{1}\left(qt + \frac{p}{2(p+q)} - 1\right)_a D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t\right\}$$

(iv) Putting λ=1, we get

$$\mu f(b) + (1-\mu)f(a) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a\, d_{p,q}x = \int_0^{\mu}(qt+\mu-1)_a D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t$$

Specially, taking $\mu = \frac{p}{p+q}$, we obtain the trapezoid-type integral identity

$$\frac{qf(a)+pf(b)}{p+q} - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)_a d_{p,q}\, x = \int_0^{\frac{p}{p+q}}\left(qt + \frac{p}{p+q} - 1\right)_a D_{p,q}f(tb+(1-t)a)_0 d_{p,q}t$$

To the best of our knowledge the above integral identites are fresh in the literature.

Lemma 3.5. Let $\mu \in [0,1]$ and $\daleth \in [0,\infty)$. Then we have

$$\int_0^{\mu} t^{\daleth} d_{p,q}t = (p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}+1\right)^{\daleth} = \frac{(p-q)\mu^{\daleth+1}}{p^{\daleth+1}-q^{\daleth+1}}$$

and

$$\int_0^{\mu}(1-t)^{\daleth} d_{p,q}t = (p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{\daleth}$$

Lemma 3.6. Let $\lambda,\mu \in [0,1]$ and $\daleth \in [0,\infty)$. Then we have

$$\int_0^{\mu} t^{\daleth}\,|qt-(\lambda-\lambda\mu)|_0 d_{p,q}t = \begin{cases} \frac{\mu^{\daleth+1}(p-q)(\lambda-\lambda\mu)}{p^{\daleth+1}-q^{\daleth+1}} - \frac{q\mu^{\daleth+2}(p-q)}{p^{\daleth+2}-q^{\daleth+2}} & (\lambda+q)\mu \le \lambda \\ \frac{2(p-q)^2(\lambda-\lambda\mu)^{J+2}\left[\left(p^{\daleth+1}+q^{\daleth+1}\right)-\frac{(1-q)p^{\daleth+1}}{(p-q)}\right]}{q^{J+1}\left(p^{\daleth+1}-q^{\daleth+1}\right)\left(p^{\daleth+2}-q^{\daleth+2}\right)} + \frac{q\mu^{\daleth+2}(p-q)}{p^{\daleth+2}-q^{\daleth+2}} & \\ -\frac{\mu^{\daleth+1}(p-q)(\lambda-\lambda\mu)}{p^{\daleth+1}-q^{\daleth+1}}. & (\lambda+q)\mu > \lambda \end{cases}$$

and

$$\int_0^{\mu}(1-t)^{\tau}\,|qt-(\lambda-\lambda\mu)|_0\, d_qt =$$

$$\begin{cases} (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\lambda-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)^{\tau}, & \lambda\mu+q \le \lambda \\ \left(\begin{array}{l} 2(p-q)(\lambda-\lambda\mu)^2\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left[1-\frac{q^{n-1}}{p^{n+1}}(\lambda-\lambda\mu)\right]^{\tau} \\ -(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\lambda-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)^{\tau}, \quad \lambda\mu+q > \lambda \end{array}\right) \end{cases}$$

Proof: When $(\lambda+q)\mu < \lambda$,by use of Lemma 3.5 we get

$$\int_0^{\mu} t^{\daleth}\,|qt-(\lambda-\lambda\mu)|_0 d_{p,q}t = \int_0^{\mu}[t^{\daleth}(\lambda-\lambda\mu)-qt^{\daleth+1}]d_{p,q}t =$$

$$\frac{\mu^{\daleth+1}(p-q)(\lambda-\lambda\mu)}{p^{\daleth+1}-q^{\daleth+1}} - \frac{q\mu^{\daleth+2}(p-q)}{p^{\daleth+2}-q^{\daleth+2}}$$

