ORIGINAL PAPER

APPLICATION OF ABOODH TRANSFORM ON SOME FRACTIONAL ORDER MATHEMATICAL MODELS

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Abstract. *In this work, a new emerging analytical techniques variational iteration method combine with Aboodh transform has been applied to find out the significant important analytical and convergent solution of some mathematical models of fractional order. These mathematical models are of great interest in engineering and physics. The derivative is in Caputo's sense. These analytical solutions are continuous that can be used to understand the physical phenomena without taking interpolation concept. The obtained solutions indicate the validity and great potential of Aboodh transform with the variational iteration method and show that the proposed method is a good scheme. Graphically, the movements of some solutions are presented at different values of fractional order.*

Keywords: *Aboodh integral transform; variational iteration method; Adomian polynomials; analytical solution.*

1. INTRODUCTION

Integral transform is about two hundred years old as Laplace transform introduced in 1780s by P.S. Laplace [1]. After him many researchers and few scientists worked on transforms as it can be seen in [2-7] and researchers have seen the applications of integral transforms and fractional calculus in different fields of real life and sciences such as biology, diffusion, probability potential theory, electrochemistry, optics, mathematical physics, and engineering mathematics. When researchers realize that integral transforms play an important and vital role in solving differential equations then they turn their research attention in the development of integral in last century and more especially in this current decades. Like, the Sumudu transform introduced by Watugula in (1993) [8], the Natural transform initiated by Khan and Khan (2008) [9], T.M. Elzaki introduced the Elzaki transform [10] in (2011) and K.S. Aboodh was devised the Aboodh transform in (2013) by [11]. In the last three decades, more attention has been paid by the authors and researchers in the kingdom of nonlinear fractional PDEs due to their compact description of modeling of widespread complex phenomena in the universe. The fractional concept has become common in modeling approach and is extensively applied for real world physical problems in almost all fields of engineering and mathematical physics. Any problem can be solved by integral transform directly but to solve nonlinear FPDEs researchers emerge transforms with other method to make approximate results reliable. However, the emergence of variational iteration method with these newly introduced integral transforms remains on other hand. Recently, Cherif [11] has reported the coupling of variational iteration method with Aboodh transform for solving some one-dimensional PDEs. Integral transforms for fractional differential equation is new

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idea to work in this modern era. The primary objective of this manuscript is the application of the introduced new method [11] to some problems of fractional order arising in mathematical physics.

2. MATERIALS AND METHODS

2.1. ABOODH TRANSFORM

The definition of Aboodh transform [12] for set A

$$A=\{u:|u(t)|<Me^{|t|/k}, t\in(-1)^{j}\times[0,\infty);(M,k_1,k_2>0)\}$$

is known by

$$A[u(t)]=\frac{1}{v}\int_0^\infty u(t)e^{-vt}dt=U(v), v\in(k_1,k_2).$$

Using Aboodh transform, nth order derivative is

$$A\{u^n(t)\}=v^n U(v)-\sum_{k=0}^{n-1}\frac{u^k(0)}{v^{2-n+k}}. \tag{1}$$

We can indisputably stretch out this consequence to the nth derivative by exploiting numerical enlistment.

2.2. METHOD

To demonstrate the indication of used method, here the most general non- linear FPDE subject to constraint

$$^{C}D_\tau^\eta u(v,\tau)+Ru(v,\tau)+Nu(v,\tau)=g(v,\tau), \tag{2}$$

where $n-1<q\leq n,\ n=1,2,...$ along with initial conditions

$$\left[\frac{\partial^{n-1}u(v,\tau)}{\partial\tau^{n-1}}\right]_{\tau=0}=j_{n-1}(v). \tag{3}$$

where $^{C}D_\tau^\eta=\dfrac{\partial^\eta}{\partial\tau^\eta}$ is Caputo fractional derivative, R as linear operator, represent all nonlinear terms as well as $g(v,\tau)$ is occurs as source term.

