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AN APPROXIMATE SOLUTION FOR LORENTZIAN SPHERICAL TIMELIKE CURVES

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Abstract. *In this article, the differential equation of lorentzian spherical timelike curves is obtained in* E_1^4. *It is seen that the differential equation characterizing Lorentzian spherical timelike curves is equivalent to a linear, third order, differential equation with variable coefficients. It is impossible to solve these equations analytically. In this article, a new numerical technique based on hermite polynomials is presented using the initial conditions for the approximate solution. This method is called the modified hermite matrix-collocation method. With this technique, the solution of the problem is reduced to the solution of an algebraic equation system and the approximate solution is obtained. In addition, the validity and applicability of the technique is explained by a sample application.*

Keywords: *Minkowski space-time; modified hermite; lorentzian sphere; spherical timelike curves.*

1. INTRODUCTION

There are many studies on curves in the literature [1]. The curve was first defined in the plane. Then it was moved to the three-dimensional Euclidean space. Afterwards, space curves of constant breadth lying on the sphere was defined [2]. A general formula was obtained that makes a curve a spherical curve [3, 4]. And this formula has been included in the literature as a necessary and sufficient condition for a curve to be a spherical curve. Moreover, many studies have been done on the lorentzian spherical spacelike, timelike and null curves [5, 6]. A number of characterizations of timelike and spacelike spherical curves on the lorentzian sphere have been given by Kazaz et al. in E_1^4 [7]. Walrave presented frenet equations for a curve in Minkowski space-time. Önder et al. have worked on timelike and null curves in E_1^4 [8, 9].

Also Yüzbaşı et al. and Akgönüllü et al. have presented the matrix and collocation methods for solving linear, nonlinear, differential and integral equations in terms of special polynomials [10, 11].

In this work, firstly, differential equation, which characterizes unit speed curves on the lorentzian sphere in 4-dimensional Minkowski space, is obtained. Then on the solution of this equation is studied and this equation is solved by modified hermite collocation method. These type equations are play an important role in many science and technical applications and are essential tools for modelling problems encountered in these fields. Also, a solution for this curve type is made for the first time in this article. Therefore, the original value of this study is high.

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2. PRELIMINARIES

There hermite polynomials are defined by Poularikas as follows [12]:

$$H_n(s)=\sum_{k=0}^{n/2}(-1)^k\,\frac{n!}{(n-2k)!k!}2^{n-2k}s^{n-2k}. \tag{1}$$

These polynomials are orthogonal on $(-\infty,\infty)$ and $-\infty\le s\le\infty$.

The Minkowski space-time E_1^4 is real vector space R^4 provided with the standart flat metric given by $g=-dy_1^2+dy_2^2+dy_3^2+dy_4^2$, where (y_1,y_2,y_3,y_4) is a rectangular coordinate system of Minkowski space-time E_1^4.

An arbitrary vector $\vec{u}=(u_1,u_2,u_3,u_4)$ in E_1^4 may be one of three lorentzian causal characters;

- $g(\vec{u},\vec{u})>0$ or $\mathrm{u}=0\Rightarrow$ spacelike,
- $g(\vec{u},\vec{u})<0\ \Rightarrow$ timelike,
- $g(\vec{u},\vec{u})=0$ and $\vec{u}\neq 0\Rightarrow$ null (lightlike).