When $(\lambda+q)\mu > \lambda$, by use of Lemma 3.2 we get

$$\int_0^{\mu} t^{J}\,|qt-(\lambda-\lambda\mu)|_0 d_qt = \int_0^{\frac{\lambda-\lambda\mu}{q}}\left[(\lambda-\lambda\mu)t^{\daleth}-qt^{\daleth+1}\right]d_{p,q}t + \int_{\frac{\lambda-\lambda\mu}{q}}^{\mu}\left[qt^{\daleth+1}-(\lambda-\lambda\mu)t^{\daleth}\right]d_{p,q}t$$

$$= 2\int_0^{\frac{\lambda-\lambda\mu}{q}}\left[(\lambda-\lambda\mu)t^{\daleth}-qt^{\daleth+1}\right]d_{p,q}t + \int_0^{\mu}\left[qt^{\daleth+1}-(\lambda-\lambda\mu)t^{\daleth}\right]_0 d_{p,q}t$$

$$= [I_1 + I_2] \tag{3.5}$$

Considering that,

$$I_1 = 2\int_0^{\frac{\lambda-\lambda\mu}{q}} [(\lambda-\lambda\mu)t^{\mathrm{J}} - qt^{\mathrm{J}+1}]_o \, d_{p,q}t$$

$$=2(p-q)\left(\frac{\lambda-\lambda\mu}{q}\right)(\lambda-\lambda\mu)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\frac{\lambda-\lambda\mu}{q}\right)^{\mathrm{J}}$$

$$-(p-q)q\left(\frac{\lambda-\lambda\mu}{q}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}\frac{\lambda-\lambda\mu}{q}\right)^{\mathrm{J}+1}$$

$$=\frac{2(p-q)(\lambda-\lambda\mu)^{\mathrm{J}+2}}{q^{\mathrm{J}+1}}\left[\left(\frac{1}{p}\right)^{\mathcal{J}+1}+\left(\frac{q}{p^2}\right)^{\mathcal{J}+1}+\left(\frac{q^2}{p^3}\right)^{\mathcal{J}+1}+\cdots\ldots\ldots\ldots\right]$$

$$-\frac{(p-q)(\lambda-\lambda\mu)^{\mathrm{J}+2}}{q^{\mathrm{J}+1}}\times\left(\frac{1}{p^{\mathrm{J}+2}-q^{\mathrm{J}+2}}\right)$$

$$=\frac{2(p-q)^2(\lambda-\lambda\mu)^{\mathrm{J}+2}\times\left[p^{\mathrm{J}+1}+q^{\mathrm{J}+1}-(1-q)\frac{p^{\mathcal{J}+1}}{p-q}\right]}{q^{\mathcal{J}+1}(p^{\mathcal{J}+2}-q^{\mathrm{J}+2})(p^{\mathrm{J}+1}-q^{\mathrm{J}+1})}$$

$$I_2 = \int_0^{\mu} t^{\mathcal{J}}\,|\mathrm{qt}-(\lambda-\lambda\mu)|_0 \mathrm{d_q t}$$

$$=(p-q)q\mu^{\mathcal{J}+2}\left(\frac{1}{p^{\mathcal{J}+2}-q^{\mathrm{J}+2}}\right)-\mu^{\mathcal{J}+1}(p-q)(\lambda-\lambda\mu)\left(\frac{1}{p^{\mathcal{J}+1}-q^{\mathrm{J}+1}}\right)$$

Putting the values of I_1 *and* I_2 *in eq* (3.5) we get

$$=\begin{cases}(p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\lambda-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\mu\right)\left(1-\frac{q^n}{p^{n+1}}\mu\right)\\ \left(\begin{bmatrix}2(p-q)(\lambda-\lambda\mu)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)[1-\frac{q^n}{p^{n+1}}(\lambda-\lambda\mu)]^{\mathcal{J}}\\ -(p-q)\mu\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\lambda-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\mu\right)\left(1-\frac{q^n}{p^{n+1}}\mu\right)^{\mathcal{J}}]\end{bmatrix}\right)\end{cases}$$

This completes the proof.