Now applying M as a transform operator

$$M[{}^{C}D_{\tau}^{\eta}u(v,\tau)]+M[Ru(v,\tau)]+M[Nu(v,\tau)]=M[g(v,\tau)]. \tag{4}$$

Now, we can apply the differential properties of transformation. Here M is invertible, and its equivalent can be expressed as

$$U(v,\tau)=H(v,\tau)-M^{-1}\big[PM[Ru(v,\tau)+Nu(v,\tau)]\big], \tag{5}$$

where P comes due to initial conditions and $H(v,\tau)$, is resulting source term

By taking partial derivative of Eq. (5), we get

$$\frac{\partial U(v,\tau)}{\partial \tau}+\frac{\partial}{\partial \tau}M^{-1}\big[PM[Ru(v,\tau)+Nu(v,\tau)]\big]-\frac{\partial H(v,\tau)}{\partial \tau}=0. \tag{6}$$

The correction functional is read as

$$u_{n+1}(v,\tau)=u_n(v,\tau)+\int_0^{\tau}\lambda(\zeta)\begin{bmatrix}\frac{\partial U_n(v,\zeta}{\partial \zeta}+\frac{\partial}{\partial \zeta}M^{-1}\left(\frac{1}{v^q}M\big[Ru_n(v,\zeta)+Nu_n(v,\zeta)\big]\right)\\ -\frac{\partial H(v,\zeta)}{\partial \zeta}\end{bmatrix}d\zeta.. \tag{7}$$

here λ is the general Lagrange multiplier. Finally, the solution is $u(v,\tau)=\lim_{n\to\infty}u_n(v,\tau)$.

The variational iteration method's convergence is introduced by Tatari. et al in [13].

3. RESULTS AND DISCUSSION

Problem 1: Consider one-dimensional nonlinear homogeneous time-fractional Boussinesq-like equation [14]

$$D_t^{\alpha}u+(u^2)_{xx}-(u^2)_{xxxx}=0,\ t>0,\ x\in R, 1<\alpha\le 2, \tag{8}$$

subject to constraints

$$u(x,0)=\frac{4}{3}\sinh^2\left(\frac{x}{4}\right),\ u_t(x,0)=-\frac{1}{3}\sinh\left(\frac{x}{2}\right). \tag{9}$$

Using fractional Aboodh transform with initial conditions we get,

$$A\big[u(x,t)\big]=\frac{1}{v^2}\left(\frac{4}{3}\sinh^2\left(\frac{x}{4}\right)\right)+\frac{1}{v^3}\left(-\frac{1}{3}\sinh\left(\frac{x}{2}\right)\right)-\frac{1}{v^{\alpha}}A[(A_n)_{xx}]+\frac{1}{v^{\alpha}}A\big[(B_n)_{xxxx}\big] \tag{10}$$

Using inverse Aboodh transform

$$u(x,t)=\frac{4}{3}\sinh^2\left(\frac{x}{4}\right)-\frac{1}{3}\sinh\left(\frac{x}{2}\right)t-A^{-1}\left[\frac{1}{v^\alpha}A[(A_n)_{xx}]\right]+A^{-1}\left[\frac{1}{v^\alpha}A[(B_n)_{xxxx}]\right]. \quad (11)$$

Taking partial derivative of Eq. (11)

$$\frac{\partial}{\partial t}u(x,t)=-\frac{\partial}{\partial t}\frac{1}{3}\sinh\left(\frac{x}{2}\right)t-\frac{\partial}{\partial t}A^{-1}\left[\frac{1}{v^\alpha}A[(A_n)_{xx}]\right]+\frac{\partial}{\partial t}A^{-1}\left[\frac{1}{v^\alpha}A[(B_n)_{xxxx}]\right]. \quad (12)$$

The correction functional is read as

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(\zeta)\begin{pmatrix}\frac{\partial u_n(x,\zeta)}{\partial\zeta}-\frac{1}{3}\sinh\left(\frac{x}{2}\right)+\frac{\partial}{\partial\zeta}\left\{A^{-1}\left[\frac{1}{v^\alpha}A[(A_n)_{xx}]\right]\right\}- \\ \frac{\partial}{\partial\zeta}\left\{A^{-1}\left[\frac{1}{v^\alpha}A[(B_n)_{xxxx}]\right]\right\}\end{pmatrix}d\zeta \quad (13)$$

Here the value of Lagrange multiplier is -1 via variational theory.

