ORIGINAL PAPER

ON RICCI-BOURGUIGNON h-ALMOST SOLITONS IN RIEMANNIAN MANIFOLDS

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Abstract. *In this paper, we shall give some structure equations for RB h-almost solitons which generalize previous results for Ricci almost solitons. As a consequence of these equations we also derive an integral formula for the compact gradient RB h-almost solitons which enables us to show that a compact nontrivial almost Ricci soliton is isometric to a sphere provided either it has constant scalar curvature or its associated vector field is conformal.*

Keywords: *Ricci almost soliton; RB h-almost soliton; Hodge-de Rham; Scalar curvature.*

1. INTRODUCTION

Ricci solitons generate self-similar solutions of the Hamilton's Ricci flow, play a fundamental role in the formation of singularities of the flow. Therefore, considering a geometric flow, it is natural to analyze the solitons associated with this flow.

Let (M,g) be an n-dimensional, compact, smooth, Riemannian manifold whose metric $g=g(t)$ is evolving according to the flow equation

$$\frac{\partial g}{\partial t} = -2(Ric - \rho R g) \tag{1}$$

where Ric is a Ricci tensor of the manifold, R its scalar curvature and ρ is a real constant. The flow in equation (1) is called a Ricci-Bourguignon flow (RB flow for short) [1]. This family of geometric flows contains, as a special case, the Ricci flow, setting $\rho = 0$. In particular, for some values of the constant ρ, we have

- the Ricci tensor, Ric, $(\rho = 0)$
- the Einstein tensor, $Ric - \frac{R}{2}g$, $(\rho = \frac{1}{2})$
- the traceless Ricci tensor, $Ric - \frac{R}{n}g$, $(\rho = \frac{1}{n})$
- the Schouten tensor, $Ric - \frac{R}{2(n-1)}g$, $(\rho = \frac{1}{2(n-1)})$.

The study of the concept RB almost soliton is introduced in a recent paper due to Dwivedi [2], which extends the almost Ricci solitons by Pigola et al. [3]. The author establish some results for the solitons of the RB flow which generalize some results of Ricci solitons. He also derive integral formulas for compact gradient RB solitons and compact gradient RB almost solitons. Using the integral formula he show that a compact gradient RB almost soliton is isometric to an Euclidean sphere.

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The RB almost soliton is a modification of RB soliton

$$Ric + \frac{1}{2} L_X g = \lambda g + \rho R g \tag{2}$$

by adding the condition on the parameter λ to be a variable function rather than a constant. If X becomes the gradient of a smooth function $f \in \mathbb{C}^\infty$ then the RB almost soliton is called gradient RB almost soliton and is shrinking, steady or expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively.

On the other hand, Gomes et al. [4] give an extension of the concept of almost Ricci soliton as h-almost Ricci soliton

$$Ric + \frac{h}{2} L_X g = \lambda g \tag{3}$$

where $\lambda, h: M \to \mathbb{R}$ are two smooth functions. They prove that a compact nontrivial h-almost Ricci soliton of dimension $n \geq 3$ with h having defined signal and constant scalar curvature is isometric to a standard sphere. Moreover, they also give characterizations for a special class of gradient h-Ricci solitons. Recently, some equations of structure for h-almost Ricci soliton are shown by Ghahremani-Gol [5]. The author obtain an integral formula for the compact h-almost Ricci soliton and prove that a compact nontrivial h-almost Ricci soliton is isometric to a Euclidean sphere. It should be noted here that a h-almost soliton is shrinking, steady or expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$. If λ has no define sign then soliton is undefined.

Inspired by the above definitions of the RB almost soliton and h-almost soliton, we consider an extension of the concept of Ricci soliton as follows:

Definition 1 *A complete Riemannian manifold (M^n, g) is called a Ricci-Bourguignon h-almost soliton (RB h-almost soliton for short) if there exist a vector field X, a soliton function $\lambda: M \to \mathbb{R}$ and a function $h: M \to \mathbb{R}$ such that*

$$Ric + \frac{h}{2} L_X g = \lambda g + \rho R g, \tag{4}$$

where $L_X g$ denotes the Lie derivative of the metric g along X and ρ is a constant.

