ORIGINAL PAPER

RULED SURFACES AND TANGENT BUNDLE OF PSEUDO-SPHERE OF NATURAL LIFT CURVES

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Abstract. *In this article, firstly, the isomorphism between the subset of the tangent bundle of Lorentzian unit sphere,* $T\tilde{M}$, *and Lorentzian unit sphere,* $\mathbb{S}_1^2$ *is represented. Secondly, the isomorphism between the subset of hyperbolic unit sphere,* $T\hat{M}$, *and hyperbolic unit sphere,* $\mathbb{H}^2$ *is given. According to E. Study mapping, any curve on* $\mathbb{S}_1^2$ *or* $\mathbb{H}^2$ *corresponds to a ruled surface in* $\mathbb{R}_1^3$. *By constructing these isomorphisms, we correspond to any natural lift curve on* $T\tilde{M}$ *or* $T\hat{M}$ *a unique ruled surface in* $\mathbb{R}_1^3$. *Then we calculate striction curve, shape operator, Gaussian curvature and mean curvature of these ruled surfaces. We give developability condition of these ruled surfaces. Finally, we give examples to support the main results.*

Keywords: *tangent bundle; ruled surface; natural lift curve; study's map.*

1. INTRODUCTION

Quite recently, considerable attention has been paid to the theories of curves and surfaces in differential geometry. In literature, definition and properties of natural lift curve that is the curve drawn by the end points of the tangents of the main curve were first encountered in J.A. Thorpe's book [1]. In $\mathbb{R}^3$, the properties of natural lift curve of a given curve were studied [2]. Frenet vector fields, the first, and second curvatures of the natural lift curve of a given curve were calculated by using the angle between Darboux vector field and the binormal vector field of the given curve [3].

Ruled surfaces have been extensively investigated in geometry, surface design, computer-aided geometric design, physics, etc. In $\mathbb{R}^3$, a ruled surface is a curved surface represented by moving a straight line along a space curve called the base curve [4]. In literature, ruled surfaces have been widely studied by different authors [5-10].

Dual numbers were introduced by W. K. Clifford in 1873. Then E. Study constructed the correspondence between the geometry of lines in $\mathbb{R}^3$ and unit dual sphere, DS^2 by defining a mapping called E. Study mapping. This mapping offers that there exists a one-to-one correspondence between the oriented straight lines in $\mathbb{R}^3$ and the points on DS^2, see [11].

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The relation among DS^2, TS^2 and non-cylindirical ruled surfaces was given, see [12]. In the light of this study, the relationship between tangent bundle of unit 2-sphere, TS^2, and unit dual sphere, DS^2 was given [13]. In that study, the correspondence between an each curve on TS^2 and ruled surface in $\mathbb{R}^3$ was given. The striction curve and the consequences of developability condition were presented. Then the isomorphism between tangent bundle of pseudo-sphere and ruled surface in Minkowski space was given by the same authors [14]. In that article, the developability condition, involute-evolute curve couples on $T\mathbb{S}_1^2$ and TH^2 were presented.

However, there is limited research into ruled surfaces generated by the natural lift curves in $\mathbb{R}_1^3$ with the isomoprhisms between the subsets of pseudo-sphere and Lorentzian unit sphere, $\mathbb{S}_1^2$ or hyperbolic unit sphere, $\mathbb{H}^2$.

The remainder of this paper is divided into four sections: Section 2 presents basic definitions and theorems about the topic. Section 3 covers the properties of dual vectors, unit dual pseudo-sphere and the isomorphisms between $T\hat{M}$ or $T\tilde{M}$ and $\mathbb{S}_1^2$ or $\mathbb{H}^2$, ruled surfaces generated by the natural lift curves on $T\hat{M}$ or $T\tilde{M}$. Section 4 deals with the developability condition for the ruled surfaces generated by the natural lift curves. In the same section, we give examples to illustrate the main results. Finally, Section 5 is about conclusion.

2. PRELIMINARIES

This section describes concepts of natural lift curve of a given curve, the tangent bundle of pseudo-sphere of natural lift curve, the theory of ruled surfaces, Weingarten map and Gauss curvature of spacelike or timelike ruled surfaces generated by the natural lift curves.