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\begin{pmatrix}\frac{\partial u_n(x,\zeta)}{\partial\zeta}-\frac{1}{3}\sinh\left(\frac{x}{2}\right)+\frac{\partial}{\partial\zeta}\left\{A^{-1}\left[\frac{1}{v^\alpha}A[(A_n)_{xx}]\right]\right\}- \\ \frac{\partial}{\partial\zeta}\left\{A^{-1}\left[\frac{1}{v^\alpha}A[(B_n)_{xxxx}]\right]\right\}\end{pmatrix}d\zeta \quad (14)$$

Using the defned polynomials,

$$\begin{aligned}&A_n=u_n^2, && B_n=u_n^2,\\ &A_0=u_0^2, && B_0=u_0^2,\\ &A_1=2u_0u_1, && B_n=2u_0u_1,\\ &\vdots && \vdots\end{aligned}$$

Consequently, from Eq. (14) we have

$$u_1(x,t)=u_0(x,t)-\int_0^t\begin{pmatrix}\frac{\partial u_0(x,\zeta)}{\partial\zeta}+\frac{1}{3}\sinh\left(\frac{x}{2}\right) \\ +\frac{\partial}{\partial\zeta}\left\{A^{-1}\left[\frac{1}{v^\alpha}A[(A_0)_{xx}]\right]\right\}-\frac{\partial}{\partial\zeta}\left\{A^{-1}\left[\frac{1}{v^\alpha}A[(B_0)_{xxxx}]\right]\right\}\end{pmatrix}d\zeta.$$

$$u_1(x,t)=\frac{4}{3}\sinh^2\left(\frac{x}{4}\right)-\frac{1}{3}\sinh\left(\frac{x}{2}\right)t+\frac{1}{6}\cosh\left(\frac{x}{2}\right)\frac{t^{\alpha}}{(\alpha)!}-\frac{1}{12}\sinh\left(\frac{x}{2}\right)\frac{t^{1+\alpha}}{(1+\alpha)!}, \tag{15}$$

On same scenario,

$$\begin{aligned}u_2(x,t)=&\frac{4}{3}\sinh^2\left(\frac{x}{4}\right)-\frac{1}{3}\sinh\left(\frac{x}{2}\right)t+\frac{1}{6}\cosh\left(\frac{x}{2}\right)\frac{t^{\alpha}}{(\alpha)!}-\frac{1}{12}\sinh\left(\frac{x}{2}\right)\frac{t^{1+\alpha}}{(1+\alpha)!}\\&+\frac{1}{24}\cosh\left(\frac{x}{2}\right)\frac{t^{2\alpha}}{(2\alpha)!}-\frac{1}{48}\sinh\left(\frac{x}{2}\right)\frac{t^{1+2\alpha}}{(1+2\alpha)!},\end{aligned} \tag{16}$$

$$\begin{aligned}u_3(x,t)=&\frac{4}{3}\sinh^2\left(\frac{x}{4}\right)-\frac{1}{3}\sinh\left(\frac{x}{2}\right)t+\frac{1}{6}\cosh\left(\frac{x}{2}\right)\frac{t^{\alpha}}{(\alpha)!}-\frac{1}{12}\sinh\left(\frac{x}{2}\right)\frac{t^{1+\alpha}}{(1+\alpha)!}\\&+\frac{1}{24}\cosh\left(\frac{x}{2}\right)\frac{t^{2\alpha}}{(2\alpha)!}-\frac{1}{48}\sinh\left(\frac{x}{2}\right)\frac{t^{1+2\alpha}}{(1+2\alpha)!}+\frac{1}{96}\cosh\left(\frac{x}{2}\right)\frac{t^{3\alpha}}{(3\alpha)!}\\&-\frac{1}{192}\sinh\left(\frac{x}{2}\right)\frac{t^{1+3\alpha}}{(1+3\alpha)!},\end{aligned} \tag{17}$$

