ORIGINAL PAPER

IDENTIFYING CANAL SURFACES IN $\mathbb{E}^4$: CHARACTERIZATION OF A SPECIAL CASE

NURTEN GURSES[1]

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Abstract. *This study, develops a way for representation of canal surfaces in 4-dimensional Euclidean space* $\mathbb{E}^4$*. It examines the fundamental forms, Gaussian and mean curvature for a special type canal surface. Moreover, the conditions of both Weingarten and linear Weingarten canal surfaces are given for this new special type. Finally, the graphs of the projections of the canal surfaces using different radius functions in* $\mathbb{E}^4$ *are presented.*

Keywords: *canal surface; tube surface; Weingarten surface; Gaussian curvature; mean curvature.*

1. INTRODUCTION

The envelope of a one-parameter set of spheres, centred at the spine curve (centre curve) $\alpha(u)$ with a radius function $r(u)$ is called a *canal surface*. If the radius function $r(u)$ is constant, the canal surface is called *tube (pipe) surface*. Canal surfaces have extensive applications in computer aided geometric design [1, 2]. There are many studies in the literature, related to canal and tube surfaces [3-15].

The *tube surface with variable radii* u constructed by a unit speed spine curve (centre curve) $\alpha(u)=(\alpha_1(u),\alpha_2(u),0)$ is defined by the following parametrization:

$$X(u,v)=\alpha(u)+r(u)\left(\cos v\mathfrak{n}(u)+\sin v\mathfrak{b}(u)\right),$$

where $\{\mathfrak{t}(u),\mathfrak{n}(u),\mathfrak{b}(u)\}$ is the Frenet frame of $\alpha(u)$ in 3-dimensional Euclidean space $\mathbb{E}^3$. It is a surface generated by a family of spheres of arbitrary radii $r(u)$ [16]. The geometrical properties of this type of surfaces are examined in [17]. In [18, 19], the concept of generalised tubes *(tube surface with variable radii* v*)* in $\mathbb{E}^3$ are examined and given by the following parametrisation

$$X(u,v)=\alpha(u)+r(v)\left(\cos v\mathfrak{n}(u)+\sin v\mathfrak{b}(u)\right),\ 0\le v\le 2\pi.$$

Moreover, in [20], the canal surface with $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u))$ in $\mathbb{E}^3$ is parameterised by:

$$X(u,v)=\alpha(u)-r(u)r'(u)\mathfrak{t}(u)\pm r(u)\sqrt{1-r'(u)^2}\left(\cos v\mathfrak{n}(u)+\sin v\mathfrak{b}(u)\right).$$

[1] Yildiz Technical University, Faculty of Art and Sciences, Department of Mathematics, 34220 Istanbul, Turkey.
E-mail: nbayrak@yildiz.edu.tr.

If the radius function $r(u)$ is constant, then $X(u,v)=\alpha(u)\pm r(\cos v\mathfrak{n}(u)+\sin v\mathfrak{b}(u))$ is the parametrization of tube surface [20].

In 4-dimensional Euclidean space $\mathbb{E}^4$, *tube surface with variable radii* u constructed by spine curve $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u),0)$ with arc-length parameterisation is given by [16]:

$$X(u,v)=\alpha(u)+r(u)(\cos vB_1(u)+\sin vB_2(u)),$$

where $\{T(u),N(u),B_1(u),B_2(u)\}$ is the Frenet frame of $\alpha(u)$. The investigation of the curvature properties of these surfaces are examined in [21]. Also in [22], loxodromes on tube surface with variable radii u are studied. By considering the curve $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u),\alpha_4(u))$ and the equivalent parametrization, tube surfaces with variable radii u with respect to the Bishop frame in $\mathbb{E}^4$ are examined in [23].

This paper is dedicated to construct canal surfaces in $\mathbb{E}^4$ with the help of the method given in [20]. Two types of canal surfaces are obtained by taking the radius function $r=r(u)$ and $r=r(v)$ for the spine curve $\alpha(u)=(\alpha_1(u),\alpha_2(u),\alpha_3(u),\alpha_4(u))$. By considering special conditions, the coefficients of the fundamental forms, Gaussian curvature, mean curvature vector and mean curvature of this special canal surface are investigated. Also, the conditions both the Weingarten and linear Weingarten canal surface are examined by using this approach. Finally, by taking different radius functions, the projections of canal surfaces in $\mathbb{E}^4$ are illustrated.

2. PRELIMINARIES

In this section, the basic concepts related to curves and surfaces in $\mathbb{E}^4$ are given.