Assume that $\mathbb{R}_1^3$ is a 3-dimensional Lorentzian space. The Lorentzian scalar product is defined as

$$\langle , \rangle_L = x_1^2 + x_2^2 + x_3^2, \tag{2.1}$$

where $x = (x_1, x_2, x_3)$ is vector in $\mathbb{R}_1^3$. Additionaly, the Lorentzian product of the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given as

$$x \times_L y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1).$$

In $\mathbb{R}_1^3$, any vector has three Lorentzian characters:

(i) If $\langle x, x\rangle_L \rangle 0$ or $x = 0$, the vector x is spacelike vector,

(ii) If $\langle x, x\rangle_L \langle 0$, the vector x is timelike vector,

(iii) If $\langle x, x\rangle_L = 0$ for $x \neq 0$, the vector x is ligtlike vector.

Let $\mathbb{S}_1^2$ and $\mathbb{H}^2$ be Lorentzian unit sphere and hyperbolic unit sphere in $\mathbb{R}_1^3$, respectively. The sets $\mathbb{S}_1^2$ and $\mathbb{H}^2$ are given in the following equations [7]:

$$\mathbb{S}_1^2 = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}_1^3 : \langle x, x \rangle_L = 1 \right\},$$

$$\mathbb{H}^2 = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}_1^3 : \langle x, x \rangle_L = -1 \right\}.$$

Tangent bundles of the Lorentzian sphere, $\mathbb{S}_1^2$ and hyperbolic unit sphere, $\mathbb{H}^2$ are presented as follows [7]:

$$T\mathbb{S}_1^2 = \left\{ (\gamma, v) \in \mathbb{S}_1^2 \times \mathbb{R}_1^3 : \langle \gamma, v \rangle_L = 0 \right\},$$

$$T\mathbb{H}^2 = \left\{ (\gamma, v) \in \mathbb{H}^2 \times \mathbb{R}_1^3 : \langle \gamma, v \rangle_L = 0 \right\}.$$

Definition 2.1. Let $\Gamma : I \to \bar{M}$ be a curve. Here $\bar{M}$ represents a hypersurface on the Lorentzian sphere or hyperbolic unit sphere. Γ is called an integral curve of X

$$\frac{d(\Gamma(t))}{dt} = X(\Gamma(t)), \tag{2.2}$$

where X is smooth tangent vector field on $\bar{M}$ [5].

Definition 2.2. For the curve Γ, $\bar{\Gamma}$ is defined as the natural lift of $\bar{\Gamma}$ on $T\bar{M}$, which produces in the following equation:

$$\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t)) = (\gamma'(t)_{\gamma(t)}, v'(t)_{v(t)}). \tag{2.3}$$

$\gamma'(t)_{\gamma(t)}$ and $v'(t)_{v(t)}$ are the derivatives of $\gamma(t)$ and $v(t)$, respectively, see [5]. Here D refers the Levi-Civita connection in $\mathbb{R}_1^3$. We have

$$T\bar{M} = \bigcup T_p\bar{M}, \quad p \in \bar{M}, \tag{2.4}$$

where $T_p\bar{M}$ is the tangent space of $\bar{M}$ at p and $\chi(\bar{M})$ is the space of vector fields on the hypersurface $\bar{M}$. Furthermore, we can write

$$\frac{d(\bar{\Gamma}(t))}{dt} = \frac{d(\Gamma'(t)_{\Gamma(t)})}{dt} = D_{\Gamma'(t)}\Gamma(t). \tag{2.5}$$

Let $T\tilde{M}$ and $T\hat{M}$ be the subsets of $T\mathbb{S}_1^2$ and $T\mathbb{H}^2$, respectively. $T\tilde{M}$ and $T\hat{M}$ are defined as follows:

$$T\tilde{M} = \{(\bar{\gamma}, \bar{v}) \in \mathbb{R}_1^3 \times \mathbb{R}_1^3 : \langle \bar{\gamma}, \bar{\gamma} \rangle_L = 1, \langle \bar{\gamma}, \bar{v} \rangle_L = 0\},$$

$$T\hat{M}=\{(\overline{\gamma},\overline{v})\in\mathbb{R}_1^3\times\mathbb{R}_1^3:\langle\overline{\gamma},\overline{\gamma}\rangle_L=-1,\langle\overline{\gamma},\overline{v}\rangle_L=0\}.$$

Here $\overline{\gamma}(t)$ and $\overline{v}(t)$ present the derivatives of $\gamma(t)$ and $v(t),$ respectively.