$\vdots,$

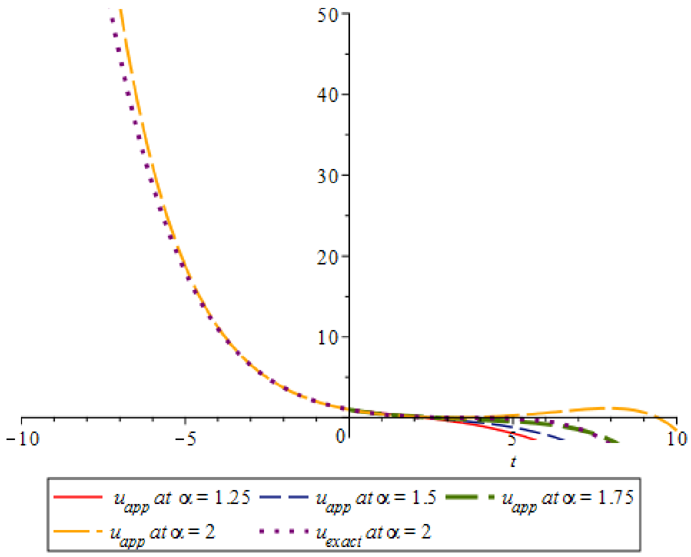

Figure 1. Solution of fractional Boussinesq-like equation and gives the comparison of approximate solution $u(x,t)\approx u_3(x,t)$ in Eq. (17) for different values of α with the exact solution for $\alpha=2$.

Fig. 1 depicts the solution of fractional Boussinesq-like equation and gives the comparison of approximate solution $u(x,t)\approx u_3(x,t)$ in Eq. (17) for different values of α with the exact solution for $\alpha=2$. The approximate values of the parameters are used for better understanding of the physical facets of the problem (8-9). From the figure, a very good agreement can be seen between these results that reflects the validity of the determined analytical solutions by the new coupled technique VIMAT. To enhance the compatibility of solution, higher order solution can be considered.

Problem 2: Consider well known Nonlinear Newell-Whitehead-Segel equation [15]

$$D_t^{\alpha}u=u_{xx}+2u-3u^2,\ \ 0<\alpha\le 1, \tag{18}$$

subjected to the following constraints

$$u(x,0)=\lambda. \tag{19}$$

Using fractional Aboodh transform on (18) along Eq. (19),

$$A[u(x,t)]=\frac{1}{v^2}\lambda+\frac{1}{v^\alpha}A[u_{xx}]+\frac{1}{v^\alpha}A[2u]-\frac{1}{v^\alpha}A[3A_n]. \tag{20}$$

By applying inverse transform, one get

$$u(x,t)=\lambda+A^{-1}\left[\frac{1}{v^\alpha}A[u_{xx}]\right]+A^{-1}\left[\frac{1}{v^\alpha}A[2u]\right]-A^{-1}\left[\frac{1}{v^\alpha}A[3A_n]\right]. \tag{21}$$

We have

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial}{\partial t}A^{-1}\left[\frac{1}{v^\alpha}A[u_{xx}]\right]+\frac{\partial}{\partial t}A^{-1}\left[\frac{1}{v^\alpha}A[2u]\right]-\frac{\partial}{\partial t}A^{-1}\left[\frac{1}{v^\alpha}A[3A_n]\right]. \tag{22}$$

The CF for Eq. (22) can formed as

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left\{\begin{array}{l}\frac{\partial u_n(x,\zeta)}{\partial\zeta}-\frac{\partial}{\partial\zeta}\left(A^{-1}\left[\frac{1}{v^\alpha}A[u_{nxx}]\right]\right)-\frac{\partial}{\partial\zeta}\left(A^{-1}\left[\frac{1}{v^\alpha}A[2u_n]\right]\right)\\+\frac{\partial}{\partial\zeta}\left(A^{-1}\left[\frac{1}{v^\alpha}A[3A_n]\right]\right)\end{array}\right\}d\zeta. \tag{23}$$

Using polynomials, as a result from Eq. (23) we hane

$$u_1(x,t)=\lambda+(2\lambda-3\lambda^2)\frac{t^\alpha}{(\alpha)!}, \tag{24}$$

$$u_2(x,t)=\lambda+(2\lambda-3\lambda^2)\frac{t^\alpha}{(\alpha)!}+2(1-3\lambda)(2\lambda-3\lambda^2)\frac{t^{2\alpha}}{(2\alpha)!}, \tag{25}$$

$$u_3(x,t)=\lambda+(2\lambda-3\lambda^2)\frac{t^\alpha}{(\alpha)!}+2\,(1-3\lambda)(2\lambda-3\lambda^2)\frac{t^{2\alpha}}{(2\alpha)!}+6\lambda(2\lambda-3\lambda^2)\frac{t^{3\alpha}}{(3\alpha)!}. \tag{26}$$

$$\vdots$$

Fig. 2 depicts the solution of fractional Newell-Whitehead-Segel equation and gives the comparison of approximate solution $u(x,t)\approx u_3(x,t)$ in Eq. (26) for different values of α with the exact solution for $\alpha=1$. Approximate values of the parameters are used for better understanding of the physical facets of the problem in (18-19). From the figure, a very good agreement can be seen between these results that reflects the validity of the determined analytical solutions by the new coupled technique VIMAT. To enhance the compatibility of solution, higher order solution can be considered.

