ORIGINAL PAPER

TIMELIKE SURFACES OF EVOLUTION IN MINKOWSKI 3-SPACE

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Abstract. *The present paper presents evolution of spherical indicatrix of a space curve according to the quasi frame in Minkowski 3-space. Some geometric properties such as fundamental forms and curvatures of the surfaces constructed by evolutions are obtained.*
Keywords: *spherical ındicatrix; quasi frame; evolution; Minkowski 3-space.*

1. INTRODUCTION

The theory of curves is a fundamental area in differential geometry. Especially, the curves obtained with the help of a given space curve and special curve couples are studied by many researchers. Bertrand curve couples, involute-evolute curve couples, Mannheim curve couples and spherical indicatrices of a space curve can be given as examples of these curves, [1].

Surfaces constructed by evolution of a space curve are studied widely among researchers. An evolving curve can be thought as a collection of curves parameterized by time. This means that each curve in the collection has a space parameter s and a time parameter t, [2]. For example, Korpinar [3] investigated the surfaces constructed by the binormal spherical indicatrix of a space curve. They derived the time evolution equations for the Frenet frame of binormal spherical indicatrix and gave some geometric properties of these surfaces. In [1], Soliman studied surfaces constructed by the evolution of the spherical indicatrices of a space curve. In that study, they used the Frenet frame of the curves in Euclidean 3-space. In [4], Soliman studied evolution of ruled surfaces with the help of their directrix using the quasi frame along a space curve in Euclidean 3-space. In that study, he gave the time evolution equations of a space curve given with the quasi frame. In study [5], they give surfaces constructed by evolution according to quasi frame, they studied spherical indicatrices of a space curve given with the quasi frame and obtained some geometric properties of the surfaces constructed by the evolution of them in Minkowski 3-space.

In this paper, we study the surfaces constructed by the evolution of the spherical indicatrices of a space curve given with the quasi frame in Minkowski 3-space. We derive some geometric properties of these surfaces such as fundamental forms and curvatures.

2. PRELIMINARIES

In this section, we mention about some basic concepts of Minkowski 3-space and we present the quasi frame of spacelike curves and timelike curves in Minkowski 3-space.

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Let $\mathbb{R}^3$ be the 3-dimensional standart real vector space. For every $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, the non-degenerate metric tensor $\langle , \rangle$ of index 1 defined by

$$\langle , \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$$

is called the Lorentzian metric. Then, the obtained $(\mathbb{R}^3, \langle , \rangle)$ space is called Minkowski 3-space and denoted by $\mathbb{R}_1^3$, [6].

Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}_1^3$ be a curve in Minkowski 3-space and $\boldsymbol{T}$ be the tangent vector field of α. If $\langle \boldsymbol{T}, \boldsymbol{T} \rangle > 0$, then α is called a spacelike curve. If $\langle \boldsymbol{T}, \boldsymbol{T} \rangle < 0$, then α is called a timelike curve. If $\langle \boldsymbol{T}, \boldsymbol{T} \rangle = 0$ and $\boldsymbol{T} \neq 0$, then α is called a null curve, [7].

Let $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{R}_1^3$. Then, the vector

$$(x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1)$$

is called the Lorentzian vector product of the vectors x and y and denoted by $x \wedge y$, [7].

Let $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ and $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, then

$$x \wedge y = -det \begin{bmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$= det \begin{bmatrix} -e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Note that $e_1 \wedge e_2 = e_3,\ e_2 \wedge e_3 = -e_1$ and $e_3 \wedge e_1 = -e_2$, [7].

In [7], considering the projection vector $\vec{k}$ is timelike or spacelike, the quasi frame of spacelike curves is constructed. According to Frenet normal and binormal vector fields are assumed timelike or spacelike, quasi formulas are obtained for this kind of curves. For timelike curves, no matter if the projection vector $\vec{k}$ is timelike or spacelike, because the same results are obtained in both cases. So, choosing the projection vector $\vec{k}$ is spacelike, the quasi frame of timelike curves is constructed.

Considering these constructions in [7], we present the quasi frame of spacelike curves and the quasi frame of timelike curves in Minkowski 3-space. Let $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ denote the Frenet vector fields of the curves and $\{\boldsymbol{t_q}, \boldsymbol{n_q}, \boldsymbol{b_q}\}$ denote the quasi vector fields of the curves. Let κ and τ be the Frenet curvature and the Frenet torsion of the curves, respectively and k_1, k_2, k_3 be the quasi curvatures of the curves.

