ORIGINAL PAPER

SOLVING SOME IMPORTANT NONLINEAR TIME-FRACTIONAL EVOLUTION EQUATIONS BY USING THE (G'/G)-EXPANSION METHOD

MEDJAHED DJILALI[1, 2], ALI HAKEM[2]

Manuscript received: 19.06.2020; Accepted paper: 02.10.2020;
Published online: 30.12.2020.

Abstract. *The aim of the present study is to find the exact solutions for three generalized nonlinear time-fractional evolution equations, Kaup-Kupershmidt equation, Burgers-Fisher equation and, Shallow Water Wave equation. By using the (G'/G)-expansion method and depending on second order linear ODE as well as the complex transformation, three kinds of solutions (hyperbolic, trigonometric, and rational) are obtained. With the help of Mathematica software package, difficult algebraic systems are solved and surfaces of some particular solutions of the equations under study are plotted.*

Keywords: *Burgers-Fisher equation; Kaup-Kupershmidt equation; time-fractional; Shallow Water Wave equation; traveling wave solutions.*

1. INTRODUCTION

In recent years, the problem of research and the construction of exact solution to nonlinear evolutionary equations has been among the first concerns of many mathematicians.

Due to the complexity of the nonlinear system, finding explicit analytical solutions to this type of equations is often a difficult process. For this, a number of powerful and important methods have been developed, like inverse scattering transform method, Exp-function method, Kudryashov method, sine–cosine method, Tanh function method, extended Tanh (or tanh-coth) method, extended F-expansion method and others.

More recently, the (G'/G)-expansion method (see [1-10]) has been proposed to obtain traveling wave solutions. This method is firstly proposed by Wang et al. [10] for which the traveling wave solutions of the nonlinear evolution equations are obtained.

In the present article, by using this well known method, exact solutions of the following three nonlinear evolution equations will be sought, first the generalized time-fractional Kaup-Kupershmidt equation in the form

$$D_t^{\alpha}u + 20a^2bu_{5x} + 10abuu_{3x} + 25abu_xu_{2x} + bu^2u_x = 0,$$

where $a,b \in \mathbb{R}^*$, $D_t^{\alpha}u := \dfrac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}$ in the sense of Caputo derivative and $0 < \alpha \le 1$.

[1] Oran Higher School of Economics, 31000 Oran, Algeria. E-mail: djilalimedjahed@yahoo.fr.
[2] Djillali Liabes University, Laboratory ACEDP, 22000 Sidi-Bel-Abbes, Algeria.
E-mail: hakemali@yahoo.com.

In the particular case $\alpha = 1$ (see [11]): It is a class of the better-known standard fifth-order KdV equation and has properties similar (but not identical) to those of fKdV(see [12]), may be used to model dispersive phenomena such as plasma waves.

As the constants $a \neq 0, b \neq 0$ take different values, different types of Kaup-Kupershmidt equation are retrieved. For examples, in the case $a = \frac{1}{20},\ b = 30$ (see [13-15]), for $a = \frac{1}{60}, b = 180$, see [16], Reyes [17] studied the case $a = \frac{1}{10}, b = -5$, the case $a = -\frac{1}{30}, b = 45$ is studied by Parker [18, 19], and when $a = \frac{1}{30}, b = 5$, yields the equation treated in [20] and [21]. While, the Kupershmidt equation is obtained by taking $a = \frac{1}{5}, b = -\frac{5}{4}$ (see [22-24]).

The second considerable equation is the time-fractional generalized Burgers-Fisher equation in the form

$$D_t^{\alpha} u - u_{xx} + auu_x - bu(1-u) = 0,$$

where $a, b \in \mathbb{R}^*$ and $0 < \alpha \leq 1, D_t^{\alpha} u := \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$ in the sense of Caputo derivative.

In the particular case $\alpha = 1$ (see [25-32]): This equation shows a prototypical model for describing the interaction between the reaction mechanism, covection effect, and diffusion transport. When b=0, the generalized Burgers– Fisher equation becomes the Burgers equation

$$u_t - u_{xx} + auu_x = 0.$$

The Burgers equation is an important nonlinear diffusion equation in physics, which describes the far field of wave propagation in the corresponding dissipative systems, also arises in a variety of physical contexts and has been studied by many authors. For examples (see [31, 32]).

