ORIGINAL PAPER

ON CAUCHY-RIEMANN STRUCTURES AND THE GENERAL EVEN ORDER STRUCTURE

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Abstract. *The present paper aims to study the Cauchy-Riemann structures and the general even order structure and find the general even order structure that acts on complementary distributions* D_l *and* Dm *as an almost complex structure and a null operator, respectively. We also discuss integrability conditions and prove certain theorems on the Cauchy-Riemann structures and the general even order structure. Moreover, we construct examples of it.*

Keywords: *Cauchy-Riemann structures; general even order equation; Nijenhuis tensor.*

1. INTRODUCTION

The study of Cauchy-Riemann manifold is a primary topic in differential geometry. Let M be a real manifold and T_pM the tangent space at point p of M. Then there exist a subspace T_p^cM and complex structure on it. The set $T^cM = \bigcup_{p\in M} T_p^cM$ is a Cauchy-Riemann structure or CR-structure over manifold M [1]. Such manifold is called Cauchy-Riemann manifold. Many authors made effective contributions to CR-structures and CR-submanifolds [2-5].

Let M be k-dimanesional manifold and a tensor field $F(\neq 0)$ of type (1,1) such that

$$F^{2n} + \alpha F^n + \beta I_{2n} = 0 \tag{1}$$

where n is a positive integer $(n>1), \alpha, \beta$ scalars not equal to zero, I_{2n} denotes the unit tensor field and being of constant rank r everywhere in M [6]. Then we say that the manifold M is equipped with the general even order structure of the rank r and M is called the general even order manifold.

The projection operators l and m are given by [7]

$$l = -\frac{F^{2n} + \alpha F^n}{\beta}, \quad m = I_{2n} + \frac{F^{2n} + \alpha F^n}{\beta}. \tag{2}$$

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Theorem 1.1. Let M be the general even order manifold. Then

$$l+m=I,\ l^2=l,\ and\ m^2=m. \tag{3}$$

Proof: In the view of equation (1), the proof is trivial.

The equations (2) and (3) show that D_l and D_m are complementary distributions with respect to the projection operators l and m, respectively. The rank of F is r implies that dimension of D_l is r and dimension of D_m is $n-r$.

Theorem 1.2. Let M be the general even order manifold. Then

$$Fl=lF=F,\ \ Fm=mF=0 \tag{4}$$

$$\frac{F^{2n}+\alpha F^n}{\beta}=-l,\quad \frac{F^{2n}+\alpha F^n}{\beta}l=-l,\quad \frac{F^{2n}+\alpha F^n}{\beta}m=0. \tag{5}$$

Thus $\left(\frac{F^{2n}+\alpha F^n}{\beta}\right)^{\frac{1}{2}}$ acts on D_l as an almost complex structure and on D_m as a null operator.

Proof: In the view of equation (1), the proof is trivial.

2. NIJENHUIS TENSOR

Let U and V be vector fields on M. The Nijenhuis tensor $N(U,V)$ of F satisfying equatin (1) in M is given by

$$N(U,V)=[FU,FV]-F[FU,V]-F[XU,FV]+F^2[U,V] \tag{6}$$

where Lie bracket $[U,V]=UV-VU$.

First we state the proposition for later use [5]:

Proposition 2.1. Let U and V be vector fields on M. The general even order structure F to be integrable ifand only if $N(U,V)=0$.

3. CAUCHY-RIEMANN STRUCTURE

Let T_cM be complexified tangent bundle of differentiable manifold M. A CR-structure on M is a complex subbundle H of T_cM such that $H_p \cap \overline{H}_p = 0$ and H is involutive, i.e., for complex vector fields U and V in H, $[U,V]$ is in H. Then M is a CR-manifold. Let M be a differentiable manifold and the general even order integrable structure

F satisfying equation (1) of rank $r = 2m$ on it. The complex subbundle H of $T_c M$ is given as $H_p = \{U - \sqrt{-1}FU, U \in \chi(D_l)\}$ such that $\mathrm{Real}(H) = D_l$ and $H_p \cap \overline{H}_p = 0$, where H_p is the complex conjugate of H and $\chi(D_l)$ the $\Gamma(D_m)$ module of all differentiable sections of D_l [5].