Moreover, if X is a gradient vector field of a smooth function f, i.e. $X = \nabla f$, an RB h-almost soliton is called a gradient RB h-almost soliton

$$Ric + h\nabla^2 f = \lambda g + \rho R g. \tag{5}$$

The RB h-almost soliton is said to be shrinking, steady and expanding according as $\lambda > 0, \lambda = 0$ or $\lambda < 0$. Also if λ has no definitive sign, the RB h-almost soliton will be called indefinite.

For some computations it will be more convenient to use the classical tensorial form of the gradient RB h-almost soliton equation as follows:

$$R_{ij} + h\nabla_i \nabla_j f = \lambda g_{ij} + \rho R g_{ij}. \tag{6}$$

On the other hand, given a vector field X on a compact oriented Riemannian manifold, the Hodge-de Rham decomposition theorem [6] shows that we can decompose X as $X =$

$\nabla u + Y$, where $divY = 0$ and u is a smooth function on M. The function u is called the Hodge-de Rham potential.

The first aim of this paper is to give some structure equations for gradient RB h-almost solitons. Here, we obtain extra terms in computations since in our more general setting λ and h are functions. Later, as a main result we obtain that a compact nontrivial RB h-almost soliton is isometric to the Euclidean sphere. Moreover, we get an integral formula for the compact gradient RB h-almost Ricci soliton.

Now, we show that the same conclusion obtained in [7] for compact Ricci solitons and in [8] for compact Ricci almost solitons also holds for compact RB h-almost solitons.

Theorem 2 *Let $(M^n, g, X, h, \lambda, \rho)$ be a compact RB h-almost soliton with h having defined signal. If M^n is also a gradient RB h-almost soliton with potential f, then, up to a constant, it agrees with the Hodge–de Rham potential u.*

Taking the Hodge–de Rham decomposition we have the next theorem. This theorem generalizes Theorem 2 in [8] for RB h-almost soliton.

Theorem 3 *If $(M^n, g, X, h, \lambda, \rho)$ is a compact RB h-almost soliton with $n \geq 3$ and X is a nontrivial conformal vector field, then M^n is isometric to an Euclidean sphere $\mathbb{S}^n$.*

Finally, for compact gradient RB h-almost solitons, we present an integral formula which is a generalization of similar one obtained in [2] and [5].

Theorem 4 *Let $(M^n, g, X, h, \lambda, \rho)$ be a gradient RB h-almost soliton. Then*

$$h^2 \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 = \frac{(2\rho(n-1)-1)}{2}\Delta R + (n-1)\Delta\lambda - \Delta h \Delta f + \frac{h}{2} g(\nabla R, \nabla f) + \frac{h}{n} R\Delta f + div(g^{jk}(\nabla_k h)\nabla_i \nabla_j f). \quad (7)$$

In particular, if M^n is compact, we have

$$\int_M h^2 \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 dM = \frac{n-2}{2n}\int_M hg(\nabla R, \nabla f)dM. \quad (8)$$

The next corollary is a consequence of this theorem, Theorem 3 and a classical theorem due to Tashiro [9].

Corollary 5 *Let $(M^n, g, \nabla f, h, \lambda, \rho)$ be a nontrivial compact gradient RB h-almost soliton, with h having defined signal and $n \geq 3$. M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$ if any one of the following conditions holds:*

1. *M^n has constant scalar curvature.*
2. *$\int_M hg(\nabla R, \nabla f)dM \leq 0$.*
3. *M^n is a homogenous manifold.*

2. PRELIMINARIES

In this section we shall present some basic formulas for RB h-almost solitons. On a Riemannian manifold (M^n, g) we can define the divergence of a (1,1)-tensor S to be the (0,1)-tensor

$$(divS)(v)(p) = tr(w \to (\nabla_w S)(v)(p)), \tag{9}$$

where $v, w \in T_pM$ and $p \in M^n$. If S is a $(0,2)$-tensor associated with the $(1,1)$-tensor S, then for all $X, Y \in \chi(M)$,

$$g(S(X), Y) = S(X, Y). \tag{10}$$

Thus, for all $\phi \in C^\infty(M)$, we have

- $div(\phi S) = \phi divS + S(\nabla\phi, .)$,
- $\nabla(\phi S) = \phi\nabla S + d\phi \otimes S$,
- $d|\nabla\phi|^2 = 2\nabla^2\phi(\nabla\phi, .)$
- $div\nabla^2\phi = Ric(\nabla\phi, .) + d\Delta\phi$

Moreover, $div(\phi g)(X) = g(\nabla\phi, X) = d\phi(X)$ for $X \in \chi(M)$ and the twice contracted second Bianchi identity is $2div(Ric) = \nabla R$.