When a=0, b=1, this equation reads

$$u_t - u_{xx} - u(1-u) = 0.$$

This important nonlinear diffusion equation in nonlinear science called Fisher equation (or KPP equation), it was first studied by Fisher, Kolmogorov, Petrovski and Pisconov as a model in biology and is in close connection with some important physical phenomena, such as neu-tron action, wave motion in liquid crystals, nerve signal propagation in biophysics.

As for Wazwaz [33], he was interested in studying the specific cases a=0 and a=-1, b=1.

Finally, let's consider the time-fractional Generalized Shallow Water Wave equation in the form

$$D_t^{\nu}(u_{3x}) + \alpha u_x D_t^{\nu}(u_x) + \beta D_t^{\nu}(u) u_{2x} - D_t^{\nu}(u_x) - u_{2x} = 0,$$

where $0 < \nu \leq 1, D_t^{\nu} u := \dfrac{\partial^{\nu} u(x,t)}{\partial t^{\nu}}$ in the sense of Caputo derivative and $\alpha \neq 0, \beta \neq 0$ and $\alpha + \beta \neq 0$.

Let's point out that the shallow-water wave equations describe a thin layer of fluid of constant density in hydrostatic balance, bounded from below by the bottom topography and from above by a free surface.

In the case $\nu = 1$ (see [34-39],): Clarkson and Mansfield (see [34]) examined two special cases $\alpha = \beta$ and $\alpha = 2\beta$, Ablowitz et al. [35] studied the specific case $\alpha = -4$ and $\beta = -2$, while Hirota and Satsuma [36] treated the equation when $\alpha = \beta = -3$.

Wazwaz [37] used the tanh–coth method to obtain single-soliton solutions and also applied the Exp-function method to derive a variety of travelling wave solutions for this equation in the two cases

1.1. CAPUTO DERIVATIVE

The Caputo derivative of order α is defined by the formula (see [40-42]):

$$D_*^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & \text{if } m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \text{if } \alpha = m, \end{cases} \tag{1.1}$$

where $m \in \mathbb{N}^*$ and $\Gamma(.)$ denotes the Gamma function defined by $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt,\ x > 0$.

The important properties of the Caputo derivative that will be used in this paper are:

$$D_t^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha},\ D^{\alpha} c = 0, \tag{1.2}$$

$$D_t^{\alpha}\big[f(t)g(t)\big] = f(t) D_t^{\alpha} g(t) + g(t) D_t^{\alpha} f(t), \tag{1.3}$$

$$D_t^{\alpha}\big[f(g(t))\big] = f_g'(g(t)) D_t^{\alpha} g(t) = D_g^{\alpha} f(g(t)) \big[g_t'(t)\big]^{\alpha}, \tag{1.4}$$

$$d^{\alpha} h(t) = \Gamma(1+\alpha) d\, h(t). \tag{1.5}$$

2. DESCRIPTION OF THE $(\frac{G'}{G})$ -EXPANSION METHOD

The general nonlinear Time-Fractional evolution equation, say in two independent variables x and t, is given by

$$P(u, D_t^{\alpha}u, u_x, D_t^{2\alpha}u, u_{xx}, D_t^{\alpha}u_x, \ldots) = 0, \quad 0 < \alpha \leq 1, \tag{2.1}$$

where $u = u(x, t)$ is a nunknown function, P is a polynomial of u and its partial fractional derivatives, in which the nonlinear terms and the highest order derivatives are included.

To find the traveling wave solution of Eq. (2.1) by $(\frac{G'}{G})$-expansion method, the following steps should be taken:

- **Step 1**: To obtain exact traveling wave solution, the following fractional complex transformation has been applied

$$u(x,t) = U(\xi), \ \xi = kx - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.2}$$

Where k, ω are constants to be determined latter. Then, the Eq (2.1) is reduced to the following nonlinear ordinary differential equation

$$P(U, -\omega U', kU, \omega^2 U'', k^2 U'', -\omega k U', \ldots) = 0, \tag{2.3}$$

where $U^{(i)} = U_{i\xi}$.

