ON RICCI-BOURGUIGNON h-ALMOST SOLITONS IN RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we shall give some structure equations for RB $h$-almost solitons which generalize previous results for Ricci almost solitons. As a consequence of these equations we also derive an integral formula for the compact gradient RB $h$-almost solitons which enables us to show that a compact nontrivial almost Ricci soliton is isometric to a sphere provided either it has constant scalar curvature or its associated vector field is conformal.

Keywords: Ricci almost soliton; RB $h$-almost soliton; Hodge-de Rham; Scalar curvature.

1. INTRODUCTION

Ricci solitons generate self-similar solutions of the Hamilton’s Ricci flow, play a fundamental role in the formation of singularities of the flow. Therefore, considering a geometric flow, it is natural to analyze the solitons associated with this flow.

Let $(M,g)$ be an $n$-dimensional, compact, smooth, Riemannian manifold whose metric $g = g(t)$ is evolving according to the flow equation

$$\frac{\partial g}{\partial t} = -2(Ric - \rho Rg)$$

(1)

where $Ric$ is a Ricci tensor of the manifold, $R$ its scalar curvature and $\rho$ is a real constant. The flow in equation (1) is called a Ricci-Bourguignon flow (RB flow for short) [1]. This family of geometric flows contains, as a special case, the Ricci flow, setting $\rho = 0$. In particular, for some values of the constant $\rho$, we have

- the Ricci tensor, $Ric$, ($\rho = 0$)
- the Einstein tensor, $Ric - \frac{R}{2}g$, ($\rho = \frac{1}{2}$)
- the traceless Ricci tensor, $Ric - \frac{R}{n}g$, ($\rho = \frac{1}{n}$)
- the Schouten tensor, $Ric - \frac{R}{2(n-1)}g$, ($\rho = \frac{1}{2(n-1)}$).

The study of the concept $RB$ almost soliton is introduced in a recent paper due to Dwivedi [2], which extends the almost Ricci solitons by Pigola et al. [3]. The author establishes some results for the solitons of the $RB$ flow which generalize some results of Ricci solitons. He also derives integral formulas for compact gradient $RB$ solitons and compact gradient $RB$ almost solitons. Using the integral formula he show that a compact gradient $RB$ almost soliton is isometric to an Euclidean sphere.

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The $RB$ almost soliton is a modification of $RB$ soliton
\begin{equation}
Ric + \frac{1}{2} L_X g = \lambda g + \rho R g
\end{equation}
by adding the condition on the parameter $\lambda$ to be a variable function rather than a constant. If $X$ becomes the gradient of a smooth function $f \in C^\infty$ then the $RB$ almost soliton is called gradient $RB$ almost soliton and is shrinking, steady or expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively.

On the other hand, Gomes et al. [4] give an extension of the concept of almost Ricci soliton as $h$-almost Ricci soliton
\begin{equation}
Ric + \frac{h}{2} L_X g = \lambda g
\end{equation}
where $\lambda, h: \mathbb{M} \to \mathbb{R}$ are two smooth functions. They prove that a compact nontrivial $h$-almost Ricci soliton of dimension $n \geq 3$ with $h$ having defined signal and constant scalar curvature is isometric to a standard sphere. Moreover, they also give characterizations for a special class of gradient $h$-Ricci solitons. Recently, some equations of structure for $h$-almost Ricci soliton are shown by Ghahremani-Gol [5]. The author obtain an integral formula for the compact $h$-almost Ricci soliton and prove that a compact nontrivial $h$-almost Ricci soliton is isometric to a Euclidean sphere. It should be noted here that a $h$-almost soliton is shrinking, steady or expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$. If $\lambda$ has no defined sign then soliton is undefined.

Inspired by the above definitions of the $RB$ almost soliton and $h$-almost soliton, we consider an extension of the concept of Ricci soliton as follows:

**Definition 1** A complete Riemannian manifold $(\mathbb{M}^n, g)$ is called a Ricci-Bourguignon $h$-almost soliton ($RB$ $h$-almost soliton for short) if there exist a vector field $X$, a soliton function $\lambda: \mathbb{M} \to \mathbb{R}$ and a function $h: \mathbb{M} \to \mathbb{R}$ such that
\begin{equation}
Ric + \frac{h}{2} L_X g = \lambda g + \rho R g,
\end{equation}
where $L_X g$ denotes the Lie derivative of the metric $g$ along $X$ and $\rho$ is a constant.