Next we recall that

Lemma 6 *Let S be a $(0,2)$-tensor on a Riemannian manifold (M, g). Then*

$$div(S(\phi X)) = \phi(divS)(X) + \phi g(\nabla X, S) + S(\nabla\phi, X) \tag{11}$$

for all $X \in \chi(M)$ and any function ϕ on M^n.

We start by proving that RB h-almost solitons provide the following:

Proposition 7 *Let $(M^n, g, \nabla f, h, \lambda, \rho)$ be a gradient RB h-almost soliton, then the following formulas hold:*

1. $(1 - n\rho)R + h\Delta f = n\lambda$,
2. $(1 - 2\rho(n-1))\nabla_i R = 2hR_{ij}\nabla^j f + 2(n-1)\nabla_i\lambda - 2\nabla_i h\Delta f + 2g^{jk}(\nabla_k h)\nabla_i\nabla_j f$,
3. $\nabla_j R_{ik} - \nabla_i R_{jk} - hR_{ijks}\nabla^s f \quad = (\nabla_j\lambda)g_{ik} - (\nabla_i\lambda)g_{jk} - (\nabla_j h)\nabla_i\nabla_k f$ $\quad + (\nabla_i h)\nabla_j\nabla_k f + \rho[(\nabla_j R)g_{ik} - (\nabla_i R)g_{jk}]$,
4. $(1 - 2\rho(n-1))\nabla R + h^2\nabla|\nabla f|^2 - 2(n-1)\nabla\lambda \quad = 2h(\lambda + \rho R)\nabla f - 2\nabla h\Delta f$ $\quad + 2g^{jk}(\nabla_k h)\nabla_i\nabla_j f$.

Proof: Taking trace of the equation (5) is enough to conclude (i), i.e.,

$$tr(Ric) + tr(h\nabla^2 f) = \mathrm{tr}\big((\lambda + \rho R)g\big) \Rightarrow R + h\Delta f = n\lambda + n\rho R.$$

For a gradient RB h-almost soliton we have

$$R_{ij} + h\nabla_i\nabla_j f = \lambda g_{ij} + \rho R g_{ij}. \tag{12}$$

If we think of the Ricci identity, the twice contracted Bianchi identity and the equation (12), then we get

$$\begin{aligned}
\tfrac{1}{2}\nabla_i R &= divRic_i \\
&= g^{jk}\nabla_k R_{ij} \\
&= g^{jk}\nabla_k(-h\nabla_i\nabla_j f + \lambda g_{ij} + \rho R g_{ij}) \\
&= -g^{jk}(\nabla_k h)\nabla_i\nabla_j f - hg^{jk}\nabla_k\nabla_i\nabla_j f + g^{jk}(\nabla_k\lambda)g_{ij} + \rho g^{jk}(\nabla_k R)g_{ij}
\end{aligned}$$

$$= -g^{jk}(\nabla_k h)\nabla_i\nabla_j f - hg^{jk}\nabla_i\nabla_k\nabla_j f - hg^{jk}R_{kijs}\nabla^s f + \nabla_i\lambda + \rho\nabla_i R$$
$$= -g^{jk}(\nabla_k h)\nabla_i\nabla_j f - h\nabla_i\Delta f - hR_{is}\nabla^s f + \nabla_i\lambda + \rho\nabla_i R$$
$$= -g^{jk}(\nabla_k h)\nabla_i\nabla_j f + (1+\rho(1-n))\nabla_i R + (1-n)\nabla_i\lambda$$
$$+(\nabla_i h)\Delta f - hR_{is}\nabla^s f. \quad (13)$$

and hence

$$(1-2\rho(n-1))\nabla_i R = 2hR_{ij}\nabla^j f + 2(n-1)\nabla_i\lambda - 2\nabla_i h\Delta f + 2g^{jk}(\nabla_k h)\nabla_i\nabla_j f. \quad (14)$$