Case 1. Spacelike Curves, $\vec{k}$ is timelike, $\boldsymbol{N}$ is spacelike:

The Frenet formulas are

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}.$$

Let θ be the hyperbolic angle between $\boldsymbol{B}$ and $\boldsymbol{b_q}$, then the quasi frame can be given according to the Frenet frame as follows:

$$\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

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The quasi curvatures can be given according to the Frenet curvatures as follows:

$$k_1 = \kappa\cosh\theta, \qquad k_2 = \kappa\sinh\theta\,, \qquad k_3 = -\theta' - \tau.$$

The quasi formulas are

$$\begin{bmatrix} \boldsymbol{t'_q} \\ \boldsymbol{n'_q} \\ \boldsymbol{b'_q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix}$$

Case 2. Spacelike Curves, $\vec{k}$ is timelike, $\boldsymbol{N}$ is timelike:
The Frenet formulas are

$$\begin{bmatrix} \boldsymbol{T'} \\ \boldsymbol{N'} \\ \boldsymbol{B'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

Let θ be the hyperbolic angle between $\boldsymbol{N}$ and $\boldsymbol{b_q}$, then the quasi frame can be given according to the Frenet frame as follows:

$$\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sinh\theta & -\cosh\theta \\ 0 & \cosh\theta & \sinh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

The quasi curvatures can be given according to the Frenet curvatures as follows:

$$k_1 = \kappa\sinh\theta, \qquad k_2 = -\kappa\cosh\theta\,, \qquad k_3 = \theta' + \tau.$$

The quasi formulas are

$$\begin{bmatrix} \boldsymbol{t'_q} \\ \boldsymbol{n'_q} \\ \boldsymbol{b'_q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix}$$

Case 3. Spacelike Curves, $\vec{k}$ is spacelike, $\boldsymbol{N}$ is timelike:
The Frenet formulas are

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

Let θ be the hyperbolic angle between $\boldsymbol{N}$ and $\boldsymbol{n_q}$, then the quasi frame can be given according to the Frenet frame as follows:

$$\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

The quasi curvatures can be given according to the Frenet curvatures as follows:

$$k_1 = -\kappa\cosh\theta, \qquad k_2 = -\kappa\sinh\theta\,, \qquad k_3 = \theta' + \tau.$$

The quasi formulas are

$$\begin{bmatrix} \boldsymbol{t}_q' \\ \boldsymbol{n}_q' \\ \boldsymbol{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix}$$

Case 4. Spacelike Curves, $\vec{k}$ is spacelike, $\boldsymbol{B}$ is timelike:
The Frenet formulas are

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

Let θ be the hyperbolic angle between $\boldsymbol{B}$ and $\boldsymbol{n_q}$, then the quasi frame can be given according to the Frenet frame as follows:

$$\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\theta & \cosh\theta \\ 0 & -\cosh\theta & -\sinh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

The quasi curvatures can be given according to the Frenet curvatures as follows:

$$k_1 = \kappa\sinh\theta, \qquad k_2 = -\kappa\cosh\theta\,, \qquad k_3 = -\theta' - \tau.$$

The quasi formulas are

$$\begin{bmatrix} \boldsymbol{t}_q' \\ \boldsymbol{n}_q' \\ \boldsymbol{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix}$$

Case 5. Timelike Curves, $\vec{k}$ is spacelike:
The Frenet formulas are

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

Let θ be the angle between $\boldsymbol{N}$ and $\boldsymbol{n_q}$, then the quasi frame can be given according to the Frenet frame as follows:

$$\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}$$

The quasi curvatures can be given according to the Frenet curvatures as follows:

$$k_1 = \kappa\cos\theta, \qquad k_2 = -\kappa\sin\theta\,, \qquad k_3 = \theta' + \tau.$$

The quasi formulas are

$$\begin{bmatrix} \boldsymbol{t_q'} \\ \boldsymbol{n_q'} \\ \boldsymbol{b_q'} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix}$$

Let S be a surface in $\mathbb{R}_1^3$. If the reduced metric on S is positive defined, then S is called a spacelike surface. If the reduced metric on S is Lorentzian metric, then S is called a timelike surface. So, the unit normal vector field of a spacelike surface is timelike, the unit normal vector field of a timelike surface is spacelike. If all tangent planes of S is null, in this case, the unit normal vector field of S is null, then S is called a null surface [8].

3. TIME EVOLUTION EQUATIONS IN $\mathbb{R}_1^3$

The following definitions can be given according to quasi frame for evolving curves in $\mathbb{R}_1^3$ considering the references [1, 4, 7]. From now on we study timelike surfaces in $\mathbb{R}_1^3$.