At the same time, recall that the pseudo-norm an arbitrary vector $\vec{u}\in E_1^4$ is given as $\|\vec{u}\|=\sqrt{|g(\vec{u},\vec{u})|}$. It is clear that if $g(\vec{u},\vec{u})=\pm 1$, $\vec{\alpha}$ will be a unit vector. Next, if $g(\vec{u},\vec{w})=0$, the vectors $\vec{u}$ and $\vec{w}$ are orthogonal in E_1^4. The velocity of the curve $\vec{\alpha}(s)$ is given by $\|\vec{\alpha}(s)\|$. The lorentzian sphere of center $o=(o_1,o_2,o_3,o_4)$ and radius $r\in R^+$ in the space E_1^4 is defined by $S_1^3(o,r)=\left\{x\in E_1^4; g(x-o,x-o)=r^2\right\}$, $x=(x_1,x_2,x_3,x_4)$. Furthermore, for a timelike curve $\vec{\alpha}(s)$ in space E_1^4, the following frenet formulas are given:

$$\begin{bmatrix}\vec{T}'\\ \vec{N}'\\ \vec{B}_1'\\ \vec{B}_2'\end{bmatrix}=\begin{bmatrix}0 & k_1 & 0 & 0\\ k_1 & 0 & k_2 & 0\\ 0 & -k_2 & 0 & k_3\\ 0 & 0 & -k_3 & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}_1\\ \vec{B}_2\end{bmatrix},$$

where $\vec{T},\vec{N},\vec{B}_1,\vec{B}_2$ are mutually orthogonal and these vectors meet equalities $g(\vec{T},\vec{T})=-1$, $g(\vec{N},\vec{N})=g(\vec{B}_1,\vec{B}_1)=g(\vec{B}_2,\vec{B}_2)=1$. k_1,k_2,k_3 are the curvature of curve $\vec{\alpha}$ [8,9].

3. MODIFIED HERMITE COLLOCATION METHOD

In this chapter, we explain the mentioned matrix and collocation method for solving the third-order ordinary differential equation with variable coefficients in $\sum_{k=0}^{3}P_k(s)y^{(k)}(s)=g(s)$ form. In this method, the approximate solution of the given equation is obtained depending on the initial conditions given as $y^{(\mathrm{k})}(\mathrm{a})=\mathrm{c}_k$, $(k=0,1,2)$ and this solution is in the form of a truncated hermite series, expressed as $y(\mathrm{s}) \cong \mathrm{y}_N(\mathrm{s}) = \sum_{n=0}^{N} a_n H_n(\mathrm{s})$. Wherein said $P_k(s)$ and $g(s)$ are the functions defined on the interval $-\infty < a \le s \le b < +\infty; c_k (k = 0,1,2)$ are appropriate constants; $a_n (n = 0,1,..., N \ge 3)$ are unknown hermite coefficients to be determined; $H_n(s)$ are the hermite polynomials defined by Eq.(1). Also the sequence of hermite polynomials satifies the recursion relation $H_n^{'}(s) = 2nH_{n-1}(s) \quad (n \ge 2)$ with $H_0^{'}(s) = 0, H_1^{'}(s) = 2H_0(s).$

When solving the equation, firstly, the matrix form of each term in the equation is found. The approximate solution, which is assumed to be in the form of a truncated hermite series, is converted to matrix form. Finally, depending on the collocation points in the given range, approximate solution is obtained by means of matrices.

4. THE CHARACTERIZATION OF LORENTZIAN SPHERICAL TIMELIKE CURVES

In this chapter, we obtain necessary and sufficient condition for an unit speed timelike curve in E_1^4 to be on the S_1^3 lorentzian sphere. After, we arrive at the conclusion that the position vector of any unit speed timelike curve on the S_1^3 lorentzian sphere meets the third order, linear, differential equation with variable coefficient.

4.1. THEOREM

Let $\vec{\alpha}(s)$ be a unit speed regular curve with smooth non-zero curvature functions k_1, k_2 and k_3. Then the condition for the curve $\vec{\alpha}(s)$ to be a S_1^3 spherical curve is that $\rho = \frac{1}{k_1}, k_2$ and k_3 satisfy differential equation as follow:

$$\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}\left(\frac{1}{k_2}\frac{d\rho}{ds}\right) + \frac{k_2}{k_3}\rho\right] + \frac{k_3}{k_2}\frac{d\rho}{ds} = 0 .$$

Let us prove this theory by using Dannon's idea of moving the spherical curves of E^3 into the space of E^4 [13].