Now we need to use (12) and by covariant derivatives of R_{ik} and R_{jk}, we obtain

$$\nabla_j R_{ik} - \nabla_i R_{jk} = (\nabla_i h)\nabla_j\nabla_k f - (\nabla_j h)\nabla_i\nabla_k f + h[-\nabla_j\nabla_i\nabla_k f + \nabla_i\nabla_j\nabla_k f]$$
$$+(\nabla_j\lambda)g_{ik} - (\nabla_i\lambda)g_{jk} + \rho[(\nabla_j R)g_{ik} - (\nabla_i R)g_{jk}]$$
$$= (\nabla_i h)\nabla_j\nabla_k f - (\nabla_j h)\nabla_i\nabla_k f + hR_{ijks}\nabla^s f$$
$$+(\nabla_j\lambda)g_{ik} - (\nabla_i\lambda)g_{jk} + \rho[(\nabla_j R)g_{ik} - (\nabla_i R)g_{jk}]. \quad (15)$$

Finally, we prove (iv). By equation (ii) and the fundamental equation as a $(1,1)-$tensor, we get

$$\frac{1}{2}(1-2\rho(n-1))\nabla R + \frac{1}{2}h^2\nabla|\nabla f|^2 = hRic(\nabla f) + h^2\nabla_{\nabla f}\nabla f + (n-1)\nabla\lambda$$
$$-\nabla h\Delta f + g^{jk}(\nabla_k h)\nabla_i\nabla_j f$$
$$= h(\lambda+\rho R)\nabla f - \nabla h\Delta f + (n-1)\nabla\lambda$$
$$+g^{jk}(\nabla_k h)\nabla_i\nabla_j f, \quad (16)$$

where

$$\frac{1}{2}h^2\nabla|\nabla f|^2 = h^2\nabla_{\nabla f}\nabla f. \quad (17)$$

Thus, we finish the proof of this proposition. □

With the help of the next theorem (Theorem 4.2 in [10]), we shall show that actually the above Theorem 3 is provided.

Theorem 8 *Let (M^n, g) be a compact Riemannian manifold with constant scalar curvature and M^n admits a nontrivial conformal vector field X. If $L_X Ric = \alpha g$, where α is a function, then M^n is isometric to the Euclidean sphere.*

3. PROOFS OF THE RESULTS

Firstly we shall give the proof of Theorem 2.

Proof: Since $(M^n, g, \nabla f, h, \lambda, \rho)$ is a compact RB h-almost soliton, so taking trace of (4), we obtain

$$(1-n\rho)R + hdivX = n\lambda. \quad (18)$$

Hodge-de Rham decomposition implies that $divX = \Delta u$, hence from the above equation, we get

$$(1 - n\rho)R + h\Delta u = n\lambda. \tag{19}$$

Again if M^n is also a compact gradient RB h-almost soliton, then from equation (i) of Proposition 7 we have

$$(1 - n\rho)R + h\Delta f = n\lambda. \tag{20}$$

Equating the last two equations, we get $h\Delta(f - u) = 0$. Hence it is enough to use Hopf's theorem to conclude $f = u + c$, which completes the proof of theorem. □

Now we present the proof of Theorem 3.

Proof: If X is a nontrivial conformal vector field then we have $L_X g = 2\xi g$, $\xi \neq 0$. Thus the equation (4) becomes

$$Ric = (\lambda - h\xi + \rho R)g. \tag{21}$$

Now, tracing (21) and taking the covariant derivative we obtain

$$(1 - n\rho)\nabla R = n\nabla(\lambda - h\xi). \tag{22}$$

On the other hand, taking the divergence of (21) and using the equality $2div(Ric) = \nabla R$ we get

$$(\tfrac{1}{2} - \rho)\nabla R = \nabla(\lambda - h\xi). \tag{23}$$

Therefore, (22) and (23) together imply that

$$(1 - n\rho)\nabla R = n(\tfrac{1}{2} - \rho)\nabla R. \tag{24}$$

So, the scalar curvature R is a constant and hence $(\lambda - h\xi)$ is a constant. From Lemma 2.3 in [10] we conclude that $R \neq 0$; otherwise $\xi = 0$. Also since $(\lambda - h\xi)$, ρ and R are all constant we have

$$L_X Ric = 2(\lambda - h\xi + \rho R)\xi g. \tag{25}$$

Now we can apply Theorem 8 (Theorem 4.2 in [10]) to conclude that M^n is isometric to an Euclidean sphere, the term $2(\lambda - h\xi + \rho R)\xi$ is corresponding to α in Theorem 8, which completes the proof of the theorem. □

Using Proposition 7, we shall prove the Theorem 4.