Case 1. Spacelike Curves, $\vec{k}$ is timelike, $\boldsymbol{N}$ is spacelike:

$$\frac{\partial}{\partial s}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} \tag{3.1}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & -\lambda_2 \\ -\lambda_1 & 0 & -\lambda_3 \\ -\lambda_2 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} \tag{3.2}$$

Applying the compatibility condititon

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}$$

in the light of the equations (3.1) and (3.2) one can easily get

$$\begin{bmatrix} 0 & \frac{\partial\lambda_1}{\partial s}-\frac{\partial k_1}{\partial t}+\lambda_2 k_3-k_2\lambda_3 & \frac{\partial k_2}{\partial t}-\frac{\partial\lambda_2}{\partial s}+k_1\lambda_3-\lambda_1 k_3\\ \frac{\partial k_1}{\partial t}-\frac{\partial\lambda_1}{\partial s}+k_2\lambda_3-\lambda_2 k_3 & 0 & \frac{\partial k_3}{\partial t}-\frac{\partial\lambda_3}{\partial s}+\lambda_1 k_2-k_1\lambda_2\\ \frac{\partial k_2}{\partial t}-\frac{\partial\lambda_2}{\partial s}+k_1\lambda_3-\lambda_1 k_3 & \frac{\partial k_3}{\partial t}-\frac{\partial\lambda_3}{\partial s}+\lambda_1 k_2-k_1\lambda_2 & 0\end{bmatrix}=0_{3\times 3}$$

Thus, the compatibility condition becomes

$$\frac{\partial k_1}{\partial t}=\frac{\partial\lambda_1}{\partial s}+\lambda_2 k_3-k_2\lambda_3$$

$$\frac{\partial k_2}{\partial t}=\frac{\partial\lambda_2}{\partial s}+\lambda_1 k_3-k_1\lambda_3$$

$$\frac{\partial k_3}{\partial t}=\frac{\partial\lambda_3}{\partial s}+k_1\lambda_2-\lambda_1 k_2$$

Case 2. Spacelike Curves, $\vec{k}$ is timelike, $\boldsymbol{N}$ is timelike:

$$\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\begin{bmatrix}0 & k_1 & -k_2\\ -k_1 & 0 & -k_3\\ -k_2 & -k_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix} \tag{3.3}$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\begin{bmatrix}0 & \nu_1 & -\nu_2\\ -\nu_1 & 0 & -\nu_3\\ -\nu_2 & -\nu_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix} \tag{3.4}$$

Applying the compatibility condititon

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}$$

in the light of the equations (3.3) and (3.4) one can easily get

$$\begin{bmatrix} 0 & \frac{\partial \nu_1}{\partial s}-\frac{\partial k_1}{\partial t}+\nu_2 k_3-k_2\nu_3 & \frac{\partial k_2}{\partial t}-\frac{\partial \nu_2}{\partial s}+k_1\nu_3-\nu_1 k_3 \\ \frac{\partial k_1}{\partial t}-\frac{\partial \nu_1}{\partial s}+k_2\nu_3-\nu_2 k_3 & 0 & \frac{\partial k_3}{\partial t}-\frac{\partial \nu_3}{\partial s}+\nu_1 k_2-k_1\nu_2 \\ \frac{\partial k_2}{\partial t}-\frac{\partial \nu_2}{\partial s}+k_1\nu_3-\nu_1 k_3 & \frac{\partial k_3}{\partial t}-\frac{\partial \nu_3}{\partial s}+\nu_1 k_2-k_1\nu_2 & 0 \end{bmatrix}=0_{3\times3}$$

Thus, the compatibility condition becomes

$$\frac{\partial k_1}{\partial t}=\frac{\partial \nu_1}{\partial s}+\nu_2 k_3-k_2\nu_3$$

$$\frac{\partial k_2}{\partial t}=\frac{\partial \nu_2}{\partial s}+\nu_1 k_3-k_1\nu_3$$

$$\frac{\partial k_3}{\partial t}=\frac{\partial \nu_3}{\partial s}+k_1\nu_2-\nu_1 k_2$$

Case 3. Spacelike Curves, $\vec{k}$ is spacelike, $\boldsymbol{N}$ is timelike:

$$\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\begin{bmatrix}0 & -k_1 & k_2\\ -k_1 & 0 & k_3\\ -k_2 & k_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix} \tag{3.5}$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\begin{bmatrix}0 & -\delta_1 & \delta_2\\ -\delta_1 & 0 & \delta_3\\ -\delta_2 & \delta_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix} \tag{3.6}$$