- **Step 2**: Assuming that the solution of Eq. (2.3) can be expressed as a finite power series of the form

$$U(\xi) = \sum_{n=0}^{N} a_n \left(\frac{G'}{G}\right)^n, \tag{2.4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{2.5}$$

λ, μ are constants to be discuss later and also $a_0, a_1, \ldots a_N \ (a_N \neq 0)$ are constants to be determined later.

The general solutions of (2.5) can be written in the forms as follow

$$G(\xi) = \begin{cases} e^{\frac{1}{2}(-\lambda)\xi}\left(A_2 \sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu}\right) + A_1 \cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu}\right)\right), & \lambda^2 - 4\mu > 0, \\ e^{\frac{1}{2}(-\lambda)\xi}\left(A_2 \sin\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right) + A_1 \cos\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right)\right), & \lambda^2 - 4\mu < 0, \\ \left(A_2\xi + A_1\right)e^{\frac{1}{2}(-\lambda)\xi}, & \lambda^2 - 4\mu = 0, \end{cases} \tag{2.6}$$

it yields

$$\frac{G'}{G}=\begin{cases}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right)-\frac{\lambda}{2},\ \lambda^2-4\mu>0,\\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{A_2\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)-A_1\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}{A_2\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_1\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}\right)-\frac{\lambda}{2},\ \lambda^2-4\mu<0,\\ \frac{A_2}{A_2\xi+A_1}-\frac{\lambda}{2},\quad \lambda^2-4\mu=0,\end{cases}\tag{2.7}$$

where A_1 and A_2 are arbitrary constants.

- **Step 3**: The degree N of the power series (2.4) is determined by considering the homoge-neous balance between the nonlinear term in Eq. (2.3) and the highest-order derivative.
- **Step 4**: Substituting Eq. (2.4) using Eq. (2.5) into Eq. (2.3). Then collecting the coefficients of like powers of $(\frac{G'}{G})^n, (n=0,1,2,\ldots,N)$. A set of nonlinear algebraic equations is obtained, by equating each coefficient to zero. The resulting algebraic system is solved with the help of *Mathematica* to get the values of unknown constants $a_0, a_1, \ldots a_N$ and, k, ω.
- **Step 5**: Since the general solution of (2.5) has been well known for us, then substituting a_n, k, ω and (2.7) into (2.4), three types of the Exact traveling wave solutions of the time-fractional nonlinear evolution equation are obtained (2.1).

3. APPLICATIONS

Here, three examples to illustrate the applicability of the $(\frac{G'}{G})$ -expansion method to solve nonlinear time-fractional evolution equations.

3.1. EXACT SOLUTIONS OF TIME-FRACTIONAL GENERALIZED KAUP-KUPERSHMIDT EQUATION

Let's start with (TFKKE), this equation can be written as the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+20a^2b\,\frac{\partial^5 u(x,t)}{\partial x^5}+10abu(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}+25ab\,\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}+bu(x,t)^2\,\frac{\partial u(x,t)}{\partial x}=0\tag{3.1}$$

Using the fractional complex transformation $u(x,t)=U(\xi),\ \xi = x-\dfrac{\omega t^{\alpha}}{\Gamma(1+\alpha)}$, the (TFKKE) (3.1) is converted to the (NLODE)

$$20a^2 b U^{(5)}+25abU'U''+10abUU^{(3)}+bU^2U'-\omega U'=0. \qquad (3.2)$$

Balancing $U^{(5)}$ with $UU^{(3)}$ in (3.2) gives N+5=3N+1 , hence N=2. Then, let's suppose that (3.2) has the following formal solutions:

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2,\quad a_2\neq 0. \qquad (3.3)$$

Substituting Equation (3.3) into Equation (3.2) and collecting all term with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand sides of Equation (3.2) are converted into a polynomial in $\left(\frac{G'}{G}\right)$.