Moreover, if $X$ is a gradient vector field of a smooth function $f$, i.e. $X = \nabla f$, an $RB$ $h$-almost soliton is called a gradient $RB$ $h$-almost soliton
\begin{equation}
Ric + h \nabla^2 f = \lambda g + \rho R g.
\end{equation}

The $RB$ $h$-almost soliton is said to be shrinking, steady and expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$. Also if $\lambda$ has no definitive sign, the $RB$ $h$-almost soliton will be called indefinite.

For some computations it will be more convenient to use the classical tensorial form of the gradient $RB$ $h$-almost soliton equation as follows:
\begin{equation}
R_{ij} + h \nabla_i \nabla_j f = \lambda g_{ij} + \rho R g_{ij}.
\end{equation}

On the other hand, given a vector field $X$ on a compact oriented Riemannian manifold, the Hodge-de Rham decomposition theorem [6] shows that we can decompose $X$ as $X = $
\( \nabla u + Y \), where \( \text{div} Y = 0 \) and \( u \) is a smooth function on \( M \). The function \( u \) is called the Hodge-de Rham potential.

The first aim of this paper is to give some structure equations for gradient RB \( h \)-almost solitons. Here, we obtain extra terms in computations since in our more general setting \( \lambda \) and \( h \) are functions. Later, as a main result we obtain that a compact nontrivial RB \( h \)-almost soliton is isometric to the Euclidean sphere. Moreover, we get an integral formula for the compact gradient RB \( h \)-almost Ricci soliton.

Now, we show that the same conclusion obtained in [7] for compact Ricci solitons and in [8] for compact Ricci almost solitons also holds for compact RB \( h \)-almost solitons.

**Theorem 2** Let \((M^n, g, X, h, \lambda, \rho)\) be a compact RB \( h \)-almost soliton with \( h \) having defined signal. If \( M^n \) is also a gradient RB \( h \)-almost soliton with potential \( f \), then, up to a constant, it agrees with the Hodge–de Rham potential \( u \).

Taking the Hodge–de Rham decomposition we have the next theorem. This theorem generalizes Theorem 2 in [8] for RB \( h \)-almost soliton.

**Theorem 3** If \((M^n, g, X, h, \lambda, \rho)\) is a compact RB \( h \)-almost soliton with \( n \geq 3 \) and \( X \) is a nontrivial conformal vector field, then \( M^n \) is isometric to an Euclidean sphere \( S^n \).

Finally, for compact gradient RB \( h \)-almost solitons, we present an integral formula which is a generalization of similar one obtained in [2] and [5].

**Theorem 4** Let \((M^n, g, X, h, \lambda, \rho)\) be a gradient RB \( h \)-almost soliton. Then

\[
\frac{h^2}{n} \left \| \nabla^2 f - \frac{\Delta f}{n} g \right \|^2 = \frac{2\rho(n-1)}{n} \Delta R + (n-1)\Delta \lambda - \Delta h \Delta f + \frac{h}{n} g(\nabla R, \nabla f) + \frac{h}{n} R \Delta f + \text{div}(g^{jk}(\nabla_k h)\nabla_j f). \tag{7}
\]

In particular, if \( M^n \) is compact, we have

\[
\int_M h^2 \left \| \nabla^2 f - \frac{\Delta f}{n} g \right \|^2 dM = \frac{n-2}{2n} \int_M h g(\nabla R, \nabla f) dM. \tag{8}
\]

The next corollary is a consequence of this theorem, Theorem 3 and a classical theorem due to Tashiro [9].

**Corollary 5** Let \((M^n, g, \nabla f, h, \lambda, \rho)\) be a nontrivial compact gradient RB \( h \)-almost soliton, with \( h \) having defined signal and \( n \geq 3 \). \( M^n \) is isometric to an Euclidean sphere \( S^n(r) \) if any one of the following conditions holds:

1. \( M^n \) has constant scalar curvature.
2. \( \int_M h g(\nabla R, \nabla f) dM \leq 0 \).
3. \( M^n \) is a homogenous manifold.