Given a one-parameter family lines $\{\overline{\beta}(t),w(t)\}$, the ruled surface generated by the family $\{\overline{\beta}(t),w(t)\}$ is

$$\Phi(t,u)=\overline{\beta}(t)+uw(t),\ \ t\in I,u\in\mathbb{R}. \tag{2.6}$$

The striction curve of the ruled surface is defined as

$$\breve{\beta}(t)=\overline{\beta}(t)-\frac{\langle\overline{\beta}'(t),w'(t)\rangle}{\langle w'(t),w'(t)\rangle}.w(t). \tag{2.7}$$

The striction curve $\breve{\beta}(t)$ coincides the base curve $\overline{\beta}(t)$ if the multiplication $\overline{\beta}'(t)$ and $w'(t)$ is equal to zero [13].

Definition 2.3. Let $\overline{M}$ be a hypersurface in $\mathbb{R}^3.$ N is the unit normal vector of $\overline{M}.$ For every $x\in\chi(\overline{M}),$

$$S:\chi(\overline{M})\to\chi(\overline{M}),$$

$$X\mapsto D_XN$$

is called Weingarten map (shape operator) on $\overline{M}.$Here D is Riemannian connection in $\mathbb{R}^3$ [13].

Definition 2.4. Let N be unit orthogonal vector field on the surface $\overline{M}.$

$$S_p:T_p\overline{M}\to T_p\overline{M},$$

$$v_p\mapsto S_p(v_p)=D_{v_p}N$$

is called Weingarten map at p on $\overline{M},$ see [13]. For the orthonormal base $\{\partial_1,\partial_2\}$ in $T_p\overline{M}$, N is defined in the following equation:

$$N=\frac{\partial_1\times\partial_2}{|\partial_1\times\partial_2|}.$$

Definition 2.5. Let $\overline{M}$ be a hypersurface in $\mathbb{R}^3.$

$$K:\overline{M}\to\mathbb{R},$$

$$p\mapsto K(p)=\det(S(p))$$

is called Gaussian curvature on $\bar{M}$. Here $S(p)$ is the shape operator at each point [13].

Definition 2.6. Let $\bar{M}$ be a hypersurface in $\mathbb{R}^3$.

$$H : \bar{M} \to \mathbb{R},$$

$$p \mapsto H(p) = \frac{1}{2} Tr(S(p))$$

is called mean curvature on $\bar{M}$ [13].

3. RULED SURFACES GENERATED BY NATURAL LIFT CURVES AND UNIT DUAL PSEUDO-SPHERE

This section describes some basic definitions and theorems about the dual space. Moreover, the correspondence among the subsets $T\hat{M}, T\tilde{M}$, $\mathbb{S}_1^2$ and $\mathbb{H}^2$ is represented.

The set of dual numbers is defined as

$$ID = \left\{ X = x + \varepsilon x^* : (x, x^*) \in \mathbb{R} \times \mathbb{R}, \varepsilon^2 = 0 \right\}.$$

Here x and x^* are dual and non-dual part of X, respectively. $X = x + \varepsilon x^*$ is called a dual number. The set

$$ID^3 = \left\{ (X_1, X_2, X_3) : X_i = x_i + \varepsilon x_i^* \in ID, 1 \le i \le 3 \right\} \tag{3.1}$$

is called $ID-$module. It is also a module over the ring ID. A dual vector $\vec{X} = (X_1, X_2, X_3)$ can be written in dual form as $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$, where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{x}^* = (x_1^*, x_2^*, x_3^*)$ are real vectors in $\mathbb{R}^3$. Assume that $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ and $\vec{Y} = \vec{y} + \varepsilon \vec{y}^*$ are dual vectors. The addition and Lorentzian inner product are defined as follows:

The addition is given as:

$$\vec{X} +_L \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon(\vec{x}^* + \vec{y}^*)$$

and their Lorentzian inner product is

$$\left\langle \vec{X}, \vec{Y} \right\rangle_L = \left\langle \vec{x}, \vec{y} \right\rangle_L + \varepsilon(\left\langle \vec{x}^*, y \right\rangle_L + \left\langle \vec{x}, \vec{y}^* \right\rangle).$$

A dual Lorentzian vector $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ is said to be timelike if $\vec{x}$ is timelike (i.e., $\left\langle \vec{x}, \vec{x} \right\rangle_L \langle 0$ or $\vec{x} = 0$), spacelike if $\vec{x}$ is spacelike (i.e., $\left\langle \vec{x}, \vec{x} \right\rangle_L \rangle 0$) and lightlike (null) if $\vec{x}$ is lightlike (i.e., $\left\langle \vec{x}, \vec{x} \right\rangle_L = 0$ or $\vec{x} \ne 0$), respectively. The set of all dual Lorentzian vectors is presented as ID_1^3.