4.2. PROOF

Assume that a unit speed regular $\vec{\alpha}(s)$ curve lies on a sphere S_1^3 with radius r and center o in E_1^4. In this case, it is obvious that $g\langle \alpha(s) - o, \alpha(s) - o \rangle = r^2$ for each $s \in I \subset R.$ We can arrange this expression as $f(s) = g\langle \alpha(s) - o, \alpha(s) - o \rangle - r^2 = 0$. Now let's get the repetitive differentiations of this function f. We will also use the equations given in section 2 for the derivatives of frenet vector fields. Thus, the following equalities are found:

$$(g\left\langle \vec{\alpha}-o,\vec{\alpha}-o\right\rangle)'=0 \Rightarrow g\left\langle \vec{T},\vec{\alpha}-o\right\rangle=0,$$

$$(g\left\langle \vec{T},\vec{\alpha}-o\right\rangle)'=0 \Rightarrow g\left\langle \vec{N},\vec{\alpha}-o\right\rangle=\rho,$$

$$(g\left\langle \vec{N},\vec{\alpha}-o\right\rangle)'=\rho' \Rightarrow \left\langle \vec{B}_1,\vec{\alpha}-o\right\rangle=\frac{\rho'}{k_2}, \tag{2}$$

$$(g\left\langle \vec{B}_1,\vec{\alpha}-o\right\rangle)'=(\frac{\rho'}{k_2})', \quad \Rightarrow g\left\langle \vec{B}_2,\vec{\alpha}-o\right\rangle=\frac{1}{k_3}+\frac{k_2}{k_3}\rho \Big/ (\frac{\rho'}{k_2})',$$

$$(g\left\langle \vec{B}_2,\vec{\alpha}-o\right\rangle)'=-\frac{1}{k_3}[\frac{1}{k_3}(\frac{\rho'}{k_2})'+\frac{k_2}{k_3}\rho]'. \tag{3}$$

Using the equivalent of Eq.(2) and Eq.(3), the following expression is obtained:

$$\frac{d}{ds}\left[\frac{1}{k_3}\frac{d}{ds}(\frac{1}{k_2}\frac{d\rho}{ds})+\frac{k_2}{k_3}\rho\right]+\frac{k_3}{k_2}\frac{d\rho}{ds}=0 .$$

We can arrange this expression as follows:

$$(\frac{1}{k_3k_2})\rho'''+[(\frac{1}{k_3k_2})'+\frac{1}{k_3}(\frac{1}{k_2})']\rho''+[(\frac{1}{k_3}(\frac{1}{k_2})')'+\frac{k_2}{k_3}+\frac{k_3}{k_2}]\rho'+(\frac{k_2}{k_3})'\rho=0 . \tag{4}$$

This is differential equation that characterizes lorentzian spherical timelike curves in E_1^4.

5. SOLUTION OF THE DIFFERENTIAL EQUATION

$$P_0=(\frac{k_2}{k_3})', P_1=(\frac{1}{k_3}(\frac{1}{k_2})')'+\frac{k_2}{k_3}+\frac{k_3}{k_2}, P_2=(\frac{1}{k_3k_2})'+\frac{1}{k_3}(\frac{1}{k_2})', P_3=\frac{1}{k_3k_2}, y=\rho$$