### 2. PRELIMINARIES

In this section we shall present some basic formulas for RB \( h \)-almost solitons. On a Riemannian manifold \((M^n, g)\) we can define the divergence of a \((1,1)\)-tensor \( S \) to be the \((0,1)\)-tensor
Thus, for all $\phi \in C^\infty(M)$, we have

$\cdot \, \text{div} (\phi S) = \phi \text{div} S + S(\nabla \phi, \cdot)$,
$\cdot \, \nabla (\phi S) = \phi \nabla S + d\phi \otimes S$,
$\cdot \, d|\nabla \phi|^2 = 2\nabla^2 \phi(\nabla \phi, \cdot)$
$\cdot \, \text{div} \nabla^2 \phi = \text{Ric} (\nabla \phi, \cdot) + d\Delta \phi$

Moreover, $\text{div}(\phi g)(X) = g(\nabla \phi, X) = d\phi(X)$ for $X \in \chi(M)$ and the twice contracted second Bianchi identity is $2\text{div}(\text{Ric}) = \nabla R$.

Next we recall that

**Lemma 6** Let $S$ be a $(0,2)$-tensor on a Riemannian manifold $(M, g)$. Then

$$\text{div}(S(\phi X)) = \phi(\text{div} S)(X) + \phi g(\nabla X, S) + S(\nabla \phi, X)$$

for all $X \in \chi(M)$ and any function $\phi$ on $M^n$.

We start by proving that $RB$ $h$-almost solitons provide the following:

**Proposition 7** Let $(M^n, g, \nabla, h, \lambda, \rho)$ be a gradient $RB$ $h$-almost soliton, then the following formulas hold:

1. $(1 - n\rho)R + h\Delta f = n\lambda$,
2. $(1 - 2\rho(n - 1))\nabla_i R = 2hR_{ij} \nabla f + 2(n - 1)\nabla_i \lambda - 2\nabla h\Delta f + 2g^{ik}(\nabla_k h) \nabla_i \nabla_j f$,
3. $\nabla_i R_{jk} - \nabla_j R_{ik} - hR_{ijk} \nabla^2 f = (\nabla_i \lambda) g_{ik} - (\nabla_j \lambda) g_{jk} - (\nabla_i h) \nabla \nabla_k f + (\nabla_i h) \nabla_j \nabla_k f + \rho(\nabla_i R) g_{ik} - (\nabla_i R) g_{jk}$,
4. $(1 - 2\rho(n - 1)) \nabla R + h^2 \nabla |\nabla f|^2 - 2(n - 1)\nabla \lambda = 2h(\lambda + \rho R) \nabla f - 2\nabla h\Delta f + 2g^{ik}(\nabla_k h) \nabla_i \nabla_j f$.

**Proof:** Taking trace of the equation (5) is enough to conclude (i), i.e.,

$$\text{tr}(\text{Ric}) + \text{tr}(h\nabla^2 f) = \text{tr}((\lambda + \rho R) g) \Rightarrow R + h\Delta f = n\lambda + n\rho R.$$

For a gradient $RB$ $h$-almost soliton we have

$$R_{ij} + h \nabla_i \nabla_j f = \lambda g_{ij} + \rho R g_{ij}.$$  

(12)

If we think of the Ricci identity, the twice contracted Bianchi identity and the equation (12), then we get

$$\frac{1}{2} \nabla_i R = \text{div}Ric_i = g^{ik} \nabla_k R_{ij} = g^{ik} \nabla_k (-h \nabla_i \nabla_j f + \lambda g_{ij} + \rho R g_{ij}) = -g^{ik}(\nabla_k h) \nabla_i \nabla_j f - h g^{ik} \nabla_k \nabla_i \nabla_j f + g^{ik}(\nabla_k \lambda) g_{ij} + \rho g^{ik}(\nabla_k R) g_{ij}.$$
where

\[ \frac{1}{2} h^2 \| \nabla f \|^2 = h^2 \nabla f \cdot \nabla f. \]  

and hence

\[ (1 - 2\rho(n - 1))\nabla_i R = 2hR_{ij} \nabla^j f + 2(n - 1)\nabla_i \lambda - 2\nabla_i h \Delta f + 2g^{jk}(\nabla_k h)\nabla_i \nabla_j f. \]  