An arbitrary dual Lorentzian space curve

$$\hat{X}: I \subseteq \mathbb{R} \to ID_1^3,$$

$$t \mapsto \hat{X}(t) = \vec{x}(t) + \varepsilon \vec{x}^*(t)$$

is called timelike if its velocity vector $\vec{x}\,'(t)$ is timelike, spacelike if its velocity vector is spacelike and lightlike if its velocity vector is lightlike for all $t \in \mathbb{R}$.

Let $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$, and $\vec{Y} = \vec{y} + \varepsilon \vec{y}^*$ be dual Lorentzian vectors. The Lorentzian cross-product is given as

$$\vec{X} \times_L \vec{Y} = \vec{x} \times_L \vec{y} + \varepsilon(\vec{x} \times_L \vec{y}^* + \vec{x}^* \times_L \vec{y}).$$

The Lorentzian norm of non-null dual Lorentzian vecctor $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ is defined as

$$\left|X\right|_L = \sqrt{\left|\langle \vec{x}, \vec{x} \rangle\right|_L} + \frac{\langle \vec{x}, \vec{x} \rangle_L}{\sqrt{\left|\langle \vec{x}, \vec{x} \rangle\right|_L}}. \tag{3.2}$$

The sets

$$H^2 = \left\{ \vec{X} = \vec{x} + \varepsilon \vec{x}^* \in ID_1^3 : \langle \vec{x}, \vec{x} \rangle_L = -1, \langle \vec{x}, \vec{x}^* \rangle_L = 0 \right\},$$

$$S_1^2 = \left\{ \vec{X} = \vec{x} + \varepsilon \vec{x}^* \in ID_1^3 : \langle \vec{x}, \vec{x} \rangle_L = -1, \langle \vec{x}, \vec{x}^* \rangle_L = 0 \right\}$$

are called dual hyperbolic unit sphere and dual Lorentzian unit sphere, respectively [7].

Proposition 3.1. The maps

$$T\tilde{M} \to S_1^2,$$

$$\bar{\Gamma} = (\bar{\gamma}, \bar{v}) \mapsto \vec{\bar{\Gamma}} = \bar{\gamma} + \varepsilon \bar{v}$$

and

$$T\hat{M} \to H^2,$$

$$\bar{\Gamma} = (\bar{\gamma}, \bar{v}) \mapsto \vec{\bar{\Gamma}} = \bar{\gamma} + \varepsilon \bar{v}$$

are isomorphisms.

Theorem 3.1. (E. Study mapping) There exists one-to-one correspondence between the directed timelike (resp. spacelike) lines in $\mathbb{R}_1^3$ and ordered pairs of vectors on S_1^2 and H^2 [1].

In $\mathbb{R}^3$, a ruled surface is a curved surface represented by moving a straight line along a space curve called the base curve. This straight line is called generator of the surface. The ruled surface is defined as timelike if the normal vector of the ruled surface at every point is a spacelike vector and as spacelike if the normal vector of the ruled surface at every point is a timelike vector.

Consequently, we can write these isomorphisms for ruled surfaces generated by the natural lift curves by using E. Study mapping:

$$T\tilde{M} \to S_1^2 \to \mathbb{R}_1^3,$$

$$\bar{\Gamma} = (\bar{\gamma}, \bar{v}) \mapsto \vec{\bar{\Gamma}} = \bar{\gamma} + \varepsilon \bar{v} = \bar{\Phi}(t,u) = \bar{\gamma}(t) \times_L \bar{v}(t) + u\bar{\gamma}(t)$$

and

$$T\hat{M} \to H^2 \to \mathbb{R}_1^3,$$

$$\bar{\Gamma} = (\bar{\gamma}, \bar{v}) \mapsto \vec{\bar{\Gamma}} = \bar{\gamma} + \varepsilon \bar{v} = \bar{\Phi}(t,u) = \bar{\gamma}(t) \times_L \bar{v}(t) + u\bar{\gamma}(t).$$

The following propositions are about the characterizations of timelike and spacelike ruled surfaces generated by the natural lift curves:

Proposition 3.2. Let $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ be natural lift curve on the subset $T\hat{M}$. If the base curve $\bar{\beta}(t) = \bar{\gamma}(t) \times_L \bar{v}(t)$ is spacelike curve and the director curve $\bar{\gamma}(t)$ is timelike curve, then $\bar{v}(t)$ is spacelike curve. Therefore, $\bar{\Phi}(t,u)$ is timelike ruled surface on $\mathbb{R}_1^3$.