Theorem 3.1. If U and V are two vector fields in M, then

$$[P,Q] = [U,V] - [FU,FV] - \sqrt{-1}[U,FV] + [FU,V] \tag{7}$$

$\forall P,Q \in H.$

Proof: Since P and Q are two elements of H. Then we can write $P = U - \sqrt{-1}FU$ and $Q = V - \sqrt{-1}FV$. Thus

$$\begin{aligned}[P,Q] &= [X - \sqrt{-1}FU, V - \sqrt{-1}FV] \\ &= [U,V] - [FU,FV] - \sqrt{-1}([U,FV] + [FU,V]).\end{aligned}$$

Theorem 3.2. The integrable general even order structure satisfying

$$F^{2n} + \alpha F^n + \beta I_{2n} = 0$$

on M is integrable, we obtain

$$-(F^{2n-1} + \alpha F^{n-1})([FU,FV] + F^2[U,V]) = \beta l([FU,V] + [U,FV]). \tag{8}$$

Proof: The Nijenhuis tensor N of F is given by

$$N(U,V) = [FU,FV] - F[FU,V] - F[U,FV] + F^2[U,V]$$

since F is integrable then $N(U,V) = 0,$ we get

$$[FU,FV] + F^2[U,V] = F([FU,V] + [U,FV]) \tag{9}$$

multiplying equation (9) by $-\dfrac{F^{2n-1} + \alpha F^{n-1}}{\beta}$ and making use of equation (2), we obtain

$$-\frac{F^{2n-1} + \alpha F^{n-1}}{\beta}([FU,FV] + F^2[U,V]) = -\frac{F^{2n-1} + \alpha F^{n-1}}{\beta} F([FU,V] + [U,FV]) \tag{10}$$

$$-(F^{2n-1} + \alpha F^{n-1})([FU,FV] + F^2[U,V]) = \beta l([FU,V] + [U,FV]) \tag{11}$$

Hence, the theorem (3.2) is proved.

Theorem 3.3. Let N be the Nijenhuis tensor of F then

$$mN(U,V) = mN(FU,FV) \tag{12}$$

$$mN\left(\frac{F^{2n-1}+\alpha F^{n-1}}{\beta}U,V\right) = m\left(\frac{F^{2n}+\alpha F^{n}}{\beta}U,FV\right) \tag{13}$$

where U and V are vector fields on M.

Proof: The proof is obvious by using Theorem (1.1), Theorem (1.2) and equations (2).

Theorem 3.4. Let N be the Nijenhuis tensor of F. The given statements are equivalent.

i. $mN(U,V) = 0$

ii. $m[FU,FV] = 0$

iii. $mN\left(\frac{F^{2n}+\alpha F^{n}}{\beta}U,V\right) = 0$

iv. $m\left[\frac{F^{2n}+\alpha F^{n}}{\beta}FU,FV\right] = 0$

v. $m\left[\frac{F^{2n}+\alpha F^{n}}{\beta}lFU,FV\right] = 0$

where U and V are vector fields on M.

Proof: Using equations (1), (2), (6) and Theorem (1.2) and Theorem (3.3), the given statements can be showed to be equivalent.

Theorem 3.5. Let $\left(\frac{F^{2n}+\alpha F^{n}}{\beta}\right)^{\frac{1}{2}}$ acts on D_l as an almost complex structure, we have

$$m\left[\frac{F^{2n}+\alpha F^{n}}{\beta}lFU,FV\right] = m[-FU,FV] = 0. \tag{14}$$

Proof: Since $\left(\frac{F^{2n}+\alpha F^{n}}{\beta}\right)^{\frac{1}{2}}$ acts on D_l as an almost complex structure i.e. $\frac{F^{2n}+\alpha F^{n}}{\beta} = -l$. Making use of equations (3), (5) and $[U,V] = UV - VU$, we get equation (14).

Theorem 3.6. *For* $U, V \in \chi(D_l)$, *we have* $l([U, FV] + [FU, V]) = [U, FV] + [FU, V]$.

Proof: As $U \in \chi(D_l)$, using Theorem (1.2) and $[U, V] = UV - VU$, we obtain the result.

Theorem 3.7. If general even order structure satisfying $F^{2n} + \alpha F^n + \beta I_{2n} = 0$ is integrable on M and gives a CR-structure H on it implies that $\text{Real}(H) = D_l$.

Proof: Since $[U, FV]$ and $[FU, V]$ are elements of $\chi(D_l)$ and using Theorem (3.1) and Theorem (3.2) and Theorem (3.6), we obtain $[P, Q] \in \chi(D_l)$. Hence the general even order structure satisfying $F^{2n} + \alpha F^n + \beta I_{2n} = 0$ on M defines a CR-structure.