Now we need to use (12) and by covariant derivatives of \( R_{ik} \) and \( R_{jk} \), we obtain

\[
\nabla_j R_{ik} - \nabla_i R_{jk} = (\nabla_i h)\nabla_j \nabla_k f - (\nabla_j h)\nabla_i \nabla_k f + h[\nabla_j \nabla_i \nabla_k f + \nabla_i \nabla_j \nabla_k f] \\
+ (\nabla_j \lambda)g_{ik} - (\nabla_i \lambda)g_{jk} + \rho(\nabla_i R)g_{ik} - (\nabla_i R)g_{jk} \\
= (\nabla_i h)\nabla_j \nabla_k f - (\nabla_j h)\nabla_i \nabla_k f + hR_{ijks} \nabla^s f \\
+ (\nabla_j \lambda)g_{ik} - (\nabla_i \lambda)g_{jk} + \rho(\nabla_j R)g_{ik} - (\nabla_i R)g_{jk}.
\]

Finally, we prove \((iv)\). By equation \((ii)\) and the fundamental equation as a \((1,1)\) -tensor, we get

\[
\frac{1}{2} (1 - 2\rho(n - 1))\nabla R + \frac{1}{2} h^2 \nabla f = hRic(\nabla f) + h^2 \nabla f \cdot \nabla f + (n - 1) \nabla \lambda \\
- \nabla h \Delta f + g^{jk}(\nabla_k h)\nabla_i \nabla_j f \\
= h(\lambda + \rho R) \nabla f - \nabla h \Delta f + (n - 1) \nabla \lambda \\
+ g^{jk}(\nabla_k h)\nabla_i \nabla_j f.
\]

Thus, we finish the proof of this proposition. \(\Box\)

With the help of the next theorem (Theorem 4.2 in [10]), we shall show that actually the above Theorem 3 is provided.

**Theorem 8** Let \((M^n, g)\) be a compact Riemannian manifold with constant scalar curvature and \(M^n\) admits a nontrivial conformal vector field \(X\). If \(L_XRic = \alpha g\), where \(\alpha\) is a function, then \(M^n\) is isometric to the Euclidean sphere.

### 3. PROOFS OF THE RESULTS

Firstly we shall give the proof of Theorem 2.

**Proof:** Since \((M^n, g, \nabla f, h, \lambda, \rho)\) is a compact \(RB\) \(h\)-almost soliton, so taking trace of (4), we obtain

\[(1 - n\rho)R + hdivX = n\lambda.\]
Hodge-de Rham decomposition implies that $\text{div} X = \Delta u$, hence from the above equation, we get

$$
(1 - n\rho)R + h\Delta u = n\lambda. 
$$

(19)

Again if $M^n$ is also a compact gradient $RB$ $h$-almost soliton, then from equation $(i)$ of Proposition 7 we have

$$
(1 - n\rho)R + h\Delta f = n\lambda. 
$$

(20)

Equating the last two equations, we get $h\Delta(f - u) = 0$. Hence it is enough to use Hopf’s theorem to conclude $f = u + c$, which completes the proof of theorem. □

Now we present the proof of Theorem 3.

Proof: If $X$ is a nontrivial conformal vector field then we have $L_X g = 2\xi g$, $\xi \neq 0$. Thus the equation (4) becomes

$$
Ric = (\lambda - h\xi + \rho R) g. 
$$

(21)

Now, tracing (21) and taking the covariant derivative we obtain

$$
(1 - n\rho)\nabla R = n\nabla (\lambda - h\xi). 
$$

(22)

On the other hand, taking the divergence of (21) and using the equality $2\text{div}(Ric) = \nabla R$ we get

$$
(\frac{1}{2} - \rho)\nabla R = \nabla (\lambda - h\xi). 
$$

(23)

Therefore, (22) and (23) together imply that

$$
(1 - n\rho)\nabla R = n(\frac{1}{2} - \rho)\nabla R. 
$$

(24)

So, the scalar curvature $R$ is a constant and hence $(\lambda - h\xi)$ is a constant. From Lemma 2.3 in [10] we conclude that $R \neq 0$; otherwise $\xi = 0$. Also since $(\lambda - h\xi)$, $\rho$ and $R$ are all constant we have

$$
L_X Ric = 2(\lambda - h\xi + \rho R)\xi g. 
$$

(25)

Now we can apply Theorem 8 (Theorem 4.2 in [10]) to conclude that $M^n$ is isometric to an Euclidean sphere, the term $2(\lambda - h\xi + \rho R)\xi$ is corresponding to $\alpha$ in Theorem 8, which completes the proof of the theorem. □

Using Proposition 7, we shall prove the Theorem 4.