Definition 2.1 Let $x=\sum_{i=1}^{4}x_ie_i$, $y=\sum_{i=1}^{4}y_ie_i$ and $z=\sum_{i=1}^{4}z_ie_i$ be vectors in $\mathbb{E}^4$, where $\{e_1,e_2,e_3,e_4\}$ is the standard basis of $\mathbb{E}^4$. The standard inner product is given by:

$$\langle x,y\rangle=x_1y_1+x_2y_2+x_3y_3+x_4y_4.$$

The ternary product (or vector product) of the vectors x,y and z is defined by [24-26]:

$$x\otimes y\otimes z=\begin{vmatrix} e_1 & e_2 & e_3 & e_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix}. \tag{2.1}$$

Let $\alpha:I\to\mathbb{E}^4$ be a curve with arc-length parameterisation. Then, the Frenet vectors and the curvatures of α are given by:

$$T=\alpha', \quad N=\frac{\alpha''}{\|\alpha''\|}, \quad B_1=B_2\otimes T\otimes N, \quad B_2=-\frac{\alpha'\otimes\alpha''\otimes\alpha'''}{\|\alpha'\otimes\alpha''\otimes\alpha'''\|},$$
$$k=\|\alpha''\|, \quad \tau=\frac{\langle B_1,\alpha'''\rangle}{k}, \quad \rho=\frac{\langle B_2,\alpha^{(4)}\rangle}{k\tau}. \tag{2.2}$$

Moreover, the Frenet formulae of the arc-length curve $\alpha:I\to\mathbb{E}^4$ with Frenet frame $\{T,N,B_1,B_2\}$ are given by [27]:

$$T'=kN, \quad N'=-kT+\tau B_1, \quad B_1'=-\tau N+\rho B_2, \quad B_2'=-\rho B_1. \tag{2.3}$$

On the other hand, let $X(u,v)$ represents the parametrisation of a regular surface M in $\mathbb{E}^4$. Let X_u and X_v be the basis vectors of the tangent space of M at an arbitrary point $p=X(u,v)$, namely, $T_p(M)=sp\{X_u,X_v\}$. Then, the coefficients of the first fundamental form can be given by:

$$E=<X_u,X_u>, F=<X_u,X_v>, G=<X_v,X_v>. \tag{2.4}$$

The surface patch is regular if and only if $EG-F^2\neq 0$. For an arbitrary point $p\in M$, $T_p(\mathbb{E}^4)=T_p(M)\oplus T_p^{\perp}(M)$, where $T_p^{\perp}(M)$ is the orthogonal component of $T_p(M)$ with the Reimannian connection $\tilde{D}$ in $\mathbb{E}^4$. For any vector fields X_i and X_j on M, the induced Reimannian connection D on M can be given by

$$D_{X_i}X_j=\left(\tilde{D}_{X_i}X_j\right)^t, \tag{2.5}$$

where $D_X Y=\sum_{k=1}^{2}\Gamma_{ij}^k X_k,\ 1\le i,j\le 2$. In the equation (2.5), the superscript t represents the tangential part and Γ_{ij}^k are known as Christoffel symbols calculated by the following formulae [20]:

$$\Gamma_{11}^1=\frac{GE_u-2FF_u+FE_v}{2(EG-F^2)}, \qquad \Gamma_{11}^2=\frac{2EF_u-EE_v-FE_u}{2(EG-F^2)},$$
$$\Gamma_{12}^1=\frac{GE_v-FG_u}{2(EG-F^2)}, \qquad \Gamma_{12}^2=\frac{EG_u-FE_v}{2(EG-F^2)}, \tag{2.6}$$
$$\Gamma_{22}^1=\frac{2GF_v-GG_u-FG_v}{2(EG-F^2)}, \qquad \Gamma_{22}^2=\frac{EG_v-2FF_v+FG_u}{2(EG-F^2)}.$$

It is well known that $\Gamma_{21}^1=\Gamma_{12}^1$, $\Gamma_{21}^2=\Gamma_{12}^2$. By considering the space of the tangent vector field $\chi(M)$ and normal vector field $\chi^{\perp}(M)$ of M, the well-defined, symmetric and bilinear map (which is called second fundamental form map) can be written as follows:

$$h:\chi(M)\times\chi(M)\to\chi^{\perp}(M)$$
$$(X_i,X_j)\to h(X_i,X_j)=\tilde{D}_{X_i}X_j-D_{X_i}X_j, \tag{2.7}$$

where $1 \le i, j \le 2$. The equation (2.7) is known as Gauss equation, [28]. Also $h(X_i, X_j)$ can be written such that

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \ 1 \le i, j \le 2,$$

where c_{ij}^k are called the coefficients of the second fundamental form. By taking the orthonormal frame field $\{N_1, N_2\}$ of M, the shape operator is defined as follows:

$$\begin{aligned} S_k : \chi(M) &\to \chi(M) \\ X_i &\to S_k(X_i) = -\tilde{D}_{X_i} N_k{}^t = -D_{X_i} N_k, 1 \le i, k \le 2. \end{aligned}$$

So, the following equality can be written:

$$\langle S_k, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k.$$

Also, for X_u and X_v the following equations hold [20]:

$$\begin{cases} \tilde{D}_{X_u} X_u = X_{uu} = D_{X_u} X_u + h(X_u, X_u), \\ \tilde{D}_{X_u} X_v = X_{uv} = D_{X_u} X_v + h(X_u, X_v), \\ \tilde{D}_{X_v} X_v = X_{vv} = D_{X_v} X_v + h(X_v, X_v). \end{cases} \tag{2.8}$$

Moreover, the Gaussian curvature and the mean curvature vector are given by:

$$K = \frac{1}{EG - F^2} \langle h(X_u, X_u), h(X_v, X_v) \rangle - \| h(X_u, X_v) \|^2, \tag{2.9}$$

$$\vec{H} = \frac{1}{2(EG - F^2)^2} \big(h(X_u, X_u)G - 2h(X_u, X_v)F + h(X_v, X_v)E \big). \tag{2.10}$$

If the mean curvature vanishes: $H = \| \vec{H} \| = 0$, then M is said to be minimal, [28].