Setting each coefficient of each term to zero, a set of algebraic equations for a_0, a_1, a_2, ω is derived.

$$\begin{aligned}
&-20a^2b\mu a_1\lambda^4-600a^2b\mu^2a_2\lambda^3-440a^2b\mu^2a_1\lambda^2-10ab\mu a_0a_1\lambda^2-25ab\mu^2a_1^2\lambda-2400a^2b\mu^3a_2\lambda\\
&-60ab\mu^2a_0a_2\lambda-320a^2b\mu^3a_1-b\mu a_0^2a_1+\mu\omega a_1-20ab\mu^2a_0a_1-50ab\mu^3a_1a_2=0,\\
&-20a^2ba_1\lambda^5-1240a^2b\mu a_2\lambda^4-1040a^2b\mu a_1\lambda^3-10aba_0a_1\lambda^3-60ab\mu a_1^2\lambda^2-11680a^2b\mu^2a_2\lambda^2\\
&-140ab\mu a_0a_2\lambda^2-2720a^2b\mu^2a_1\lambda-ba_0^2a_1\lambda+\omega a_1\lambda-80ab\mu a_0a_1\lambda-310ab\mu^2a_1a_2\lambda\\
&-70ab\mu^2a_1^2-2b\mu a_0a_1^2-100ab\mu^3a_2^2-5440a^2b\mu^3a_2-2b\mu a_0^2a_2+2\mu\omega a_2-160ab\mu^2a_0a_2=0,\\
&-640a^2ba_2\lambda^5-620a^2ba_1\lambda^4-35aba_1^2\lambda^3-17680a^2b\mu a_2\lambda^3-80aba_0a_2\lambda^3-5840a^2b\mu a_1\lambda^2\\
&-70aba_0a_1\lambda^2-500ab\mu a_1a_2\lambda^2-230ab\mu a_1^2\lambda-2ba_0a_1^2\lambda-460ab\mu^2a_2^2\lambda-34240a^2b\mu^2a_2\lambda\\
&-2ba_0^2a_2\lambda+2\omega a_2\lambda-520ab\mu a_0a_2\lambda-b\mu a_1^3-2720a^2b\mu^2a_1-ba_0^2a_1+\omega a_1\\
&-80ab\mu a_0a_1-530ab\mu^2a_1a_2-6b\mu a_0a_1a_2=0,\\
&-8440a^2ba_2\lambda^4-3600a^2ba_1\lambda^3-240aba_1a_2\lambda^3-170aba_1^2\lambda^2-640ab\mu a_2^2\lambda^2-62080a^2b\mu a_2\lambda^2\\
&-380aba_0a_2\lambda^2-ba_1^3\lambda-9600a^2b\mu a_1\lambda-120aba_0a_1\lambda-1500ab\mu a_1a_2\lambda-6ba_0a_1a_2\lambda\\
&-180ab\mu a_1^2-2ba_0a_1^2-660ab\mu^2a_2^2-4b\mu a_0a_2^2-24640a^2b\mu^2a_2-2ba_0^2a_2-4b\mu a_1^2a_2\\
&+2\omega a_2-400ab\mu a_0a_2=0,\\
&-280aba_2^2\lambda^3-34200a^2ba_2\lambda^3-7800a^2ba_1\lambda^2-1000aba_1a_2\lambda^2-245aba_1^2\lambda-1720ab\mu a_2^2\lambda\\
&-4ba_0a_2^2\lambda-4ba_1^2a_2\lambda-79200a^2b\mu a_2\lambda-540aba_0a_2\lambda-ba_1^3-5b\mu a_1a_2^2-4800a^2b\mu a_1\\
&-60aba_0a_1-1030ab\mu a_1a_2-6ba_0a_1a_2=0,\\
&-2b\mu a_2^3-1080ab\lambda^2a_2^2-1100ab\mu a_2^2-4ba_0a_2^2-5b\lambda a_1a_2^2-60000a^2b\lambda^2a_2-4ba_1^2a_2\\
&-33600a^2b\mu a_2-240aba_0a_2-1310ab\lambda a_1a_2-110aba_1^2-7200a^2b\lambda a_1=0,.
\end{aligned} \qquad (3.4)$$