Proof: From the Lemma 6 we have

$$div(Ric(\nabla f)) = (divRic)(\nabla f) + g(\nabla^2 f, Ric). \tag{26}$$

Using the second contracted second Bianchi identity, the equation (5) and the equality $|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n}$, we get

$$\begin{aligned} div(Ric(\nabla f)) &= \frac{1}{2} g(\nabla R, \nabla f) + (\lambda + \rho R)\Delta f - h|\nabla^2 f|^2 \\ &= \frac{1}{2} g(\nabla R, \nabla f) + (\lambda + \rho R)\Delta f - h \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \frac{h}{n}(\Delta f)^2 \\ &= \frac{1}{2} g(\nabla R, \nabla f) + \frac{R}{n}\Delta f - h \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2. \end{aligned} \tag{27}$$

Since

$$hRic(\nabla f) = \frac{(1-2\rho(n-1))}{2}\nabla R - (n-1)\nabla\lambda + \nabla h \Delta f - g^{jk}(\nabla_k h)\nabla_i \nabla_j f \tag{28}$$

by (ii) of Proposition 7, we also have

$$\begin{aligned} hdivRic(\nabla f) = &\frac{(1-2\rho(n-1))}{2}\Delta R - (n-1)\Delta\lambda + \Delta h \Delta f \\ &- div(g^{jk}(\nabla_k h)\nabla_i \nabla_j f). \end{aligned} \tag{29}$$

Comparing the last equation with (27) yields

$$\begin{aligned} h^2 \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 = &\frac{(2\rho(n-1)-1)}{2}\Delta R + (n-1)\Delta\lambda - \Delta h \Delta f + \frac{h}{2} g(\nabla R, \nabla f) \\ &+ \frac{h}{n} R \Delta f + div(g^{jk}(\nabla_k h)\nabla_i \nabla_j f). \end{aligned} \tag{30}$$

Integrating both sides of the above equation over compact M^n, we obtain

$$\begin{aligned} \int_M h^2 \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 dM &= \frac{1}{2}\int_M hg(\nabla R, \nabla f) dM + \frac{1}{n}\int_M hR\Delta f dM \\ &= \frac{n-2}{2n}\int_M hg(\nabla R, \nabla f) dM. \end{aligned} \tag{31}$$

This completes the proof of theorem. □

Finally, we shall give the proof of Corollary 5 by using Theorem 4.

Proof: First we notice that the equation (5) and the equation (i) in Proposition 7 yield

$$Ric - \frac{R}{n} g = -h\nabla^2 f + (\lambda + \rho R - \frac{R}{n}) g = -h(\nabla^2 f - \frac{\Delta f}{n} g). \tag{32}$$

On the other hand, if any one of the conditions of Corollary 5 holds, then the term of $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dM$ is equal to zero. Hence, we have $Ric = \frac{R}{n} g$. Considering this with the equation (5) gives us to

$$\nabla^2 f = \frac{1}{h}(\lambda + R(\rho - \frac{1}{n}))g, \tag{33}$$

which implies ∇f is a nontrivial conformal vector field. So from Theorem 3 we obtain that M^n is isometric to an Euclidean sphere $\mathbb{S}^n(r)$. This completes the proof of corollary. □

4. CONCLUSIONS

We developed a new extension of the concept of Ricci soliton and gave some structure equations for this RB h-almost soliton in Riemannian manifolds. We obtained extra terms in computations since in our more general setting λ and h are functions. Later, as a main result we obtained that a compact nontrivial RB h-almost soliton is isometric to the Euclidean sphere. Finally, we had an integral formula for the compact gradient RB h-almost soliton.

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