3. STUDY ON CANAL SURFACES IN $\mathbb{E}^4$

In this section, subsection 3.1 concerns the new definitions of the canal surfaces in $\mathbb{E}^4$ with the help of the method given in [20]. Two types of canal surface are obtained by taking the radius function $r = r(u)$ and $r = r(v)$ for the spine curve $\alpha(u) = \big(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)\big)$ in Theorem 3.1 and Theorem 3.2, respectively.

In subsection 3.2, canal surface with special parametrisation (with $b(u,v) = 0$) is given and differential geometric properties of this special canal surface are investigated for the spine $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u), 0)$ under this special parametrization. Further, the conditions of Weingarten and linear Weingarten canal surfaces are examined by this approach.

3.1. DETERMINING CANAL SURFACES IN $\mathbb{E}^4$

Theorem 3.1 Let $\alpha : I \to \mathbb{E}^4$ be a curve with arc-length parameterisation. Then, the canal surface constructed by the spine curve $\alpha(u) = \left(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)\right)$ is parameterized by:

$$\begin{aligned} X(u,v) &= \alpha(u) - r(u)r'(u)T(u) + b(u,v)N(u) \\ &\pm \sqrt{r(u)^2(1-r'(u)^2) - b(u,v)^2}\left(\cos vB_1(u) + \sin vB_2(u)\right), \end{aligned} \tag{3.1}$$

where $\{T(u), N(u), B_1(u), B_2(u)\}$ is the Frenet frame of $\alpha(u)$.

Proof: Let X be a parametrisation of the envelope of spheres which define the canal surface and $\alpha(u)$ be a unit speed centre curve of canal surface with non-zero curvatures. Then, we have:

$$X(u,v) - \alpha(u) = a(u,v)T(u) + b(u,v)N(u) + c(u,v)B_1(u) + d(u,v)B_2(u), \tag{3.2}$$

where $a(u,v)$, $b(u,v)$, $c(u,v)$ and $d(u,v)$ are differentiable on the interval at which α is defined. Since $X(u,v)$ lies on the a sphere of radius $r(u)$ centred at $\alpha(u)$, then the following equation holds:

$$\left\| X(u,v) - \alpha(u) \right\|^2 = r(u)^2. \tag{3.3}$$

Because of the fact that $X(u,v) - \alpha(u)$ is normal to the canal surface, the following equations can be written:

$$\begin{cases} \langle X(u,v) - \alpha(u), X_u(u,v) \rangle = 0 \\ \langle X(u,v) - \alpha(u), X_v(u,v) \rangle = 0. \end{cases} \tag{3.4}$$

From the equation (3.3) we have:

$$\begin{cases} a^2 + b^2 + c^2 + d^2 = r(u)^2, \\ aa_u + bb_u + cc_u + dd_u = r(u)r'(u), \\ aa_v + bb_v + cc_v + dd_v = 0, \end{cases} \tag{3.5}$$

where $a = a(u,v)$, $b = b(u,v)$, $c = c(u,v)$ and $d = d(u,v)$. By deriving the equation (3.2) and using the Frenet formulae given in the equation (2.3), the following equation can be given:

$$X_u = \left(1 + a_u - bk\right)T + \left(b_u + ak - c\tau\right)N + \left(c_u + b\tau - d\rho\right)B_1 + \left(d_u + c\rho\right)B_2. \tag{3.6}$$

Then, by using the equations (3.2), (3.4), (3.5), (3.6):

$$\begin{cases} a = -r(u)r'(u), \\ b^2 + c^2 + d^2 = r(u)^2(1 - r'(u)^2). \end{cases}$$

By taking the parametrisation:

$$\begin{cases} b = b(u,v), \\ c(u,v) = \pm\sqrt{r(u)^2(1-r'(u)^2) - b(u,v)^2}\cos v, \\ d(u,v) = \pm\sqrt{r(u)^2(1-r'(u)^2) - b(u,v)^2}\sin v, \end{cases} \quad (3.7)$$

the representation of canal surface given in the equation (3.1) is obtained.

Further if $r = r(u)$ is constant, then the parametrization of tube surface can be given by

$$X(u,v) = \alpha(u) + b(u,v)N(u) \pm \sqrt{r^2 - b(u,v)^2}\left(\cos vB_1(u) + \sin vB_2(u)\right) \quad (3.8)$$

Note 3.1 The special type tube surface $X(u,v) = \alpha(u) \pm r\left(\cos vB_1(u) + \sin vB_2(u)\right)$ is determined under the condition $b(u,v) = 0$ in the equation (3.8) for $\alpha(u) = \left(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)\right)$. In [16], *tube surface with variable radii* u is defined as $X(u,v) = \alpha(u) + r(u)\left(\cos vB_1(u) + \sin vB_2(u)\right)$ in $\mathbb{E}^4$ for $\alpha(u) = \left(\alpha_1(u), \alpha_2(u), \alpha_3(u), 0\right)$. In [21], the geometric properties of these type of surfaces are examined. Also, equivalent parametrization is studied with respect to the Bishop frame in $\mathbb{E}^4$ considering $\alpha(u) = \left(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)\right)$ in [23].