The resulting algebraic system (3.4) is solved with the help of *Mathematica* to get the values of unknown constants $a_0; a_1; a_2; \omega$

$$\left\{a_0 \to -20\left(a\lambda^2+8a\mu\right), a_1 \to -240a\lambda, a_2 \to -240a, \omega \to 220a^2b\left(\lambda^2-4\mu\right)^2\right\},$$
$$\left\{a_0 \to \frac{1}{2}(-5)\left(a\lambda^2+8a\mu\right), a_1 \to -30a\lambda, a_2 \to -30a, \omega \to \frac{5}{4}a^2b\left(\lambda^2-4\mu\right)^2\right\}, \tag{3.5}$$

where λ and μ are arbitrary constants.

By using Eqs. (3.5) , expression (3.3) can be written as

$$U_1(\xi) = -20\left(a\lambda^2+8a\mu\right) - 240a\lambda\left(\frac{G'}{G}\right) - 240a\left(\frac{G'}{G}\right)^2, \tag{3.6}$$

where $\xi = x - 220a^2b\left(\lambda^2-4\mu\right)^2 \dfrac{t^\alpha}{\Gamma(1+\alpha)}$, or

$$U_2(\xi) = \frac{1}{2}(-5)\left(a\lambda^2+8a\mu\right) - 30a\lambda\left(\frac{G'}{G}\right) - 30a\left(\frac{G'}{G}\right)^2, \tag{3.7}$$

where $\xi = x - \dfrac{5}{4}a^2b\left(\lambda^2-4\mu\right)^2 \dfrac{t^\alpha}{\Gamma(1+\alpha)}$.

Using the general solutions of Eq. (2.5) into (3.6)-(3.7), three kinds of traveling wave solutions are obtained.

- **Case 1**: $\lambda^2-4\mu>0$, hyperbolic function solutions of Eq. (3.1) are obtained.

$$u_{1,1}(x,t) = U_{1,1}(\xi)$$
$$= 40a(\lambda^2-4\mu) - 60a(\lambda^2-4\mu)\left(\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right)^2 \tag{3.8}$$

where $\xi = x - 220a^2b\left(\lambda^2-4\mu\right)^2 \dfrac{t^\alpha}{\Gamma(1+\alpha)}$, or

$$u_{1,2}(x,t) = U_{1,2}(\xi)$$
$$= 5a(\lambda^2-4\mu) - \frac{15}{2}a(\lambda^2-4\mu)\left(\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right)^2 \tag{3.9}$$

where $\xi = x - \dfrac{5}{4}a^2b\left(\lambda^2-4\mu\right)^2 \dfrac{t^\alpha}{\Gamma(1+\alpha)}$.

In particular, if $a = -\dfrac{1}{30}, b = 45, \lambda = 1, \mu = 0, \alpha = \dfrac{3}{4}, A_1 \neq 0, A_2 = 0$, then

$$\xi = x - 11\frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})},$$

and (3.8) becomes

$$u_{1,1}(x,t) = -\frac{4}{3} + 2\tanh^2\left(x - 11\frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})}\right). \tag{3.10}$$

Also, if $a = \frac{1}{10}, b = -5, \lambda = 1, \mu = 0, \alpha = \frac{1}{2}, A_1 \neq 0, A_2 = 0$, then $\xi = x - \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$ and (3.9) becomes

$$u_{1,2}(x,t) = \frac{1}{2} - \frac{3}{4}\tanh^2\left(x + \frac{3}{16}\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right). \tag{3.11}$$