Proof: From the Lemma 6 we have

$$
\text{div}(Ric(\nabla f)) = (\text{div} Ric)(\nabla f) + g(\nabla^2 f, Ric). 
$$

(26)
Using the second contracted second Bianchi identity, the equation (5) and the equality $|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n}$, we get

$$
div(Ric(\nabla f)) = \frac{1}{2} g(\nabla R, \nabla f) + (\lambda + \rho R) \Delta f - h|\nabla^2 f|^2 = \frac{1}{2} g(\nabla R, \nabla f) + \frac{R}{n} \Delta f - h \frac{\Delta f}{n} g - \frac{h}{n} (\Delta f)^2
$$

(27)

Since

$$
hRic(\nabla f) = \frac{1-2\rho(n-1)}{2} \nabla R - (n-1) \nabla \lambda + \nabla h \Delta f - g^{jk}(\nabla_k h)\nabla_j \nabla f
$$

by (ii) of Proposition 7, we also have

$$
hd\text{div}Ric(\nabla f) = \frac{1-2\rho(n-1)}{2} \Delta R - (n-1) \Delta \lambda + \Delta h \Delta f
$$

$$
+ \frac{h}{n} R \Delta f - \text{div}(g^{jk}(\nabla_k h)\nabla_j \nabla f).
$$

(29)

Comparing the last equation with (27) yields

$$
h^2 \left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 = \frac{(2\rho(n-1)-1)}{2} \Delta R + (n-1) \Delta \lambda - \Delta h \Delta f + \frac{h}{2} g(\nabla R, \nabla f)
$$

$$
+ \frac{h}{n} R \Delta f + \text{div}(g^{jk}(\nabla_k h)\nabla_j \nabla f).
$$

(30)

Integrating both sides of the above equation over compact $M^n$, we obtain

$$
\int_M h^2 \left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 \, dM = \frac{1}{2} \int_M h g(\nabla R, \nabla f) \, dM + \frac{1}{n} \int_M h R \Delta f \, dM
$$

$$
= \frac{n-2}{2n} \int_M h g(\nabla R, \nabla f) \, dM.
$$

(31)

This completes the proof of theorem. □

Finally, we shall give the proof of Corollary 5 by using Theorem 4.

**Proof:** First we notice that the equation (5) and the equation (i) in Proposition 7 yield

$$
Ric - \frac{R}{n} g = -h \nabla^2 f + (\lambda + \rho R - \frac{R}{n}) g = -h(\nabla^2 f - \frac{\Delta f}{n} g).
$$

(32)

On the other hand, if any one of the conditions of Corollary 5 holds, then the term of $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 \, dM$ is equal to zero. Hence, we have $Ric = \frac{R}{n} g$. Considering this with the equation (5) gives us to

$$
\nabla^2 f = \frac{1}{h}(\lambda + R(\rho - \frac{1}{n})) g,
$$

(33)
which implies $\nabla f$ is a nontrivial conformal vector field. So from Theorem 3 we obtain that $M^n$ is isometric to an Euclidean sphere $S^n(r)$. This completes the proof of corollary.

\[ \square \]

4. CONCLUSIONS

We developed a new extension of the concept of Ricci soliton and gave some structure equations for this $RB$-$h$-almost soliton in Riemannian manifolds. We obtained extra terms in computations since in our more general setting $\lambda$ and $h$ are functions. Later, as a main result we obtained that a compact nontrivial $RB$-$h$-almost soliton is isometric to the Euclidean sphere. Finally, we had an integral formula for the compact gradient $RB$-$h$-almost soliton.

REFERENCES