Theorem 3.2 Let $\alpha : I \to \mathbb{E}^4$ be a curve with arc-length parameterisation and $\{T(u), N(u), B_1(u), B_2(u)\}$ be the Frenet frame of $\alpha(u)$. Then, the canal surface constructed by the spine curve $\alpha(u) = \left(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)\right)$ is parameterized by:

$$X(u,v) = \alpha(u) + b(u,v)N(u) \pm \sqrt{r(v)^2 - b(u,v)^2}\left(\cos vB_1(u) + \sin vB_2(u)\right). \quad (3.9)$$

Proof: As in the proof of Theorem 3.1, the following equations can be written:

$$X(u,v) - \alpha(u) = a(u,v)T(u) + b(u,v)N(u) + c(u,v)B_1(u) + d(u,v)B_2(u),$$

$$\left\| X(u,v) - \alpha(u) \right\|^2 = r(v)^2,$$

and

$$\begin{cases} \langle X(u,v) - \alpha(u), X_u \rangle = 0, \\ \langle X(u,v) - \alpha(u), X_v \rangle = 0. \end{cases}$$

Besides,

$$\begin{cases} a^2 + b^2 + c^2 + d^2 = r(v)^2, \\ aa_u + bb_u + cc_u + dd_u = 0, \\ aa_v + bb_v + cc_v + dd_v = r(v)r'(v), \end{cases}$$

and

$$X_u = \left(1 + a_u - bk\right)T + \left(b_u + ak - c\tau\right)N + \left(c_u + b\tau - d\rho\right)B_1 + \left(d_u + c\rho\right)B_2.$$

An easy computation shows that:

$$\begin{cases} a = 0, \\ b^2 + c^2 + d^2 = r(v)^2. \end{cases}$$

By taking the parametrisation

$$\begin{cases} b = b(u,v), \\ c(u,v) = \pm\sqrt{r(v)^2 - b(u,v)^2}\cos v, \\ d(u,v) = \pm\sqrt{r(v)^2 - b(u,v)^2}\sin v, \end{cases}$$

the canal surface given in the equation (3.9) can be obtained.

Moreover if $r = r(v)$ is constant, then the parametrization of tube surface can be given by

$$X(u,v) = \alpha(u) + b(u,v)N(u) \pm \sqrt{r^2 - b(u,v)^2}\left(\cos vB_1(u) + \sin vB_2(u)\right). \tag{3.10}$$

3.2. GEOMETRIC PROPERTIES OF NEW TYPE CANAL SURFACES IN $\mathbb{E}^4$

In this subsection, the special parametrization is addressed. By taking $b = b(u,v) = 0$ in the parametrisation (3.9), we have the following canal surface:

$$X(u,v) = \alpha(u) + r(v)\left(\cos vB_1(u) + \sin vB_2(u)\right). \tag{3.11}$$

For the sake of clarity, after this phase, the calculations are given for the curve $\alpha : I \to \mathbb{E}^4$, $\alpha(u) = \left(\alpha_1(u), \alpha_2(u), \alpha_3(u), 0\right)$ with arc-length parameterisation.

Proposition 3.1 The canal surface parameterized by (3.11) is regular if and only if $\left(1 + r^2\tau^2\cos^2 v\right)\left(r^2 + r'^2\right) \neq 0.$

Proof: The calculation of the frame field of the tangent space of the canal surface given in (3.11) can be calculated as follows:

$$\begin{cases} X_u = T - r\tau\cos vN, \\ X_v = \left(r'\cos v - r\sin v\right)B_1 + \left(r'\sin v + r\cos v\right)B_2. \end{cases} \tag{3.12}$$

By using the equations (2.4), the coefficients of first fundamental form of $X(u,v)$ can be obtained such that:

$$\begin{cases} E = 1 + r^2\tau^2\cos^2 v, \\ F = 0, \\ G = r^2 + r'^2. \end{cases} \tag{3.13}$$

Hence, $X(u,v)$ is regular if and only if $EG-F^2=\left(1+r^2\tau^2\cos^2 v\right)\left(r^2+r'^2\right)\neq 0.$

Proposition 3.2 The Christoffel symbols of the canal surface M given by (3.11) are calculated by:

$$\Gamma_{11}^1=\frac{r^2\tau\tau'\cos^2 v}{1+r^2\tau^2\cos^2 v},\qquad \Gamma_{11}^2=\frac{r\tau^2\cos v(r\sin v-r'\cos v)}{r^2+r'^2},$$
$$\Gamma_{12}^1=\frac{r\tau^2\cos v(-r\sin v+r'\cos v)}{1+r^2\tau^2\cos^2 v},\quad \Gamma_{12}^2=0, \tag{3.14}$$
$$\Gamma_{22}^1=0,\qquad \Gamma_{22}^2=\frac{r'(r+r'')}{r^2+r'^2}.$$

Proof: By using the equations (3.13), the partial derivatives of the coefficients of the first fundamental form can be calculated by:

$$E_u=2r^2\tau\tau'\cos^2 v,\quad E_v=-2r^2\tau^2\cos v\sin v+2rr'\tau^2\cos^2 v,$$
$$F_u=0,\qquad F_v=0,$$
$$G_u=0,\qquad G_v=2rr'+2r'r''.$$

From the formulae given in the equation (2.6), the results in the equation (3.14) are obtained.