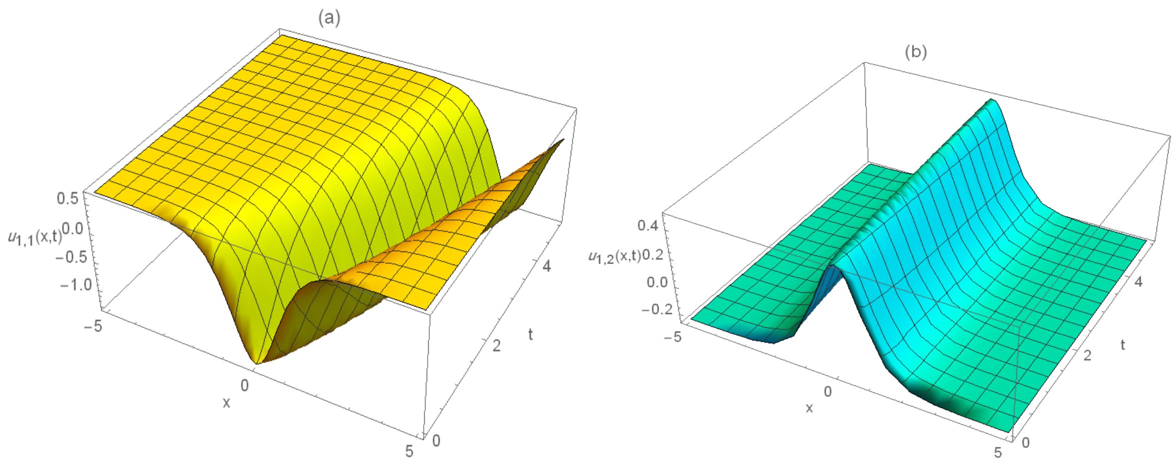

Figure 1. The 3D surfaces of the exact solutions of Eq. (3.1), (a) given by (3.10) and (b) (3.11), for $(x,t) \in [-5,5] \times [0,5]$

Case 2: $\lambda^2 - 4\mu < 0$, trigonometric function solutions of Eq. (3.1) are obtained.

$$u_{2,1}(x,t) = U_{2,1}(\xi)$$

$$= -40a(4\mu - \lambda^2) - 60a(4\mu - \lambda^2)\left(\frac{A_2\cos\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right) - A_1\sin\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right)}{A_2\sin\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right) + A_1\cos\left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right)}\right)^2 \tag{3.12}$$

where $\xi = x - 220a^2b\left(\lambda^2 - 4\mu\right)^2 \frac{t^\alpha}{\Gamma(1+\alpha)}$, or

$$u_{2,2}(x,t) = U_{2,2}(\xi)$$

$$=-5a(4\mu-\lambda^2)-\frac{15}{2}a(4\mu-\lambda^2)\left(\frac{A_2\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)-A_1\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}{A_2\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_1\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}\right)^2 \quad (3.13)$$

where $\xi=x-\frac{5}{4}a^2b\left(\lambda^2-4\mu\right)^2\frac{t^\alpha}{\Gamma(1+\alpha)}$.

In particular, if $a=-\frac{1}{30}, b=45, \lambda=0, \mu=\frac{1}{4}, \alpha=\frac{3}{4}, A_1\neq 0, A_2=0$, then $\xi=x-11\frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})}$ and (3.12) becomes

$$u_{2,1}(x,t)=\frac{4}{3}+2\tan^2\left(x-11\frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})}\right). \quad (3.14)$$

Also, if $a=\frac{1}{10}, b=-5, \lambda=0, \mu=\frac{1}{4}, \alpha=\frac{1}{2}, A_1\neq 0, A_2=0$, then $\xi=x-\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$ and (3.13) becomes

$$u_{2,2}(x,t)=-\frac{1}{2}-\frac{3}{4}\tan^2\left(x+\frac{3}{16}\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right) \quad (3.15)$$