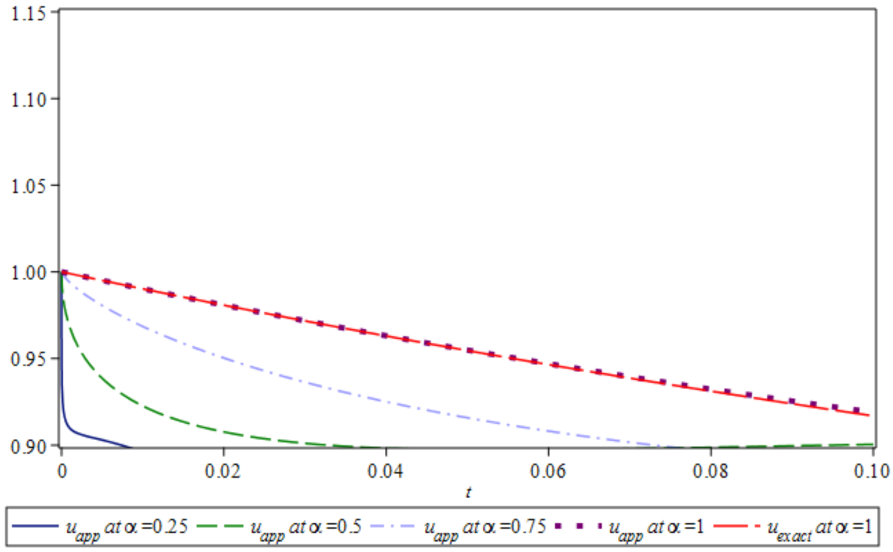

Figure 2. Solution of fractional Newell-Whitehead-Segel equation and gives the comparison of approximate solution $u(x,t) \approx u_3(x,t)$ in Eq. (26) for different values of α with the exact solution for $\alpha = 1$.

Problem 3: Consider the system of nonlinear PDEs with time-fractional derivatives [16]

$$
\begin{aligned}
&{}^{C}D_t^{\alpha} u(x,t) + w(x,t)u_x(x,t) + u(x,t) = 1, \quad 0 \le \alpha < 1, \\
&{}^{C}D_t^{\beta} w(x,t) - u(x,t)w_x(x,t) - w(x,t) = 1, \; 0 \le \beta < 1,
\end{aligned} \tag{28}
$$

subject to constraints

$$u(x,0) = e^{x}, \text{and } w(x,0) = e^{-x}. \tag{29}$$

According to above described steps, we have

$$
\begin{aligned}
u(x,t) &= \frac{t^{\alpha}}{\alpha!} + e^{x} - A^{-1}\left[\frac{1}{v^{\alpha}} A[A_n]\right] - A^{-1}\left[\frac{1}{v^{\alpha}} A[u(x,t)]\right], \\
w(x,t) &= \frac{t^{\beta}}{\beta!} + e^{-x} + A^{-1}\left[\frac{1}{v^{\beta}} A[B_n]\right] + A^{-1}\left[\frac{1}{v^{\beta}} A[w(x,t)]\right].
\end{aligned} \tag{30}
$$

The iterative CF formula is

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\zeta)}{\partial \zeta} - \frac{\partial}{\partial \zeta}\frac{\zeta^{\alpha}}{\alpha!} - \frac{\partial}{\partial \zeta} A^{-1}\left[\frac{1}{v^{\alpha}} A[A_n]\right] + \frac{\partial}{\partial \zeta} A^{-1}\left[\frac{1}{v^{\alpha}} A[u_n(x,\zeta)]\right]\right) d\zeta,$$

$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t \left(\frac{\partial w_n(x,\zeta)}{\partial \zeta} - \frac{\partial}{\partial \zeta}\frac{\zeta^{\beta}}{\beta!} - \frac{\partial}{\partial \zeta} A^{-1}\left[\frac{1}{v^{\beta}} A[B_n]\right] - \frac{\partial}{\partial \zeta} A^{-1}\left[\frac{1}{v^{\beta}} A[w_n(x,\zeta)]\right]\right) d\zeta.$$