Corollary 3.1 The second partial derivatives for parametrisation (3.11) of the canal surface M in $\mathbb{E}^4$, can be calculated as follows:

$$X_{uu}=kr\tau\cos vT+(k-\cos vr\tau')N-r\tau^2\cos vB_1,$$
$$X_{uv}=\tau(r\sin v-r'\cos v)N, \tag{3.15}$$
$$X_{vv}=(-2r'\sin v+(-r+r'')\cos v)B_1+(2r'\cos v+(-r+r'')\sin v)B_2.$$

Moreover, by using the equation (2.7), the following equations are determined:

$$\begin{cases} h(X_u,X_u)=\left(kr\tau\cos v-\dfrac{r^2\tau\tau'\cos^2 v}{1+r^2\tau^2\cos^2 v}\right)T+\left(k-\cos vr\tau'+\dfrac{r^3\tau^2\tau'\cos^3 v}{1+\cos^2 r^2\tau^2}\right)N \\ \qquad +\left(-r\tau^2\cos v-\dfrac{r\tau^2\cos v(r\sin v-r'\cos v)(-r\sin v+r'\cos v)}{r^2+r'^2}\right)B_1 \\ \qquad +\left(-\dfrac{r\tau^2\cos v(r\sin v-r'\cos v)(r\cos v+r'\sin v)}{r^2+r'^2}\right)B_2, \\ h(X_u,X_v)=\left(\dfrac{r\tau^2\cos v(r\sin v-\cos vr')}{1+r^2\tau^2\cos^2 v}\right)T+\left(\dfrac{\tau(r\sin v-\cos vr')}{1+r^2\tau^2\cos^2 v}\right)N, \\ h(X_v,X_v)=\left(-\dfrac{(r\cos v+r'\sin v)(r^2+2r'^2-rr'')}{r^2+r'^2}\right)B_1 \\ \qquad +\left(\dfrac{(-r\sin v+r'\cos v)(r^2+2r'^2-rr'')}{r^2+r'^2}\right)B_2. \end{cases} \tag{3.16}$$

Theorem 3.3 The Gaussian curvature of the canal surface M parameterized by (3.11) in $\mathbb{E}^4$ is given by:

$$K = \frac{\left(r^2 + 2r'^2 - rr''\right)\left(r^2(-1+\tau^2\cos^2 v) - 2r'^2 + r(r'\tau^2\cos v\sin v + r'')\right)}{\left(1+r^2\tau^2\cos^2 v\right)\left(r^2+r'^2\right)^2}. \quad (3.17)$$

Proof: By substituting the equations (3.13) and (3.16) into the formula given in the equation (2.9), the equation (3.17) can be calculated.

Corollary 3.2 The Gaussian curvature of the canal surface M generated by a planar curve α and parameterized by (3.11) in $\mathbb{E}^4$ is given by:

$$K = -\frac{\left(r^2 + 2r'^2 - rr''\right)^2}{\left(r^2+r'^2\right)^2}.$$

Corollary 3.3 The Gaussian curvature of the tube surface M parameterized by (3.11) in $\mathbb{E}^4$ with $r(v) = c$ (c is constant) is given by:

$$K = \frac{-1+\tau^2\cos^2 v}{1+r^2\tau^2\cos^2 v}. \quad (3.18)$$

Corollary 3.4 If M is a tube surface parameterized by (3.11) with $r(v) = c$ (c is constant), and constructed by a planar curve in $\mathbb{E}^4$, then $K = -1$.

Theorem 3.4 The mean curvature vector of the canal surface M parameterized by (3.11) in $\mathbb{E}^4$ is obtained by:

$$\begin{aligned}\vec{H} =& \left(\frac{r\tau\cos v(k + kr^2\tau^2\cos^2 v - r\tau'\cos v)}{2(1+r^2\tau^2\cos^2 v)^3(r^2+r'^2)}\right)T + \left(\frac{k + kr^2\tau^2\cos^2 v - r\tau'\cos v}{2(1+r^2\tau^2\cos^2 v)^3(r^2+r'^2)}\right)N \\ &+\left(-\frac{r\tau^2\cos v(r\cos v + r'\sin v)^2}{2(1+r^2\tau^2\cos^2 v)^2(r^2+r'^2)^2} - \frac{(r\cos v + r'\sin v)(r^2+2r'^2-rr'')}{2(1+r^2\tau^2\cos^2 v)(r^2+r'^2)^3}\right)B_1 \\ &+\left(\frac{r\tau^2\cos v(-r\sin v + r'\cos v)(r\cos v + r'\sin v)}{2(1+r^2\tau^2\cos^2 v)^2(r^2+r'^2)^2}\right. \\ &\left. - \frac{(r\sin v - r'\cos v)(r^2+2r'^2-rr'')}{2(1+r^2\tau^2\cos^2 v)(r^2+r'^2)^3}\right)B_2,\end{aligned} \quad (3.19)$$