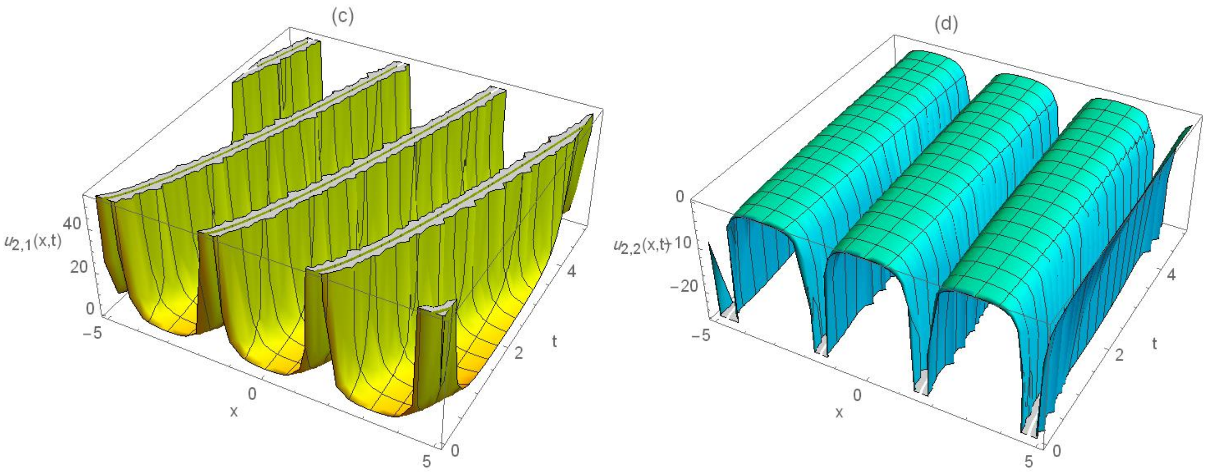

Figure 2. The 3D surfaces of the exact solutions of Eq. (3.1), (c) given by (3.14) and (d) (3.15), for $(x,t)\in[-5,5]\times[0,5]$.

- **Case 3**: $\lambda^2-4\mu=0$, so that $\xi=x$, rational function solutions of Eq. (3.1) are obtained.

$$u_{3,1}(x,t)=-240a\left(\frac{A_2}{A_1+A_2x}\right)^2 , \quad (3.16)$$

or

$$u_{3,2}(x,t)=-30a\left(\frac{A_2}{A_1+A_2x}\right)^2. \tag{3.17}$$

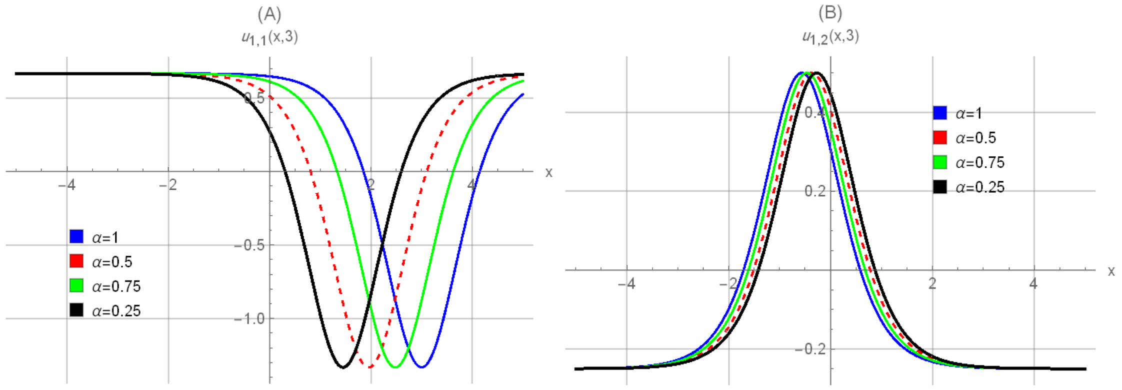

Figure 3. The 2D surfaces of the exact solutions of Eq. (3.1), (A) given by (3.10) and (B) (3.11), for $x\in[-5,5], t=3$ **and** $\alpha=1, 0.75, 0.5, 0.25$.