By using the equations, the differential Eq.(4) characterizing the lorentzian spherical timelike curves can rewrote as follows:

$$\sum_{k=0}^{3}P_k(s)y^{(k)}(s)=G(s) . \tag{5}$$

Suppose this Eq.(5) has an approximate solution, depending on the initial conditions given as

$$y^{(k)}(0)=c_k \ \ (k=0,1,2), \tag{6}$$

in the form of an truncated hermite series as

$$y(s) \cong y_N(s) = \sum_{n=0}^{N} a_n H_n(s) \ (0 \le s \le 2\pi), \tag{7}$$

where $P_k(s)$ and $g(s)$ are the functions defined on the interval $0 \le s \le 2\pi$. For simplicity we will get $N = 4$. The first four hermite polynomials can be given as

$$H_0(s) = 1, H_1(s) = 2s,$$

$$H_2(s) = 4s^2 - 2, H_3(s) = 8s^3 - 12s, H_4(s) = 16s^4 - 48s^2 + 12.$$

Now we can define the collocation points as

$$s_i = \frac{\pi}{2} i \ \ (i = 0,1,\ldots,4), \tag{8}$$

where $s_0 = 0, \mathrm{s}_1 = \dfrac{\pi}{2}, s_2 = \dfrac{3\pi}{2}, s_4 = 2\pi$. By substituting the collocation points (8) in to Eq. (5), we get the system of matrix equations as

$$\sum_{k=0}^{3} P_k(s_i) y^{(k)}(s_i) = G(s_i), \tag{9}$$

where $P_k = \begin{bmatrix} P_k(0) & 0 & 0 & 0 & 0 \\ 0 & P_k(\frac{\pi}{2}) & 0 & 0 & 0 \\ 0 & 0 & P_k(\pi) & 0 & 0 \\ 0 & 0 & 0 & P_k(\frac{3\pi}{2}) & 0 \\ 0 & 0 & 0 & 0 & P_k(2\pi) \end{bmatrix}, G(s_i) = \begin{bmatrix} g(0) \\ g(\pi/2) \\ g(\pi) \\ g(3\pi/2) \\ g(2\pi) \end{bmatrix}, (k = 0,1,2,3).$

By putting the collocation points (8), into the Eq. (7), we have the following matrix equation:

$$y(s_i) \cong y_N(s_i) = H^{(k)}(s_i) A = H(s_i) M^k A \ (i = 0,1,\ldots,4),$$

where

$$H = \begin{bmatrix} H_0(0) & H_1(0) & \cdots & H_4(0) \\ H_0(\pi/2) & H_1(\pi/2) & \cdots & H_4(\pi/2) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(2\pi) & H_1(2\pi) & \cdots & H_4(2\pi) \end{bmatrix}.$$

This matrix relation by using in Eq. (9), we obtain the fundamental matrix equation in compact form as follow:

$$\sum_{k=0}^{3} P_k H M^k A = G, \tag{10}$$

where $W=\left[w_{mn}\right]=\sum_{k=0}^{3} P_k H M^k; m,n=0,..,4.$ Also, we can write the matrix Equation (10) in the augmented form as

$$WA=G \text{ or } \left[W\ ;\ G\right]. \tag{11}$$

Besides, we obtain the equation of conditions as

$$\begin{aligned} y(0) &= H(0)M^0A=c_0, \\ y'(0) &= H(0)M^1A=c_1, \\ y''(0) &= H(0)M^2A=c_2. \end{aligned} \tag{12}$$

On the other hand, U_0, U_1 and U_2 are calculated using $U_k=H(0)M^k$, $U=\left[u_{k0}\quad u_{k1}\quad \ldots\quad u_{k4}\right]$ and thus an increased matrix of conditions is obtained as

$$U=\begin{bmatrix} 1 & 0 & -2 & 0 & 12 & ; & c_0 \\ 0 & 2 & 0 & -12 & 0 & ; & c_1 \\ 0 & 0 & 8 & 0 & -96 & ; & c_2 \end{bmatrix}.$$