Consequently, we have

$$u_1(x,t)=u_0(x,t)-\int_0^t\left(\frac{\partial u_0(x,\zeta)}{\partial\zeta}-\frac{\partial}{\partial\zeta}\frac{\zeta^\alpha}{\alpha!}+\frac{\partial}{\partial\zeta}A^{-1}\left[\frac{1}{v^\alpha}A[A_0]\right]+\frac{\partial}{\partial\zeta}A^{-1}\left[\frac{1}{v^\alpha}A[u_0(x,\zeta)]\right]\right)d\zeta,$$

$$w_1(x,t)=w_0(x,t)-\int_0^t\left(\frac{\partial w_0(x,\zeta)}{\partial\zeta}-\frac{\partial}{\partial\zeta}\frac{\zeta^\beta}{\beta!}-\frac{\partial}{\partial\zeta}A^{-1}\left[\frac{1}{v^\beta}A[B_0]\right]-\frac{\partial}{\partial\zeta}A^{-1}\left[\frac{1}{v^\beta}A[w_0(x,\zeta)]\right]\right)d\zeta.$$

$$u_1(x,t)=e^x\left(1-\frac{t^\alpha}{(\alpha)!}\right)$$
$$w_1(x,t)=e^{-x}\left(1+\frac{t^\beta}{\beta!}\right), \tag{31}$$

$$u_2(x,t)=e^x\left(1-\frac{t^\alpha}{(\alpha)!}+\frac{t^{2\alpha}}{(2\alpha)!}\right)$$
$$w_2(x,t)=e^{-x}\left(1+\frac{t^\beta}{\beta!}+\frac{t^{2\beta}}{(2\beta)!}\right) \tag{32}$$

$$u_3(x,t)=e^x\left(1-\frac{t^\alpha}{(\alpha)!}+\frac{t^{2\alpha}}{(2\alpha)!}-\frac{t^{3\alpha}}{(3\alpha)!}\right)$$
$$w_3(x,t)=e^{-x}\left(1+\frac{t^\beta}{(\beta)!}+\frac{t^{2\beta}}{(2\beta)!}+\frac{t^{3\beta}}{(3\beta)!}\right). \tag{33}$$

$\vdots$

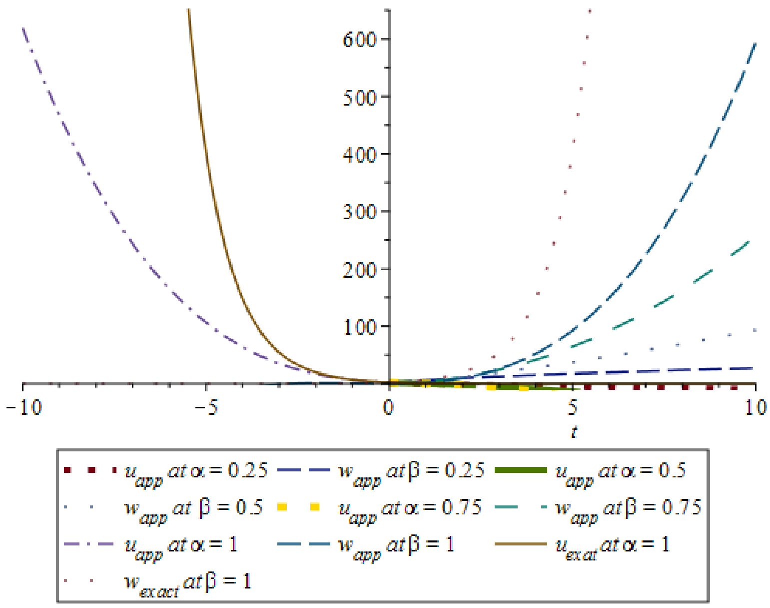

Figure 3. Comparison of approximate solution $u(x,t)\approx u_3(x,t)$ in Eq. (33) for different values of α with the exact solution for $\alpha=1$.

Fig. 3 depicts the comparison of approximate solution $u(x,t) \approx u_3(x,t)$ in Eq. (33) for different values of α with the exact solution for $\alpha = 1$. Approximate values of the parameters are used for better understanding of the physical facets of the problem in (28-29). From the figure, a very good agreement can be seen between these results that reflects the validity of the determined analytical solutions by the new coupled technique VIMAT. To enhance the compatibility obtained solution, higher order solution can be considered.