Applying the compatibility condititon

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}$$

in the light of the equations (3.5) and (3.6) one can easily get

$$\begin{bmatrix} 0 & \frac{\partial k_1}{\partial t}-\frac{\partial \delta_1}{\partial s}+\delta_2 k_3-k_2\delta_3 & \frac{\partial \delta_2}{\partial s}-\frac{\partial k_2}{\partial t}+k_1\delta_3-\delta_1 k_3 \\ \frac{\partial k_1}{\partial t}-\frac{\partial \delta_1}{\partial s}+\delta_2 k_3-k_2\delta_3 & 0 & \frac{\partial \delta_3}{\partial s}-\frac{\partial k_3}{\partial t}+k_1\delta_2-\delta_1 k_2 \\ \frac{\partial k_2}{\partial t}-\frac{\partial \delta_2}{\partial s}+\delta_1 k_3-k_1\delta_3 & \frac{\partial \delta_3}{\partial s}-\frac{\partial k_3}{\partial t}+k_1\delta_2-\delta_1 k_2 & 0 \end{bmatrix}=0_{3\times3}.$$

Thus, the compatibility condition becomes

$$\frac{\partial k_1}{\partial t}=\frac{\partial \delta_1}{\partial s}+k_2\delta_3-\delta_2 k_3$$

$$\frac{\partial k_2}{\partial t} = \frac{\partial \delta_2}{\partial s} + k_1\delta_3 - \delta_1 k_3$$

$$\frac{\partial k_3}{\partial t} = \frac{\partial \delta_3}{\partial s} + k_1\delta_2 - \delta_1 k_2$$

Case 4. Spacelike Curves, $\vec{k}$ is spacelike, $\boldsymbol{B}$ is timelike:

$$\frac{\partial}{\partial s}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} \tag{3.7}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \begin{bmatrix} 0 & -\rho_1 & \rho_2 \\ -\rho_1 & 0 & \rho_3 \\ -\rho_2 & \rho_3 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} \tag{3.8}$$

Applying the compatibility condititon

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix} \boldsymbol{t_q} \\ \boldsymbol{n_q} \\ \boldsymbol{b_q} \end{bmatrix}$$

in the light of the equations (3.7) and (3.8) one can easily get

$$\begin{bmatrix} 0 & \frac{\partial k_1}{\partial t} - \frac{\partial \rho_1}{\partial s} + \rho_2 k_3 - k_2\rho_3 & \frac{\partial \rho_2}{\partial s} - \frac{\partial k_2}{\partial t} + k_1\rho_3 - \rho_1 k_3 \\ \frac{\partial k_1}{\partial t} - \frac{\partial \rho_1}{\partial s} + \rho_2 k_3 - k_2\rho_3 & 0 & \frac{\partial \rho_3}{\partial s} - \frac{\partial k_3}{\partial t} + k_1\rho_2 - \rho_1 k_2 \\ \frac{\partial k_2}{\partial t} - \frac{\partial \rho_2}{\partial s} + \rho_1 k_3 - k_1\rho_3 & \frac{\partial \rho_3}{\partial s} - \frac{\partial k_3}{\partial t} + k_1\rho_2 - \rho_1 k_2 & 0 \end{bmatrix} = 0_{3\times 3}$$

Thus, the compatibility condition becomes

$$\frac{\partial k_1}{\partial t} = \frac{\partial \rho_1}{\partial s} + k_2\rho_3 - \rho_2 k_3$$

$$\frac{\partial k_2}{\partial t} = \frac{\partial \rho_2}{\partial s} + k_1\rho_3 - \rho_1 k_3$$

$$\frac{\partial k_3}{\partial t} = \frac{\partial \delta_3}{\partial s} + k_1\rho_2 - \rho_1 k_2$$

Case 5. Timelike Curves, $\vec{k}$ is spacelike:

$$\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\begin{bmatrix}0 & k_1 & k_2\\ k_1 & 0 & k_3\\ k_2 & -k_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix} \tag{3.9}$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\begin{bmatrix}0 & \mu_1 & \mu_2\\ \mu_1 & 0 & \mu_3\\ \mu_2 & -\mu_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix} \tag{3.10}$$