Besides, for $C=\left[c_0\quad c_1\quad c_2\right]^T$ the following equation is obvious:

$$UA=G \text{ or } \left[U\ ;\ C\right]. \tag{13}$$

We obtain $\tilde{W}A=\tilde{G}$ or

$$\left[\tilde{W}\ ;\ \tilde{G}\right]=\begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & w_{04} & ; & g(0) \\ w_{10} & w_{11} & w_{12} & w_{13} & w_{14} & ; & g(\frac{\pi}{2}) \\ 1 & 0 & -2 & 0 & 12 & ; & c_0 \\ 0 & 2 & 0 & -12 & 0 & ; & c_1 \\ 0 & 0 & 8 & 0 & -96 & ; & c_2 \end{bmatrix}$$

from the Eq. (11) and Eq. (13). Also, w_{ij} $(i=0,1 \text{ and } j=0,1,\ldots,4)$ are obtained as follows:

$$w_{00}=P_0(0), w_{01}=2P_1(0), w_{02}=-2P_0(0)+8P_2(0), w_{03}=-12P_1(0)+48P_3(0), w_{04}=12P_0(0)-96P_2(0),$$

$$w_{10}=P_0(\frac{\pi}{2}), w_{11}=2P_1(\frac{\pi}{2})+\pi P_0(\frac{\pi}{2}), w_{12}=(\pi^2-2)P_0(\frac{\pi}{2})+4\pi P_1(\frac{\pi}{2})+8P_2(\frac{\pi}{2}),$$

$$w_{13}=(\pi^3-16\pi)P_0(\frac{\pi}{2})+(6\pi^2-2)P_1(\frac{\pi}{2})+24\pi P_2(\frac{\pi}{2})+48P_3(\frac{\pi}{2}),$$

$$w_{14} = (\pi^4 - 12\pi^2 + 12)P_0(\frac{\pi}{2}) + (8\pi^3 - 48\pi)P_1(\frac{\pi}{2}) + (48\pi^2 - 96)P_2(\frac{\pi}{2}) + 192\pi P_3(\frac{\pi}{2}).$$

Thus, the matrix of unknows is obtained $\tilde{W}^{-1}\tilde{G} = A$. If we use these a_n unknowns in Eq.(7), we find the equality below:

$$\rho = y(s) = a_0 + 2sa_1 + (-2 + 4s^2)a_2 + (-12s + 8s^3)a_3 + (12 - 48s^2 + 16s^4)a_4 .$$

This expression is the radius of curvature $(\rho = 1/k_1)$ of lorentzian spherical timelike curves in E_1^4. Thus, $k_2(s)$ and $k_3(s)$ curvature functions are obtained through this function and its derivatives.

6. AN APPLICATION

We obtained the differential equation that characterizes lorentzian spherical timelike curves in E_1^4 using the system of the frenet-like equation consisting of derivatives of the frenet vector fields. By taking $\vec{B}_2 = 0$ and $k_3(s) = 0$ in this system of the frenet-like equation, the following frenet formulas are obtained:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \end{bmatrix}. \tag{14}$$

In the same way as above, let's take the repetitive derivatives of the function $f(s) = g\left\langle \alpha(s) - o, \alpha(s) - o \right\rangle - r^2 = 0$. We will use the system of Eq. (14) for the derivatives of frenet vector fields. Thus, the following statements are reached:

$$(g\left\langle \vec{\alpha} - o, \vec{\alpha} - o \right\rangle)' = 0 \Rightarrow g\left\langle \vec{T}, \vec{\alpha} - o \right\rangle = 0,$$

$$(g\left\langle \vec{T}, \vec{\alpha} - o \right\rangle)' = 0 \Rightarrow g\left\langle \vec{N}, \vec{\alpha} - o \right\rangle = \rho, \tag{15}$$

$$(g\left\langle \vec{N}, \vec{\alpha} - o \right\rangle)' = \rho' \Rightarrow g\left\langle \vec{B}_1, \vec{\alpha} - o \right\rangle = \frac{\rho'}{k_2},\ (g\left\langle \vec{B}_1, \vec{\alpha} - o \right\rangle)' = (\frac{\rho'}{k_2})'$$

$$\Rightarrow g\left\langle \vec{N}, \vec{\alpha} - o \right\rangle = -\frac{1}{k_2}[\frac{1}{k_2}(\frac{\rho'}{k_2})']'. \tag{16}$$

Using the equivalent of Eq. (15) and Eq. (16), $\frac{d}{ds}\left[\frac{1}{k_2}\frac{d}{ds}(\frac{1}{k_1})\right] + \frac{k_2}{k_1} = 0$ is obtained. Thus, we reduced the differential equation that characterizes lorentzian spherical timelike curves from E_1^4 to E_1^3. So we get the following theorem.