Proposition 3.3. Let $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ be natural lift curve on the subset $T\hat{M}$. If the base curve $\bar{\beta}(t) = \bar{\gamma}(t) \times_L \bar{v}(t)$ is spacelike curve, there are two conditions:

(i) If the director curve $\bar{\gamma}(t)$ is timelike curve, then $\bar{v}(t)$ is timelike curve. Therefore, $\bar{\Phi}(t,u)$ is timelike ruled surface on $\mathbb{R}_1^3$.

(ii) If the director curve $\bar{\gamma}(t)$ is timelike curve, then $\bar{v}(t)$ is spacelike curve. Therefore, $\bar{\Phi}(t,u)$ is timelike ruled surface on $\mathbb{R}_1^3$.

Proposition 3.4. Let $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ be natural lift curve on the subset $T\tilde{M}$. If the base curve $\bar{\beta}(t) = \bar{\gamma}(t) \times_L \bar{v}(t)$ is spacelike curve and the director curve $\bar{\gamma}(t)$ is spacelike curve, then $\bar{v}(t)$ is timelike curve. Therefore, $\bar{\Phi}(t,u)$ is spacelike ruled surface on $\mathbb{R}_1^3$.

Proposition 3.5. Let $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ be natural lift curve on the subset $T\tilde{M}$. If the base curve $\bar{\beta}(t) = \bar{\gamma}(t) \times_L \bar{v}(t)$ is timelike curve and the director curve $\bar{\gamma}(t)$ is spacelike curve, then $\bar{v}(t)$ is spacelike curve. Therefore, $\bar{\Phi}(t,u)$ is timelike ruled surface on $\mathbb{R}_1^3$.

Proposition 3.6. Let $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ be a curve on the subset $T\tilde{M}$. If the base curve $\bar{\beta}(t) = \bar{\gamma}(t) \times_L \bar{v}(t)$ is spacelike curve, there are two conditions:

(i) If the director curve $\bar{\gamma}(t)$ is timelike curve, then $\bar{v}(t)$ is spacelike curve. Therefore, $\bar{\Phi}(t,u)$ is timelike ruled surface on $\mathbb{R}_1^3$.

(ii) If the director curve $\bar{\gamma}(t)$ is timelike curve, then $\bar{v}(t)$ is timelike curve. Therefore, $\bar{\Phi}(t,u)$ is timelike ruled surface on $\mathbb{R}^3_1$.

For the base curve $\bar{\beta}(t)=\bar{\gamma}(t)\times_L \bar{v}(t),$ we get the derivative of $\bar{\beta}(t)$:

$$\bar{\beta}'(t)=\bar{\gamma}'(t)\times_L \bar{v}(t)+\bar{\gamma}(t)\times_L \bar{v}'(t).$$

By taking $\bar{\gamma}(t)$ as unit, the developability condition of the ruled surface is calculated as follows:

$$\begin{aligned}\det(\bar{\beta}'(t),\bar{\gamma}(t),\bar{\gamma}'(t)) &= (\bar{\gamma}'(t)\times_L \bar{v}(t)+\bar{\gamma}(t)\times_L \bar{v}'(t))\bar{\gamma}'(t)\\ &= \langle \bar{\gamma}'(t),\bar{v}'(t)\rangle_L = 0. \quad (3.3)\end{aligned}$$

Corollary 3.1. Assume that $\bar{\Gamma}(t)=\bar{\gamma}(t)+\varepsilon\bar{v}(t)$ is spacelike curve on S^2_1 (resp. timelike on H^2). The ruled surface generated by natural lift curve $\bar{\Gamma}(t)$ is developable if and only if

$$\langle \bar{\gamma}'(t),\bar{v}'(t)\rangle_L = 0.$$

The proof of corollary is a result of Eq.(3.3).