Applying the compatibiliy conditiiton

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}=\frac{\partial}{\partial t}\frac{\partial}{\partial s}\begin{bmatrix}\boldsymbol{t_q}\\ \boldsymbol{n_q}\\ \boldsymbol{b_q}\end{bmatrix}$$

in the light of the equations (3.9) and (3.10) one can easily get

$$\begin{bmatrix}0 & \frac{\partial\mu_1}{\partial s}-\frac{\partial k_1}{\partial t}+k_2\mu_3-\mu_2 k_3 & \frac{\partial\mu_2}{\partial s}-\frac{\partial k_2}{\partial t}+\mu_1 k_3-k_1\mu_3\\ \frac{\partial\mu_1}{\partial s}-\frac{\partial k_1}{\partial t}+k_2\mu_3-\mu_2 k_3 & 0 & \frac{\partial\mu_3}{\partial s}-\frac{\partial k_3}{\partial t}+\mu_1 k_2-k_1\mu_2\\ \frac{\partial\mu_2}{\partial s}-\frac{\partial k_2}{\partial t}+\mu_1 k_3-k_1\mu_3 & \frac{\partial k_3}{\partial t}-\frac{\partial\mu_3}{\partial s}-\mu_1 k_2+k_1\mu_2 & 0\end{bmatrix}=0_{3\times 3}$$

Thus, the compatibility condition becomes

$$\frac{\partial k_1}{\partial t}=\frac{\partial\mu_1}{\partial s}+k_2\mu_3-\mu_2 k_3$$

$$\frac{\partial k_2}{\partial t}=\frac{\partial\mu_2}{\partial s}+\mu_1 k_3-k_1\mu_3$$

$$\frac{\partial k_3}{\partial t}=\frac{\partial\mu_3}{\partial s}+\mu_1 k_2-k_1\mu_2$$

4. TIMELIKE SURFACES CONSTRUCTED BY THE EVOLUTION OF THE SPHERICAL INDICATRICES OF A SPACE CURVE IN $\mathbb{R}_1^3$

In this section, we study the timelike surfaces constructed by the evolution of the spherical indicatrix of the tangent, spherical indicatrix of the quasi normal and spherical indicatrix of the quasi binormal to a space curve in $\mathbb{R}_1^3$.

Let $\alpha: I \subset \mathbb{R} \to \mathbb{R}_1^3$ be a space curve given with quasi frame $\{\boldsymbol{t_q}, \boldsymbol{n_q}, \boldsymbol{b_q}\}$, parameterized with arc-length. The following space curves lie on a unit sphere in $\mathbb{R}_1^3$

$$\alpha_1(s) = \boldsymbol{t_q}(s)$$

$$\alpha_2(s) = \boldsymbol{n_q}(s)$$

$$\alpha_3(s) = \boldsymbol{b_q}(s)$$

and they are called the spherical indicatrix of the tangent, the quasi normal and the quasi binormal to the curve α, respectively [5].

4.1. TIMELIKE SURFACES CONSTRUCTED USING THE SPHERICAL INDICATRIX OF THE TANGENT

Let $\alpha_1(s) = \boldsymbol{t_q}(s)$ be the spherical indicatrix of the tangent to the curve α. The equation of surfaces constructed by the evolution of α_1 is given by

$$\psi = \tilde{\boldsymbol{t}}_q(s,t)$$

Theorem 4.1.1. Let α be a spacelike curve, the Frenet normal $\boldsymbol{N}$ of α be spacelike and the projection vector $\vec{k}$ be timelike. Under the assumption $k_1\lambda_2 - \lambda_1 k_2 > 0$, the Gaussian curvature K_1, the mean curvature H_1 and the principal curvatures k_{11} and k_{21} of ψ are given by

$$K_1 = 1, H_1 = -1, k_{11} = -1, k_{21} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\psi_s = k_1\boldsymbol{n_q} - k_2\boldsymbol{b_q}$$
$$\psi_t = \lambda_1\boldsymbol{n_q} - \lambda_2\boldsymbol{b_q} \qquad (4.1.1)$$

where the lower indices show partial differentiation. Then the unit normal to ψ is given by

$$\boldsymbol{N_\psi} = \frac{\psi_s \wedge \psi_t}{\|\psi_s \wedge \psi_t\|} = \boldsymbol{t_q}$$