6.1. THEOREM

Let $\vec{\alpha}(s)$ be a unit speed timelike curve, with a curvature $k_1(s) \neq 0$ and $k_2(s) \neq 0$ for $\rho = 1/k_1$. Then $\vec{\alpha}(s)$ lies on a lorentzian sphere in E_1^3 if and only if

$$k_2(s)\rho'' - (k_2(s))'\rho' + (k_2(s))^3\rho = 0 . \tag{17}$$

This equation for $P_0 = (k_2(s))^3$, $P_1 = -(k_2(s))', P_2 = k_2(s)$ and $y = \rho$ can be written as

$$\sum_{k=0}^{2} P_k(s) y^{(k)}(s) = G(s) . \tag{18}$$

This is second-order ordinary differential equation with variable coefficients and an approximate solution of this equation can easily be obtained using the modified-hermite matrix collocation method.

6.2. EXAMPLE

We can calculate the approximate value of the curvature radius at $0 \leq s \leq \pi$ of a lorentzian spherical timelike curve given as the second curve s in E_1^3 as follows. Since the given curve is a spherical curve, this curve provides the differential Eq. (17)**.** This equation for $k_2(s) = s$ and $P_0 = s^3, P_1 = -1, P_2 = s$ can be written as

$$\sum_{k=0}^{2} P_k(s) y^{(k)}(s) = G(s) , \tag{19}$$

subject to the initial conditions $y^{(k)}(0) = c_k$ $(k = 0,1)$ and apply to obtain the approximate solution in the truncated hermite series as $y(s) \cong y_N(s) = \sum_{n=0}^{N} a_n H_n(s)$ $(0 \leq s \leq \pi)$. For simplicity, we take N=3 and for k =0,1,2 we get the following matrices:

$$H(s) = \begin{bmatrix} 1 & 2s & 4s^2-2 & 8s^3-12s \end{bmatrix}, A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T ,$$

$$M^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, M^1 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M^2 = \begin{bmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we can define the collocation points as $s_i = \frac{\pi}{3} i \left(i = 0,..,3\right)$, where $s_0 = 0, s_1 = \frac{\pi}{3}$, $s_2 = \frac{2\pi}{3}, s_3 = \pi$. Using these collocation points, we obtain the following matrices:

$$P_0=\begin{bmatrix}0&0&0&0\\0&\frac{\pi^3}{27}&0&0\\0&0&\frac{8\pi^3}{27}&0\\0&0&0&\pi^3\end{bmatrix},P_1=\begin{bmatrix}-1&0&1&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},P_2=\begin{bmatrix}0&0&0&0\\0&\frac{\pi}{3}&0&0\\0&0&\frac{2\pi}{3}&0\\0&0&0&\pi\end{bmatrix},$$

$$Y^{(k)}=HM^kA,H=\begin{bmatrix}1&0&-2&0\\1&\frac{2\pi}{3}&(\frac{2\pi}{3})^2-2&(\frac{2\pi}{3})^3-4\pi\\1&\frac{4\pi}{3}&(\frac{4\pi}{3})^2-2&(\frac{4\pi}{3})^3-8\pi\\1&2\pi&(2\pi)^2-2&(2\pi)^3-12\pi\end{bmatrix}.$$