4. RULED SURFACES AND TANGENT BUNDLE OF PSUDO-SPHERE OF NATURAL LIFT CURVES

This section describes the developability condition for ruled surface $\bar{\Phi}(t,u)$ generated by a smooth curve $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))\in T\hat{M}$ or $T\tilde{M}$.

Proposition 4.1. Assume that $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))$ is a natural lift curve on $T\hat{M}$ or $T\tilde{M}$. The ruled surface generated by the curve couple $(\bar{\gamma}(t),\bar{v}(t))$ is developable if and only if

$$\langle \bar{\gamma}'(t),\bar{v}'(t)\rangle_L = 0.$$

The proof is a result of Corollary 3.1.

Proposition 4.2. Let $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))$ be a natural lift curve on $T\hat{M}$ or $T\tilde{M}$. Then the couple $(\bar{\gamma}(t),\bar{v}(t))$ is an involute-evolute curve couple if and only if the spacelike or timelike ruled surface generated by the curve $\bar{\Gamma}(t)$ is developable.

Proof: Let $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))$ be a natural lift curve on $T\hat{M}$ or $T\tilde{M}$. Using the derivatives of $\bar{\gamma}(t)$ and $\bar{v}(t),$ we obtain

$$\langle \bar{\gamma}'(t),\bar{v}'(t)\rangle_L = 0.$$

The tangent vectors of $\bar{\gamma}(t)$ and $\bar{v}(t)$ are orthogonal. So, the condition of being an involute-evolute curve couple is provided.

Proposition 4.3. Assume that $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))$ be a natural lift curve on $T\hat{M}$ or $T\tilde{M}$. If the curve couple $(\breve{\beta}(t),\bar{\gamma}(t))$ is an involute-evolute curve couple, then the striction curve $\breve{\beta}(t)$ and the base curve $\bar{\beta}(t)$ of the spacelike or timelike ruled surface $\bar{\Phi}(t,u)$ generated by the natural lift curve coincide.

Example 4.1. Assume that $\bar{\gamma}(t)=(0,\sinh t,\cosh t)$ is spacelike curve on $\hat{M}$ and $\bar{v}(t)=(0,3\cosh t,3\sinh t)$ is the timelike vector in $\mathbb{R}^3_1$. Because $\langle\bar{\gamma}(t),\bar{\gamma}(t)\rangle_L=-1$ and $\langle\bar{\gamma}(t),\bar{v}(t)\rangle_L=0$, the curve $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))$ is on $T\hat{M}$. The timelike ruled surface generated by the curve $\bar{\Gamma}(t)=(\bar{\gamma}(t),\bar{v}(t))$ is

$$\bar{\Phi}(t,u)=\bar{\gamma}(t)\times_L\bar{v}(t)+u\bar{\gamma}(t)=(-3,u\sinh t,u\cosh t).$$

Here the base curve is

$$\bar{\beta}(t)=(-3,0,0).$$

Moreover, we obtain $\langle\bar{\gamma}'(t),\bar{v}'(t)\rangle=0$. So, the ruled surface generated by the curve $\bar{\Gamma}(t)$ is developable.

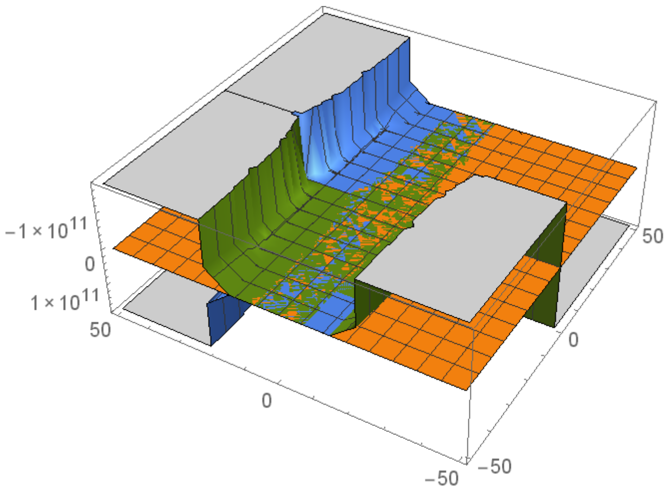

Figure 1. The timelike ruled surface generated by the natural lift curve $\bar{\Gamma}(t)$.