Using the equations (3.1), (3.2) and (4.1.1), the second order derivatives are calculated and given by

$$\psi_{ss} = (-k_1^2 + k_2^2)\boldsymbol{t_q} + ((k_1)_s + k_2k_3)\boldsymbol{n_q} - ((k_2)_s + k_1k_3)\boldsymbol{b_q}$$

$$\psi_{tt} = (-\lambda_1^2 + \lambda_2^2)\boldsymbol{t_q} + ((\lambda_1)_t + \lambda_2\lambda_3)\boldsymbol{n_q} - ((\lambda_2)_t + \lambda_1\lambda_3)\boldsymbol{b_q}$$

$$\psi_{st} = (-k_1\lambda_1 + \lambda_2k_2)\boldsymbol{t_q} + ((k_1)_t + k_2\lambda_3)\boldsymbol{n_q} - ((k_2)_t + k_1\lambda_3)\boldsymbol{b_q}$$

The components $g_{ij}, (1 \leq i, j \leq 2)$ of the first fundamental form are obtained as follows:

$$g_{11} = \langle \psi_s, \psi_s \rangle = k_1^2 - k_2^2$$

$$g_{12} = \langle \psi_s, \psi_t \rangle = k_1\lambda_1 - \lambda_2k_2$$

$$g_{22} = \langle \psi_t, \psi_t \rangle = \lambda_1^2 - \lambda_2^2$$

The components $l_{ij}, (1 \leq i, j \leq 2)$ of the second fundamental form are obtained as follows:

$$l_{11} = \langle \psi_{ss}, \boldsymbol{N_\psi} \rangle = -k_1^2 + k_2^2$$

$$l_{12} = \langle \psi_{st}, \boldsymbol{N_\psi} \rangle = -k_1\lambda_1 + \lambda_2k_2$$

$$l_{22} = \langle \psi_{tt}, \boldsymbol{N_\psi} \rangle = -\lambda_1^2 + \lambda_2^2$$

Thus, we get the following equalities:

$$K_1 = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = 1$$

$$H_1 = \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = -1$$

$$k_{11} = H_1 + \sqrt{H_1^2 - K_1} = -1$$

$$k_{21} = H_1 - \sqrt{H_1^2 - K_1} = -1$$

Theorem 4.1.2. Let α be a spacelike curve, the Frenet normal $\boldsymbol{N}$ of α be timelike and the projection vector $\vec{k}$ be timelike. Under the assumption $k_1\nu_2 - \nu_1k_2 > 0$, the Gaussian curvature K_1, the mean curvature H_1 and the principal curvatures k_{11} and k_{21} of ψ are given by

$$K_1 = 1, H_1 = -1, k_{11} = -1, k_{21} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.1.1.

Theorem 4.1.3. Let α be a spacelike curve, the Frenet normal $\boldsymbol{N}$ of α be timelike and the projection vector $\vec{k}$ be spacelike. Under the assumption $k_1\delta_2 - \delta_1 k_2 > 0$, the Gaussian curvature K_1, the mean curvature H_1 and the principal curvatures k_{11} and k_{21} of ψ are given by

$$K_1 = 1, H_1 = -1, k_{11} = -1, k_{21} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.1.1.

Theorem 4.1.4. Let α be a spacelike curve, the Frenet binormal $\boldsymbol{B}$ of α be timelike and the projection vector $\vec{k}$ be spacelike. Under the assumption $k_1\rho_2 - \rho_1 k_2 > 0$, the Gaussian curvature K_1, the mean curvature H_1 and the principal curvatures k_{11} and k_{21} of ψ are given by

$$K_1 = 1, H_1 = -1, k_{11} = -1, k_{21} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.1.1.

4.2. TIMELIKE SURFACES CONSTRUCTED USING THE SPHERICAL INDICATRIX OF THE QUASI NORMAL

Let $\alpha_2(s) = \boldsymbol{n_q}(s)$ be the spherical indicatrix of the quasi normal to the curve α. The equation of surfaces constructed by the evolution of α_2 is given by

$$\phi = \widetilde{\boldsymbol{n}}_q(s,t).$$

Theorem 4.2.1. Let α be a spacelike curve, the Frenet normal $\boldsymbol{N}$ of α be spacelike and the projection vector $\vec{k}$ be timelike. Under the assumption $k_1\lambda_3 - \lambda_1 k_3 > 0$, the Gaussian curvature K_2, the mean curvature H_2 and the principal curvatures k_{12} and k_{22} of ϕ are given by

$$K_2 = 1, H_2 = -1, k_{12} = -1, k_{22} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned} \phi_s &= -k_1\boldsymbol{t_q} - k_3\boldsymbol{b_q} \\ \phi_t &= -\lambda_1\boldsymbol{t_q} - \lambda_3\boldsymbol{b_q} \end{aligned} \tag{4.2.1}$$

where the lower indices show partial differentiation. Then the unit normal to ϕ is given by

$$\boldsymbol{N}_\phi = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|} = \boldsymbol{n_q}$$