On the other hand U_0 and U_1 are calculated, and so the expression in the form of increased matrix of matrix equation of conditions is as $U=\begin{bmatrix}1&0&-2&0&;&c_0\\0&2&0&-12&;&c_1\end{bmatrix}$. We obtain $\left[\tilde{W};\tilde{G}\right]=\begin{bmatrix}0&2&0&-12&;&c_1\\w_{10}&w_{11}&w_{12}&w_{13}&;&g(\frac{\pi}{3})\\1&0&-2&0&;&c_0\\w_{30}&w_{31}&w_{32}&w_{33}&;&g(\pi)\end{bmatrix}$, where $w_{ij}\,(i=1,3,\ j=0,1,2,3)$ obtained as follows:

$$w_{10}=\frac{\pi^3}{27},\qquad w_{11}=\frac{2\pi^4-162}{81},$$

$$w_{12}=(\frac{4\pi^2-18}{9})\frac{\pi^3}{27},\qquad w_{13}=(\frac{8\pi^3-108\pi}{27})\frac{\pi^3}{27}+\frac{42\pi^2}{9}+12,$$

$$w_{30}=\pi^3,\qquad w_{31}=2\pi^4-2,$$
$$w_{32}=(4\pi^2-2)\pi^3,\qquad w_{33}=(8\pi-12)\pi^4+24\pi^2+12.$$

Accordingly, the following equalities are obtained:

$$a_0=\frac{(6w_{31}+w_{33})(w_{11}c_1-w_{12}c_0)+(6w_{11}+w_{13})(w_{32}c_0-w_{31}c_1)}{(2w_{30}+w_{32})(6w_{11}+w_{13})-(2w_{10}+w_{12})(6w_{31}+w_{33})},$$

$$a_1=\frac{(2w_{30}+w_{32})(w_{13}c_1+6w_{12}c_0)-(2w_{10}+w_{12})(6w_{32}c_0+w_{33}c_1)}{2(2w_{30}+w_{32})(6w_{11}+w_{13})-2(2w_{10}+w_{12})(6w_{31}+w_{33})},$$

$$a_2=\frac{(6w_{31}+w_{33})(12w_{10}c_0-w_{12}c_1)+(6w_{11}+w_{13})(w_{33}c_1-12w_{30}c_0)}{12(2w_{30}+w_{32})(6w_{11}+w_{13})-12(2w_{10}+w_{12})(6w_{31}+w_{33})},$$

$$a_3 = \frac{(2w_{30} + w_{32})(w_{12}c_0 + w_{11}c_1) - (2w_{10} + w_{12})(w_{31}c_1 + w_{32}c_0)}{2(2w_{30} + w_{32})(6w_{11} + w_{13}) - 2(2w_{10} + w_{12})(6w_{31} + w_{33})}.$$

Thus the matrix of unknows is obtained $\tilde{W}^{-1}\tilde{G} = A$. If we put this a_n unknowns in the approximate solution, we obtain $\rho = a_0 + 2sa_1 + (4s^2 - 2)a_2 + (8s^3 - 12s)a_3$. This expression is the approximate value of the curvature radius at $0 \le s \le \pi$ of a lorentzian spherical timelike curve given as $k_2(s) = s$ in E_1^3. Since $\rho = 1/k_1$, the function of curvature $k_1(s)$ is obtained.

7. CONCLUSIONS AND SUGGESTIONS

In this study, differential equation characterizing lorentzian spherical timelike curves in Minkowski space is found. With the specially developed solution method, approximate solutions of the equation characterizing this curve type are found. This solution method can also be used for differential equations that characterize different curve types in different spaces.

Approximate solutions of the differential equation obtained for this curve type can be done by different matrix methods such as Taylor, Bernstein, Morgan-Voyce. Even more suitable solution method for curve type can be determined by error analysis between two different methods selected. Frenet formulas obtained for a timelike curve can be treated as a system of equations and solved by the same method. This study combines differential geometry with solutions of differential equations. This situation is important for the examination of geometric properties.

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