and the mean curvature of M is calculated by:

$$H=\frac{\begin{pmatrix}\left(r^2+r'^2\right)^4\left(k+kr^2\tau^2\cos^2 v-r\tau'\cos v\right)^2\left(1+r^2\tau^2\cos^2 v\right)\\ +\left(r^2+r'^2\right)\left(1+r^2\tau^2\cos^2 v\right)^2\left[r\tau^2\cos v\left(r\cos v+r'\sin v\right)\left(r^2+r'^2\right)\right.\\ \left.+\left(1+r^2\tau^2\cos^2 v\right)\left(r^2+2r'^2-rr''\right)\right]^2\end{pmatrix}^{\frac{1}{2}}}{2\left(1+\cos^2 vr^2\tau^2\right)^3\left(r^2+r'^2\right)^3}. \tag{3.20}$$

Proof: By substituting the equations (3.12) and (3.15) in the formula given in equation (2.10), the equations (3.19) and (3.20) can be obtained.

Corollary 3.5 The mean curvature vector and mean curvature of the canal surface M parameterized by (3.11) in $\mathbb{E}^4$ are given related to the spine curve as follows:

i) If M is constructed by a planar curve:

$$\vec{H}=\left(\frac{k}{2(r^2+r'^2)}\right)N+\left(-\frac{(r\cos v+r'\sin v)(r^2+2r'^2-rr'')}{2(r^2+r'^2)^3}\right)B_1$$
$$+\left(-\frac{(r\sin v-r'\cos v)(r^2+2r'^2-rr'')}{2(r^2+r'^2)^3}\right)B_2,$$

$$H=\frac{\left(k^2(r^2+r'^2)^4+(r^2+r'^2)(r^2+2r'^2-rr'')^2\right)^{\frac{1}{2}}}{2(r^2+r'^2)^3}. \tag{3.21}$$

ii) If M is constructed by a straight line:

$$\vec{H}=\left(-\frac{(r\cos v+r'\sin v)(r^2+2r'^2-rr'')}{2(r^2+r'^2)^3}\right)B_1+\left(-\frac{(r\sin v-r'\cos v)(r^2+2r'^2-rr'')}{2(r^2+r'^2)^3}\right)B_2,$$

$$H=\frac{r^2+2r'^2-rr''}{2(r^2+r'^2)^{\frac{5}{2}}}. \tag{3.22}$$

Corollary 3.6 The mean curvature vector and mean curvature of the tube surface with $r(v)=c$ (c is constant) M parameterized by (3.11) in $\mathbb{E}^4$ are obtained by:

$$\vec{H}=\left(\frac{\tau\cos v(k+kr^2\tau^2\cos^2 v-r\tau'\cos v)}{2r(1+r^2\tau^2\cos^2 v)^3}\right)T+\left(\frac{k+kr^2\tau^2\cos^2 v-r\tau'\cos v}{2r^2(1+r^2\tau^2\cos^2 v)^3}\right)N$$
$$+\left(-\frac{\cos v(1+2r^2\tau^2\cos^2 v)}{2r^3(1+r^2\tau^2\cos^2 v)^2}\right)B_1+\left(-\frac{\sin v(1+2r^2\tau^2\cos^2 v)}{2r^3(1+r^2\tau^2\cos^2 v)^2}\right)B_2,$$

$$H=\frac{\begin{pmatrix}1+5r^2\tau^2\cos^2 v+4r^6\tau^6\cos^6 v+k^2(r+r^3\tau^2\cos^2 v)^2\\ -2kr^3\tau'\cos v(1+r^2\tau^2\cos^2 v)+r^4\cos^2 v(8\tau^4\cos^2 v+\tau'^2)\end{pmatrix}^{\frac{1}{2}}}{2r^3(1+r^2\tau^2\cos^2 v)^{\frac{5}{2}}}. \tag{3.23}$$

Corollary 3.7 The mean curvature vector and the mean curvature of the tube surface M parameterized by (3.11) with $r(v)=c$ (c is constant) in $\mathbb{E}^4$ with respect to following spine curves can be given as follows:

i) If M is constructed by a planar curve:

$$\vec{H}=\frac{k}{2r^2}N-\frac{\cos v}{2r^3}B_1-\frac{\sin v}{2r^3}B_2,$$
$$H=\frac{1}{2r^3}\sqrt{1+k^2r^2}.$$

ii) If M is constructed by a straight line:

$$\vec{H}=-\frac{\cos v}{2r^3}B_1-\frac{\sin v}{2r^3}B_2,$$
$$H=\frac{1}{2r^3}. \tag{3.24}$$

Proposition 3.3 Let M be a tube surface parameterized by (3.11) in $\mathbb{E}^4$. Then, M is a Weingarten surface if and only if one of the conditions hold:

- M is constructed by a planar curve, i.e, $\tau=0$.
- M is constructed by a straight line, i.e, $k=0$.
- The first curvature of α is constant, i.e, $k'=0$.