Using the equations (3.1), (3.2) and (4.2.1), the second order derivatives are calculated and given by

$$\phi_{ss} = (-(k_1)_s + k_2 k_3)\boldsymbol{t_q} + (-k_1^2 + k_3^2)\boldsymbol{n_q} + (k_1 k_2 - (k_3)_s)\boldsymbol{b_q}$$

$$\phi_{tt} = (-(\lambda_1)_t + \lambda_2\lambda_3)\boldsymbol{t_q} + (-\lambda_1^2 + \lambda_3^2)\boldsymbol{n_q} + (\lambda_1\lambda_2 - (\lambda_3)_t)\boldsymbol{b_q}$$

$$\phi_{st} = (-(k_1)_t + \lambda_2 k_3)\boldsymbol{t_q} + (-k_1\lambda_1 + k_3\lambda_3)\boldsymbol{n_q} + (k_1\lambda_2 - (k_3)_t)\boldsymbol{b_q}$$

The components $g_{ij}, (1 \leq i,j \leq 2)$ of the first fundamental form are obtained as follows:

$$g_{11} = \langle \phi_s, \phi_s \rangle = k_1^2 - k_3^2$$

$$g_{12} = \langle \phi_s, \phi_t \rangle = k_1\lambda_1 - \lambda_3 k_3$$

$$g_{22} = \langle \phi_t, \phi_t \rangle = \lambda_1^2 - \lambda_3^2$$

The components $l_{ij}, (1 \leq i,j \leq 2)$ of the second fundamental form are obtained as follows:

$$l_{11} = \langle \phi_{ss}, \boldsymbol{N_\phi} \rangle = -k_1^2 + k_3^2$$

$$l_{12} = \langle \phi_{st}, \boldsymbol{N_\phi} \rangle = -k_1\lambda_1 + \lambda_3 k_3$$

$$l_{22} = \langle \phi_{tt}, \boldsymbol{N_\phi} \rangle = -\lambda_1^2 + \lambda_3^2$$

Thus, we get the following equalities:

$$K_2 = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = 1$$

$$H_2 = \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = -1$$

$$k_{12} = H_2 + \sqrt{H_2^2 - K_2} = -1$$

$$k_{22} = H_2 - \sqrt{H_2^2 - K_2} = -1$$

Theorem 4.2.2. Let α be a spacelike curve, the Frenet normal $\boldsymbol{N}$ of α be timelike and the projection vector $\vec{k}$ be timelike. Under the assumption $k_1\nu_3 - \nu_1 k_3 > 0$, the Gaussian curvature K_2, the mean curvature H_2 and the principal curvatures k_{12} and k_{22} of ϕ are given by

$$K_2 = 1, H_2 = -1, k_{12} = -1, k_{22} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.2.1.

Theorem 4.2.3. Let α be a timelike curve and the projection vector $\vec{k}$ be spacelike. Under the assumption $k_3\mu_1 - \mu_3 k_1 > 0$, the Gaussian curvature K_2, the mean curvature H_2 and the principal curvatures k_{12} and k_{22} of ϕ are given by

$$K_2 = 1, H_2 = -1, k_{12} = -1, k_{22} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.2.1.

4.3. TIMELIKE SURFACES CONSTRUCTED USING THE SPHERICAL INDICATRIX OF THE QUASI BINORMAL

Let $\alpha_3(s) = \boldsymbol{b_q}(s)$ be the spherical indicatrix of the quasi binormal to the curve α. The equation of surfaces constructed by the evolution of α_3 is given by

$$\varphi = \widetilde{\boldsymbol{b}}_q(s,t).$$

Theorem 4.3.1. Let α be a spacelike curve, the Frenet normal $\boldsymbol{N}$ of α be timelike and the projection vector $\vec{k}$ be spacelike. Under the assumption $k_2\delta_3 - \delta_2 k_3 > 0$, the Gaussian curvature K_3, the mean curvature H_3 and the principal curvatures k_{13} and k_{23} of φ are given by

$$K_3 = 1, H_3 = -1, k_{13} = -1, k_{23} = -1.$$

Proof. The tangent space to the surface is spanned by

$$\begin{aligned} \varphi_s &= -k_2\boldsymbol{t_q} + k_3\boldsymbol{n_q} \\ \varphi_t &= -\delta_2\boldsymbol{t_q} + \delta_3\boldsymbol{n_q} \end{aligned} \tag{4.3.1}$$

where the lower indices show partial differentiation. Then the unit normal to φ is given by

$$\boldsymbol{N}_\varphi = \frac{\varphi_s \wedge \varphi_t}{\|\varphi_s \wedge \varphi_t\|} = \boldsymbol{b_q}$$