Let us calculate the shape operator and Gaussian curvature of the ruled surface $\bar{\Phi}(t,u)$:

$$\bar{\Phi}(t,u) = (-3, u\sinh t, u\cosh t),$$

$$\bar{\Phi}_t(t,u) = (0, u\cosh t, u\sinh t),$$

$$\bar{\Phi}_u(t,u) = (0, \sinh t, \cosh t).$$

Since $\left\langle \bar{\Phi}_t, \bar{\Phi}_u \right\rangle = 0,$ the orthonormal base is

$$E_1 = \frac{\bar{\Phi}_t}{\left|\bar{\Phi}_t\right|} = \frac{(0, u\cosh t, u\sinh t)}{u} = (0, \cosh t, \sinh t),$$

$$E_2 = \frac{\bar{\Phi}_u}{\left|\bar{\Phi}_u\right|} = \frac{(0, \sinh t, \cosh t)}{-1} = (0, -\sinh t, -\cosh t).$$

The normal vector N of the ruled surface is

$$N = E_1 \times E_2,$$

$$N = (1, 0, 0).$$

Therefore, the shape operator of the ruled surface is

$$S(E_1) = \lambda_1 E_1 + \lambda_2 E_2,$$

$$S(E_2) = \mu_1 E_1 + \mu_2 E_2.$$

where>

$$\lambda_1 = \frac{-1}{\left|\bar{\Phi}_t\right|^3 \left|\bar{\Phi}_u\right|} \det(\bar{\Phi}_{tt}, \bar{\Phi}_t, \bar{\Phi}_u),$$

$$\lambda_2 = \mu_1 = \frac{-1}{\left|\bar{\Phi}_t\right|^2 \left|\bar{\Phi}_u\right|^2} \det(\bar{\Phi}_{tu}, \bar{\Phi}_t, \bar{\Phi}_u)$$

and

$$\mu_2 = \frac{-1}{\left|\bar{\Phi}_t\right| \left|\bar{\Phi}_u\right|^2} \det(\bar{\Phi}_{uu}, \bar{\Phi}_t, \bar{\Phi}_u)$$

are taken for calculations for shape operator. After some calculations, we obtain

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Gaussian curvature K of the ruled surface is

$$K = \det(S),$$

$$K = 0.$$

The mean curvature H of the ruled surface is

$$H = \frac{1}{2} Tr(S),$$

$$H = 0.$$

Example 4.2. Assume that $\bar{\gamma}(t) = (\sqrt{2}\sin(\frac{t}{\sqrt{2}}), -\sqrt{2}\cos(\frac{t}{\sqrt{2}}), 0)$ is spacelike curve on $\tilde{M}$ and $\bar{v}(t) = (\frac{1}{\sqrt{2}}\cos(\frac{t}{\sqrt{2}}), \frac{1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), 0)$ is the spacelike vector in $\mathbb{R}^3_1$. Since $\langle \bar{\gamma}(t), \bar{\gamma}(t) \rangle_L = 1$ and $\langle \bar{\gamma}(t), \bar{v}(t) \rangle_L = 0$, the curve $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ is in $T\tilde{M}$. The timelike ruled surface generated by the curve $\bar{\Gamma}(t) = (\bar{\gamma}(t), \bar{v}(t))$ is

$$\bar{\Phi}(t,u) = \bar{\gamma}(t) \times_L \bar{v}(t) + u\bar{\gamma}(t) = (u\sqrt{2}\sin(\frac{t}{\sqrt{2}}), -u\sqrt{2}\cos(\frac{t}{\sqrt{2}}), -1).$$

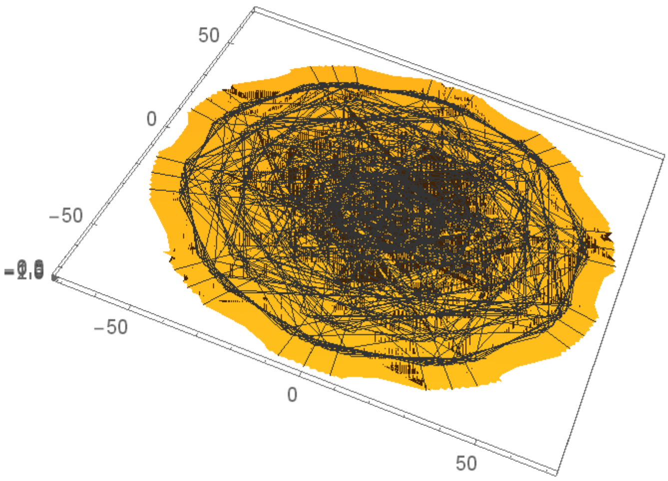

Figure 2. The timelike ruled surface generated by the natural lift curve $\bar{\Gamma}(t)$.