Definition 3.1. If M be a differentiable manifold and $\tilde{K}$ be the complementary distribution of $\text{Real}(H)$ to TM, a morphism of vector bundles $F: TM \to TM$ implies that $F(X) = 0$ for all $X \in (\tilde{K})$, such that

$$F(U) = \frac{1}{2}\sqrt{-1}(P - \overline{P}) \tag{15}$$

where $U + \sqrt{-1}V \in H_p$ and $\overline{P}$ is a complex conjugate of P [5].

Corollary 3.1. If $P = U + \sqrt{-1}V$ and $\overline{P} = U - \sqrt{-1}V \in H_p$,

then

$$F(U) = \frac{1}{2}\sqrt{-1}(P - \overline{P}),\ F(V) = \frac{1}{2}(P + \overline{P})$$

and

$$F(-V) = -\frac{1}{2}(P + \overline{P}),$$

then

$$F(U) = -V,\ F^2(U) = -U$$

and

$$F(-V) = -U\ .$$

Proof. In view of equation (15), we obtain

$$F(U) = \frac{1}{2}\sqrt{-1}(U + \sqrt{-1}V - U - \sqrt{-1}V) = \frac{1}{2}\sqrt{-1}(2\sqrt{-1}V)$$

$$F(U) = -V$$

Operating F on both sides of above equation, we obtain

$$F(F(U)) = F(-V), \tag{16}$$

But

$$F(V) = \frac{1}{2}(U + \sqrt{-1}V + U - \sqrt{-1}V) = -U,$$

Also,

$$\begin{aligned} F(-V) &= -\frac{1}{2}(U + \sqrt{-1}V + U - \sqrt{-1}V) \\ &= -U \end{aligned} \tag{17}$$

From equations (16) and (17), we obtain $F^2(U) = -U$.

Theorem 3.8. Let M be a differentiable manifold and CR-structure defined on it. Then $F^{2n} + \alpha F^n + \beta I_{2n} = 0$ and consequently the general even order structure is defined on M implies that D_l and D_m coincide respectively with $\text{Real}(H)$ and K.

Proof: Let CR-structure defined on a differentiable manifold M. Then with the help of equation (15), $F(U)$ can be written as

$$F(U) = -V \tag{18}$$

multiply equation (18) by $-\dfrac{F^{2n-1} + \alpha F^{n-1}}{\beta}$, we obtain

$$\frac{F^{2n-1} + \alpha F^{n-1}}{\beta} F(U) = \frac{F^{2n-1} + \alpha F^{n-1}}{\beta}(-V) \tag{19}$$

By straight forward calculation, we have $F^{2n} + \alpha F^n + \beta I_{2n} = 0$.

4. EXAMPLES

4.1. F(4, 2)-STRUCTURE

Suppose that a tensor field $F(\neq 0)$ of type (1,1) in M satisfying $F^4 + F^2 = 0$ with rank r.

The projection operators are given by

$$l = -F^2, \quad m = I + F^2$$

It can obtain

$$l + m = I, \qquad l^2 = l, \qquad m^2 = m,$$

$$Fl = lF = F, \quad Fm = mF = 0,$$

$$F^2 = -l, \qquad F^2 l = -l, \qquad F^2 m = 0.$$

Thus F acts on D_l as an almost complex structure and on D_m as null operator. Taking into account the above results, it is easily seen that a relationship between CR-structures and $F(4,2)$ -structure by a similar device used in the above Theorems.

4.2. F(4, -2)-*STRUCTURE*

Suppose that a tensor field $F(\neq 0)$ of type (1,1) in M satisfying $F^4 - F^2 = 0$ with rank r.

The projection operators are given by

$$l = F^2, \quad m = I - F^2$$

It can obtain

$$l + m = I, \qquad l^2 = l, \qquad m^2 = m,$$

$$Fl = lF = F, \quad Fm = mF = 0,$$

$$F^2 = l, \; F^2 l = l, \qquad F^2 m = 0.$$

Thus F acts on D_l as an almost product structure and on D_m as null operator.

5. CONCLUSION

We considered that CR-structures and the general even order structure satisfying $F^{2n} + \alpha F^n + \beta I_{2n} = 0,\; n > 1,\; \alpha, \beta$ are nonzero scalars. We have shown that the relationship between CR-structures and the general even order structure.

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