Example 4: Finally, taking the system of nonlinear FPDEs [17]

$$D_{*t}^{\alpha} u + v_x w_y - v_y w_x = -u, \quad 0 < \alpha, \beta, \gamma < 1,$$

$$D_{*t}^{\beta} v + u_x w_y + u_y w_x = v, \tag{34}$$

$$D_{*t}^{\gamma} w + u_x v_y + u_y v_x = w,$$

subject to constraints

$$u(x,y,0) = e^{x+y},\ v(x,y,0) = e^{x-y},\ w(x,y,0) = e^{-x+y}. \tag{35}$$

According to above described steps, the iteration CF formula is

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t \left(\frac{\partial u_n(x,y,\zeta)}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(A^{-1}\left[\frac{1}{v^{\alpha}} A[A_n] - \frac{1}{v^{\alpha}} A[B_n] \right] \right) + \frac{\partial}{\partial \zeta}\left(e^{x+y} \frac{\zeta^{\alpha}}{(\alpha)!} \right) \right) d\zeta,$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t \left(\frac{\partial v_n(x,y,\zeta)}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(A^{-1}\left[\frac{1}{v^{\alpha}} A[C_n] + \frac{1}{v^{\alpha}} A[D_n] \right] \right) - \frac{\partial}{\partial \zeta} e^{x-y} \frac{\zeta^{\alpha}}{(\alpha)!} \right) d\zeta,$$

$$w_{n+1}(x,y,t) = w_n(x,y,t) - \int_0^t \left(\frac{\partial w_n(x,y,\zeta)}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(A^{-1}\left[\frac{1}{v^{\alpha}} A[E_n] + \frac{1}{v^{\alpha}} A[F_n] \right] \right) - \frac{\partial}{\partial \zeta} e^{-x+y} \frac{\zeta^{\alpha}}{(\alpha)!} \right) d\zeta.$$

Consequently,

$$u_1(x,y,t) = u_0(x,y,t) - \int_0^t \left(\frac{\partial u_0(x,y,\zeta)}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(A^{-1}\left[\frac{1}{v^{\alpha}} A[A_0] - \frac{1}{v^{\alpha}} A[B_0] \right] \right) + \frac{\partial}{\partial \zeta}\left(e^{x+y} \frac{\zeta^{\alpha}}{(\alpha)!} \right) \right) d\zeta,$$

$$v_1(x,y,t) = v_0(x,y,t) - \int_0^t \left(\frac{\partial v_0(x,y,\zeta)}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(A^{-1}\left[\frac{1}{v^{\alpha}} A[C_0] + \frac{1}{v^{\alpha}} A[D_0] \right] \right) - \frac{\partial}{\partial \zeta} e^{x-y} \frac{\zeta^{\alpha}}{(\alpha)!} \right) d\zeta,$$

$$w_1(x,y,t) = w_0(x,y,t) - \int_0^t \left(\frac{\partial w_0(x,y,\zeta)}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(A^{-1}\left[\frac{1}{v^{\alpha}} A[E_0] + \frac{1}{v^{\alpha}} A[F_0] \right] \right) - \frac{\partial}{\partial \zeta} e^{-x+y} \frac{\zeta^{\alpha}}{(\alpha)!} \right) d\zeta.$$

Using

$$A_n = v_{nx} w_{ny}, \qquad B_n = v_{ny} w_{nx},$$

$$A_0 = v_{0x} w_{0y} = 1, \qquad B_0 = v_{0y} w_{0x} = 1,$$
$$\vdots \qquad \vdots$$
$$C_n = u_{nx} w_{ny}, \qquad D_n = u_{ny} w_{nx},$$

$$C_0 = u_{0x} w_{0y} = e^{2y}, \qquad D_0 = u_{0y} w_{0x} = -e^{2y},$$
$$\vdots \qquad \vdots$$
$$E_n = u_{nx} v_{ny} \qquad F_n = u_{ny} v_{nx},$$

$$E_0 = u_{0x} v_{0y} = -e^{2x} \qquad F_0 = u_{0y} v_{0x} = e^{2x},$$
$$\vdots \qquad \vdots$$