Proof: By using the equations (3.18) and (3.23) :

$$K_uH_v-K_vH_u=\frac{kk'\tau^2(1+r^2)\cos v\sin v}{r(1+r^2\tau^2\cos^2 v)^3\sqrt{(1+2r^2\tau^2\cos^2 v)^2+k^2(r^2+r^4\tau^2\cos^2 v)}}.$$

Then, $K_uH_v-K_vH_u=0$ for which $\tau=0$ or $k=0$ or $k'=0$.

Proposition 3.4 Let M be a tube surface parameterized by (3.11) with $r(v)=c$ (c is constant) in $\mathbb{E}^4$. Then, M is a linear Weingarten surface if and only the spine curve α is a straight line.

Proof: Suppose that M is a tube surface with parametrisation (3.11) with $r(v)=c$ (c is constant) and the spine curve α is a straight line. Then, $K=-1$ from Corollary 3.2 and $H=\frac{1}{2r^3}$ from the equation (3.24). So,

$$\mathfrak{a}K+\mathfrak{b}H=-\mathfrak{a}+\frac{\mathfrak{b}}{2r^3}=\mathfrak{c}.$$

The above equation has the solution $(\mathfrak{a},2(\mathfrak{a}+\mathfrak{c})r^3,\mathfrak{c})$ for non-zero real numbers $\mathfrak{a},\mathfrak{b}$ and $\mathfrak{c}$.

4. EXAMPLES ON VISUALIZATION

In this section, the examples of canal surfaces in $\mathbb{E}^4$ are presented. The projections of canal surfaces in $\mathbb{E}^4$ are plotted by using the command:

$$ParametricPlot3D[x, y, z+w, u, u_{min}, u_{max}, v, v_{min}, v_{max}]$$

in Wolfram Mathematica 9.

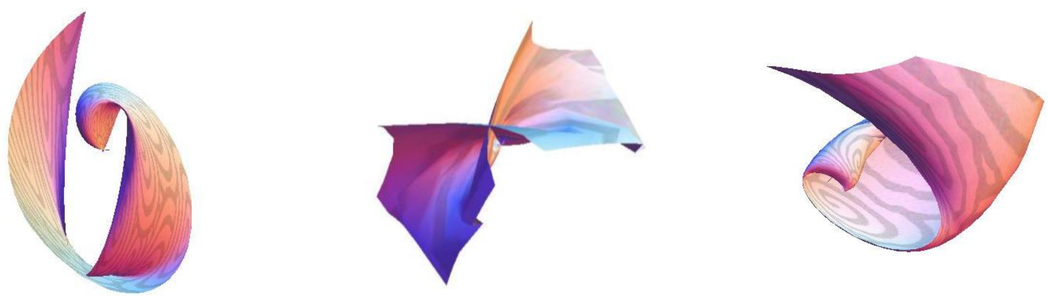
Example 4.1 Consider the curve with arc-length parameterisation:

$$\alpha(u) = \frac{1}{\sqrt{2}}\left(\cos^2 u, \frac{\sin 2u}{2}, \sin u, \cos u\right).$$

If we use the parametrisation (3.9), then the equation of the canal surface is given below (see Fig. 1):

$$\begin{aligned} X(u,v) = \Bigg(& \frac{1}{\sqrt{2}}\cos^2 u - \frac{2}{\sqrt{5}} b\cos 2u + \frac{1}{\sqrt{2}} q\cos v\sin 2u - \frac{1}{\sqrt{5}} q\sin v\cos 2u, \\ & \frac{1}{2\sqrt{2}}\sin 2u + 2b\sin 2u - \frac{1}{\sqrt{2}} q\cos v\cos 2u - \frac{1}{\sqrt{5}} q\sin v\sin 2u, \\ & \frac{1}{\sqrt{2}}\sin u + 2b\sin u + \frac{1}{\sqrt{2}} q\cos v\cos u + \frac{2}{\sqrt{5}} q\sin v\sin u, \\ & \frac{1}{\sqrt{2}}\cos u + b\cos u - \frac{1}{\sqrt{2}} q\cos v\sin u + \frac{2}{\sqrt{5}} q\sin v\cos u \Bigg), \end{aligned} \tag{4.1}$$

where $q = \sqrt{r(v)^2 - b(u,v)^2}$.



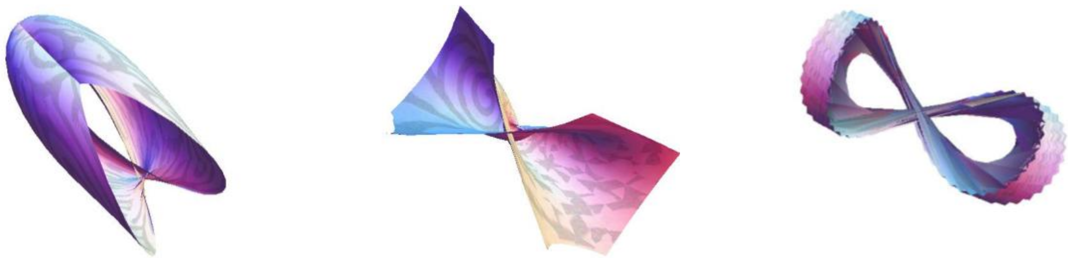
$r(v) = v$, $b(u,v) = \sin v$ $r(v) = e^v$, $b(u,v) = uv$ $r(v) = v^2$, $b(u,v) = \sinh uv$

Figure 1. Canal surfaces given in equation (4.1).