Using the equations (3.5), (3.6) and (4.3.1), the second order derivatives are calculated and given by

$$\varphi_{ss} = (-(k_2)_s - k_1k_3)\boldsymbol{t_q} + (k_1k_2 + (k_3)_s)\boldsymbol{n_q} + (k_3^2 - k_2^2)\boldsymbol{b_q}$$

$$\varphi_{tt} = (-(\delta_2)_t - \delta_1\delta_3)\boldsymbol{t_q} + (\delta_1\delta_2 + (\delta_3)_t)\boldsymbol{n_q} + (\delta_3^2 - \delta_2^2)\boldsymbol{b_q}$$

$$\varphi_{st} = (-(k_2)_t - \delta_1k_3)\boldsymbol{t_q} + (k_2\delta_1 + (k_3)_t)\boldsymbol{n_q} + (k_3\delta_3 - k_2\delta_2)\boldsymbol{b_q}$$

The components $g_{ij}, (1 \leq i,j \leq 2)$ of the first fundamental form are obtained as follows:

$$g_{11} = \langle \varphi_s, \varphi_s \rangle = k_2^2 - k_3^2$$

$$g_{12} = \langle \varphi_s, \varphi_t \rangle = k_2\delta_2 - \delta_3 k_3$$

$$g_{22} = \langle \varphi_t, \varphi_t \rangle = \lambda_2^2 - \lambda_3^2$$

The components $l_{ij}, (1 \leq i,j \leq 2)$ of the second fundamental form are obtained as follows:

$$l_{11} = \langle \varphi_{ss}, \boldsymbol{N}_\varphi \rangle = k_3^2 - k_2^2$$

$$l_{12} = \langle \varphi_{st}, \boldsymbol{N}_\varphi \rangle = \delta_3 k_3 - k_2\delta_2$$

$$l_{22} = \langle \varphi_{tt}, \boldsymbol{N}_\varphi \rangle = \delta_3^2 - \delta_2^2$$

Thus, we get the following equalities:

$$K_3 = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} = 1$$

$$H_3 = \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = -1$$

$$k_{13} = H_3 + \sqrt{H_3^2 - K_3} = -1$$

$$k_{23} = H_3 - \sqrt{H_3^2 - K_3} = -1$$

Theorem 4.3.2. Let α be a spacelike curve, the Frenet binormal $\boldsymbol{B}$ of α be timelike and the projection vector $\vec{k}$ be spacelike. Under the assumption $k_2\rho_3 - \rho_2 k_3 > 0$, the Gaussian curvature K_3, the mean curvature H_3 and the principal curvatures k_{13} and k_{23} of φ are given by

$$K_3 = 1, H_3 = -1, k_{13} = -1, k_{23} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.3.1.

Theorem 4.3.3. Let α be a timelike curve and the projection vector $\vec{k}$ be spacelike. Under the assumption $k_2\mu_3 - \mu_2 k_3 > 0$, the Gaussian curvature K_3, the mean curvature H_3 and the principal curvatures k_{13} and k_{23} of φ are given by

$$K_3 = 1, H_3 = -1, k_{13} = -1, k_{23} = -1.$$

Proof. The calculations can be made similar to the proof of Theorem 4.3.1.

5. CONCLUSIONS

Surfaces constructed by the evolution of the spherical indicatrices of a space curve are studied by Soliman in [1]. In that study, they used the Frenet frame of the curves in Euclidean 3-space. In our study [5] on surfaces constructed by evolution according to quasi frame, we studied spherical indicatrices of a space curve given with the quasi frame and obtained some geometric properties of the surfaces constructed by the evolution of them in Euclidean 3-space. There are two main advantages of the quasi frame over the Frenet frame [9]: 1) It is well defined even if the curve has vanishing second derivative, 2) it avoids the unnecessary twist around the tangent. Moreover, the computation of the quasi frame is easier than the rotation minimizing frames, for example one of them is Bishop frame. In this work, we consider timelike surfaces constructed by the evolution of the spherical indicatrices of a space curve given with quasi frame in Minkowski 3-space. We obtain some geometric properties such as fundamental forms and curvatures of these surfaces.

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