By computing, we have

$$u_1 = e^{x+y}\left(1 - \frac{t^{\alpha}}{(\alpha)!}\right),$$

$$v_1 = e^{x-y}\left(1 + \frac{t^{\beta}}{(\beta)!}\right), \tag{36}$$

$$w_1 = e^{-x+y}\left(1 + \frac{t^{\gamma}}{(\gamma)!}\right),$$

$$u_2 = e^{x+y}\left(1 - \frac{t^{\alpha}}{(\alpha)!} + \frac{t^{2\alpha}}{(2\alpha)!}\right),$$

$$v_2 = e^{x-y}\left(1 + \frac{t^{\beta}}{(\beta)!} + \frac{t^{2\beta}}{(2\beta)!}\right), \tag{37}$$

$$w_2 = e^{-x+y}\left(1 + \frac{t^{\gamma}}{(\gamma)!} + \frac{t^{2\gamma}}{(2\gamma)!}\right).$$

$$u_3 = e^{x+y}\left(1-\frac{t^{\alpha}}{(\alpha)!}+\frac{t^{2\alpha}}{(2\alpha)!}-\frac{t^{3\alpha}}{(3\alpha)!}\right),$$

$$v_3 = e^{x-y}\left(1+\frac{t^{\beta}}{(\beta)!}+\frac{t^{2\beta}}{(2\beta)!}+\frac{t^{3\beta}}{(3\beta)!}\right), \tag{38}$$

$$w_3 = e^{-x+y}\left(1+\frac{t^{\gamma}}{(\gamma)!}+\frac{t^{2\gamma}}{(2\gamma)!}+\frac{t^{3\gamma}}{(3\gamma)!}\right).$$

$\vdots$

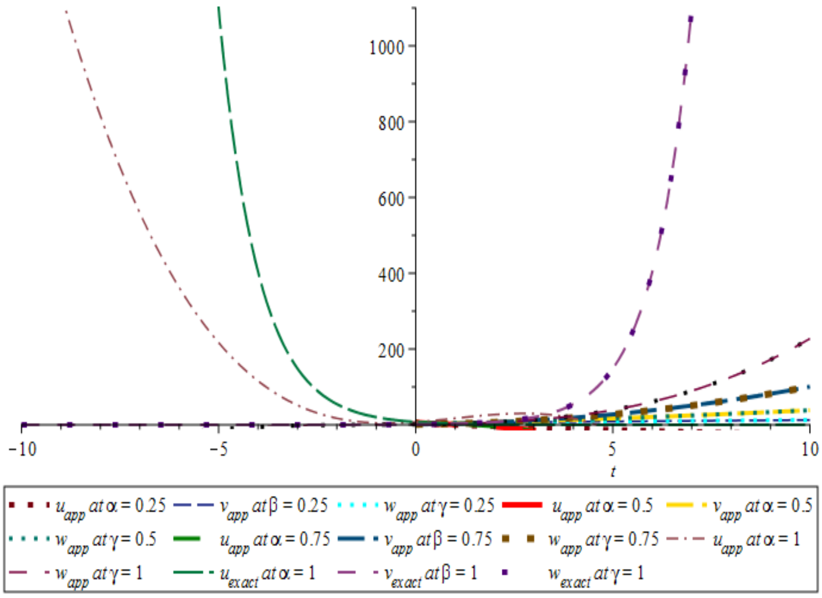

Figure 4. comparison of approximate solution $u(x,t) \approx u_3(x,t)$ in Eq. (38) for different values of α with the exact solution for $\alpha = 1$.

Fig. 4 depicts the comparison of approximate solution $u(x,t) \approx u_3(x,t)$ in Eq. (38) for different values of α with the exact solution for $\alpha = 1$. Approximate values of the parameters are used for better understanding of the physical facets of the problem in (34-35). From the figure, a very good agreement can be seen between these results that reflects the validity of the determined analytical solutions by the new coupled technique VIMAT. To enhance the compatibility of solution, higher order solution can be considered.

4. CONCLUSIONS

In this work, fractional Aboodh integral transform coupled with variational iteration method is implemented in easy way to some fractional nonlinear mathematical problems to derive the more general analytical solutions. This coupling makes the numeric calculations very easy. It gives reliable and efficient results. Figures empower that to consider the difference between two solutions graphically. From the graphs we can see that the exactitude of analytical solution can be enriched by using higher order calculations in solution.

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