Here the base curve is

$$\bar{\beta}(t) = (0, 0, -1).$$

Moreover, we obtain $\langle \bar{\gamma}'(t), \bar{v}'(t) \rangle = 0.$ So, the ruled surface generated by the curve $\bar{\Gamma}(t)$ is developable.

Let us calculate the shape operator and Gaussian curvature of the ruled surface $\bar{\Phi}(t,u)$:

$$\bar{\Phi}(t,u) = (u\sqrt{2}\sin(\frac{t}{\sqrt{2}}), -u\sqrt{2}\cos(\frac{t}{\sqrt{2}}), -1),$$

$$\bar{\Phi}_t(t,u) = (u\cos(\frac{t}{\sqrt{2}}), u\sin(\frac{t}{\sqrt{2}}), 0),$$

$$\bar{\Phi}_u(t,u) = (\sqrt{2}\sin(\frac{t}{\sqrt{2}}), -\sqrt{2}\cos(\frac{t}{\sqrt{2}}), 0).$$

Since $\langle \bar{\Phi}_t, \bar{\Phi}_u \rangle = 0,$ the orthonormal base is

$$E_1 = \frac{\bar{\Phi}_t}{|\bar{\Phi}_t|} = \frac{(u\cos(\frac{t}{\sqrt{2}}), u\sin(\frac{t}{\sqrt{2}}), 0)}{u} = (\cos(\frac{t}{\sqrt{2}}), \sin(\frac{t}{\sqrt{2}}), 0),$$

$$E_2 = \frac{\bar{\Phi}_u}{|\bar{\Phi}_u|} = \frac{(\sqrt{2}\sin(\frac{t}{\sqrt{2}}), -\sqrt{2}\cos(\frac{t}{\sqrt{2}}), 0)}{\sqrt{2}} = (\sin(\frac{t}{\sqrt{2}}), -\cos(\frac{t}{\sqrt{2}}), 0).$$

The normal vector N of the ruled surface is

$$N = E_1 \times E_2,$$

$$N = (0, 0, 1).$$

Therefore, the shape operator of the ruled surface is

$$S(E_1) = \lambda_1 E_1 + \lambda_2 E_2,$$

$$S(E_2) = \mu_1 E_1 + \mu_2 E_2.$$

where

$$\lambda_1 = \frac{-1}{|\bar{\Phi}_t|^3 |\bar{\Phi}_u|} \det(\bar{\Phi}_{tt}, \bar{\Phi}_t, \bar{\Phi}_u),$$

$$\lambda_2 = \mu_1 = \frac{-1}{\left|\bar{\Phi}_t\right|^2 \left|\bar{\Phi}_u\right|^2} \det(\bar{\Phi}_{tu}, \bar{\Phi}_t, \bar{\Phi}_u)$$

and

$$\mu_2 = \frac{-1}{\left|\bar{\Phi}_t\right| \left|\bar{\Phi}_u\right|^2} \det(\bar{\Phi}_{uu}, \bar{\Phi}_t, \bar{\Phi}_u)$$

are taken for calculations of the coefficients for shape operator. After some calculations, we obtain

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Gaussian curvature K of the ruled surface is

$$K = \det(S),$$

$$K = 0.$$

The mean curvature H of the ruled surface is

$$H = \frac{1}{2} Tr(S),$$

$$H = 0.$$

5. CONCLUSION

E. Study mapping plays fundamental role in kinematics. In this paper, the isomorphisms among the subsets for tangent bundles of pseudo-sphere, Lorentzian unit sphere and hyperbolic unit sphere are constructed. Then we obtain ruled surfaces generated by the natural lift curves in $\mathbb{R}^3_1$. Using these results, it is possible to model motions by using the natural lift curves on the subsets $T\tilde{M}$ or $T\hat{M}$ instead of Lorentzian unit sphere and hyperbolic unit sphere. Natural lift curves on $T\tilde{M}$ or $T\hat{M}$ can also be used to modeling motions in $\mathbb{R}^3_1$.

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