Remark 3.7. In Lemma 3.6, If we take $\mathrm{p} = 1$, then we recapture Lemma 2.3.

The given results of Lemma 3.8 and 3.9 are described without proof.

Lemma 3.8 Let λ, $\mu \in [0,1]$ *and* $\tau \in [0,\infty)$. Then we have

$$\int_0^1 \mathrm{t}^{\tau}\,|\mathrm{qt}-(1-\lambda\mu)|_0 \mathrm{d_{p,q}t} =$$

$$\begin{cases} \dfrac{\mu^{\tau+1}(\mathrm{p}-\mathrm{q})(1-\lambda\mu)}{\mathrm{p}^{\tau+1}-\mathrm{q}^{\tau+1}} - \dfrac{q\mu^{\tau+2}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{\tau+2}-\mathrm{q}^{\tau+2}} & (\lambda+q)\mu \le 1, \\ \dfrac{2(p-q)^2(1-\lambda\mu)^{\jmath+2}\left[\left(p^{\jmath+1}+q^{\jmath+1}\right) - \dfrac{(1-q)p^{\jmath+1}}{(p-q)}\right]}{\mathrm{q}^{\tau+1}(\mathrm{p}^{\tau+1}-q^{\jmath+1})(\mathrm{p}^{\tau+2}-q^{\jmath+2})} & \\ +\dfrac{q\mu^{\tau+2}(p-q)}{\mathrm{p}^{\tau+2}-q^{\jmath+2}} - \dfrac{\mu^{\tau+1}(p-q)(1-\lambda\mu)}{\mathrm{p}^{\tau+1}-q^{\jmath+1}} & (\lambda+q)\mu > 1 \end{cases}$$

and

$$\int_0^1 (1-\mathrm{t})^\tau\, |\mathrm{qt}-(1-\lambda\mu)|\, {}_0\mathrm{d}_{\mathrm{p,q}}\mathrm{t} =$$

$$\begin{cases} (\mathrm{p}-\mathrm{q})\mu \displaystyle\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\mu\right)\left(1-\frac{q^n}{p^{n+1}}\mu\right)^\tau & (\lambda+q)\mu \le 1 \\ \left(\begin{array}{l} 2(p-q)(1-\lambda\mu)^2 \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left[1-\frac{q^{n-1}}{p^{n+1}}(1-\lambda\mu)\right]^\tau \\ -(\mathrm{p}-\mathrm{q})\mu \sum_{n=0}^{\infty} q^n \left(1-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\mu\right)\left(1-\frac{q^n}{p^{n+1}}\mu\right)^\tau, \quad (\lambda+q)\mu > 1 \end{array}\right) & \end{cases}$$

Lemma 3.9. Let λ, $\mu \in [0,1]$ *and* $\theta \in [0,\infty)$. Then we have

$$\int_0^1 |\mathrm{qt}-(1-\lambda\mu)|^\theta\, {}_0\mathrm{d}_{\mathrm{p,q}}\mathrm{t} =$$

$$\begin{cases} (\mathrm{p}-\mathrm{q}) \displaystyle\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1-\lambda\mu-\frac{q^{n+1}}{p^{n+1}}\mu\right)^\theta, & 0 \le \lambda\mu \le p-q, \\ (p-q)\left(\begin{array}{l} (1-\lambda\mu)^{\theta+1} \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^\theta + \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}}-1+\lambda\mu\right)^\theta \\ -\ (1-\lambda\mu)^{\theta+1} \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}-1\right)^\theta \qquad (p-q) < \lambda\mu \le 1 \end{array}\right) & \end{cases}$$

4. CONCLUSION

Utilizing post quantum derivatives and post quantum integrals, we introduce some new post quantum integral identites. For different values of μ *and* λ, we obtain (p,q)-midpoint, Simpson, Averaged midpoint trapezoid, and trapezoid type integral identities. Current work has improved some results of [17] and can be reduced to the classical quantum identity formulas in special cases with p = 1. It is suggested that the ideas and technique may be applicable for (α, m) - convex functions.

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