Moreover, the equation of the canal surface with parametrisation (3.11) is given by the following equation (see Fig. 2):

$$\begin{aligned} X(u,v) = \Bigg(& \frac{1}{\sqrt{2}}\cos^2 u + \frac{1}{\sqrt{2}} r(v)\cos v\sin 2u - \frac{1}{\sqrt{5}} r(v)\sin v\cos 2u, \\ & \frac{1}{2\sqrt{2}}\sin 2u - \frac{1}{\sqrt{2}} r(v)\cos v\cos 2u - \frac{1}{\sqrt{5}} r(v)\sin v\sin 2u, \end{aligned}$$

$$\frac{1}{\sqrt{2}}\sin u+\frac{1}{\sqrt{2}}r(v)\cos v\cos u+\frac{2}{\sqrt{5}}r(v)\sin v\sin u,$$
$$\left.\frac{1}{\sqrt{2}}\cos u-\frac{1}{\sqrt{2}}r(v)\cos v\sin u+\frac{2}{\sqrt{5}}r(v)\sin v\cos u\right). \tag{4.2}$$



$r(v)=\sin v$ $r(v)=\cosh v$ $r(v)=e^{v^2}$

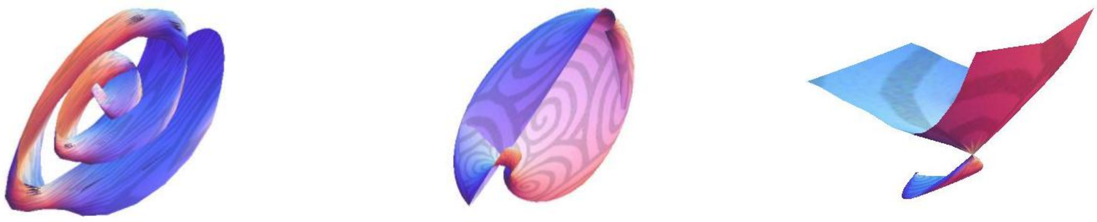
Figure 2. Canal surfaces given in equation (4.2).

Example 4.2 Consider the curve with arc-length parameterisation:

$$\alpha(u)=\left(\sin\frac{u}{2},\ \frac{\sqrt{3}}{2}u,\ \cos\frac{u}{2},\ 0\right).$$

The canal surface given by the parametrisation (3.11) is as follows (see Fig. 3):

$$X(u,v)=\left(\sin\frac{u}{2}-\frac{\sqrt{3}}{2}r(v)\cos v\cos\frac{u}{2},\frac{\sqrt{3}}{2}u+\frac{1}{2}r(v)\cos v,\cos\frac{u}{2}+\frac{\sqrt{3}}{2}r(v)\cos v\sin\frac{u}{2},r(v)\sin v\right). \tag{4.3}$$



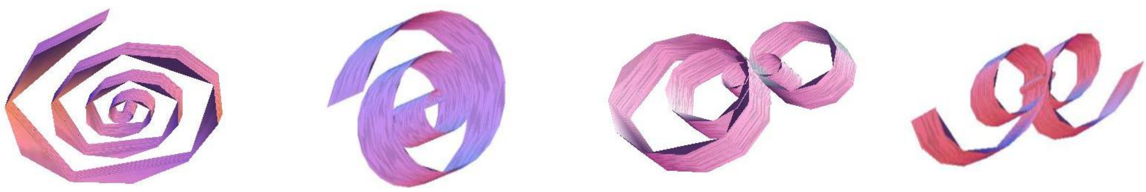
$r(v)=v$ $r(v)=v^2$ $r(v)=e^{v+3}$

Figure 3. Canal surfaces given in equation (4.3).

Example 4.3 Consider the straight line:

$$\alpha(u)=\left(\frac{\sqrt{3}}{2}u+2,\frac{1}{\sqrt{2}}u+3,\frac{1}{\sqrt{2}}u+4,0\right).$$

Then, the graph of the canal surface of the above straight line by taking parametrisation (3.11) can be seen in Fig. 4 for different radius functions.



$r(v)=v$ $r(v)=v^2$ $r(v)=v\sin v$ $r(v)=v\cos v$

Figure 4. Canal surfaces of a straight line.

4. CONCLUSION

This study deals with constructing of canal surfaces in $\mathbb{E}^4$ by using the theory given in [20]. By taking the radius function $r = r(u)$ and $r = r(v)$ for the unit speed spine curve $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u))$, the parametrization of new type of canal surfaces are obtained in the equations (3.1) and (3.9). The differential geometric properties are examined under the condition $r = r(v)$, $b(u,v) = 0$ for the curve $\alpha(u) = (\alpha_1(u), \alpha_2(u), \alpha_3(u), 0)$. Finally, graphing surfaces in examples in Section 4 makes up visuality for better understanding.

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