3.2. EXACT SOLUTIONS OF TIME-FRACTIONAL GENERALIZED BURGERS-FISHER EQUATION

Let's study the following (TFGBFE) in the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}-\frac{\partial^2 u(x,t)}{\partial x\,\partial x}+au(x,t)\frac{\partial u(x,t)}{\partial x}-bu(x,t)(1-u(x,t))=0, \tag{3.18}$$

the fractional complex transformation

$$u(x,t)=U(\xi),\ \ \xi=kx-\frac{\theta t^\alpha}{\Gamma(1+\alpha)},$$

permit to convert the (TFGBFE) (3.18) into the (NLODE)

$$-\theta U'-k^2U''+akUU'-bU(1-U)=0. \tag{3.19}$$

Now, balancing the terms of U'' with UU' gives $N+2=2N+1$, so that $N=1$ and thus the solutions are as follow.

$$U(\xi)=a_0+a_1\left(\frac{G'}{G}\right),\ \ a_1\neq 0. \tag{3.20}$$

Proceeding as above, substituting (3.20) into (3.19) and equating the coefficients of same powers of $\left(\frac{G'}{G}\right)$ to zero, yields a set of simultaneous algebraic equations among a_0, a_1, k, θ.

$$\begin{aligned}
&a_0^2 b - a_0 b + a_1\theta\mu + a_1(-\lambda)k^2\mu - aa_1a_0k\,\mu = 0,\\
&-a_1 b + 2a_0a_1 b + a_1\theta\lambda + a_1\left(-\lambda^2\right)k^2 - 2a_1k^2\mu - aa_0a_1\lambda k - aa_1^2k\,\mu = 0,\\
&a_1^2 b + a_1\theta - 3a_1\lambda k^2 - aa_1^2\lambda k - aa_0a_1k = 0,\\
&-2a_1k^2 - aa_1^2k = 0\,.
\end{aligned} \tag{3.21}$$

After solving these algebraic systems, four sets of values of arbitrary constants are obtained:

- if $\lambda^2 - 4\mu > 0$, then

$$\left\{a_0 \to \frac{1}{2}\left(1+\frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right), a_1 \to \frac{1}{\sqrt{\lambda^2-4\mu}}, k \to -\frac{a}{2\sqrt{\lambda^2-4\mu}}, \theta \to -\frac{1}{4}\left(\frac{a^2+4b}{\sqrt{\lambda^2-4\mu}}\right)\right\},$$

$$\left\{a_0 \to \frac{1}{2}\left(1-\frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right), a_1 \to -\frac{1}{\sqrt{\lambda^2-4\mu}}, k \to \frac{a}{2\sqrt{\lambda^2-4\mu}}, \theta \to \frac{1}{4}\left(\frac{a^2+4b}{\sqrt{\lambda^2-4\mu}}\right)\right\}. \tag{3.22}$$

- If $\lambda^2 - 4\mu < 0$, then

$$\left\{a_0 \to \frac{1}{2}\left(1+\frac{i\,\lambda}{\sqrt{4\mu-\lambda^2}}\right), a_1 \to \frac{i}{\sqrt{4\mu-\lambda^2}}, k \to -\frac{ia}{2\sqrt{4\mu-\lambda^2}}, \theta \to -\frac{1}{4}i\left(\frac{a^2+4b}{\sqrt{4\mu-\lambda^2}}\right)\right\},$$

$$\left\{a_0 \to \frac{1}{2}\left(1-\frac{i\,\lambda}{\sqrt{4\mu-\lambda^2}}\right), a_1 \to -\frac{i}{\sqrt{4\mu-\lambda^2}}, k \to \frac{ia}{2\sqrt{4\mu-\lambda^2}}, \theta \to \frac{1}{4}i\left(\frac{a^2+4b}{\sqrt{4\mu-\lambda^2}}\right)\right\}, \tag{3.23}$$

where $\lambda^2 - 4\mu \neq 0$.

By substituting a_0, a_1, k, θ from (3.22) -(3.23) and the general solution of second order linear ODE (2.5) into (3.20), three types of travelling wave solutions of Eq.(3.18) are derieved.

- **Case 1**: when $\lambda^2 - 4\mu > 0$, then

$$\begin{aligned}
u_{1,1}(x,t) = U_{1,1}(\xi) &= \frac{1}{2}\left(1+\frac{\lambda}{\sqrt{\lambda^2-4\mu}}\right)\\
&+\frac{1}{\sqrt{\lambda^2-4\mu}}\left(\frac{\sqrt{\lambda^2-4\mu}\left(A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)\right)}{2\left(A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)\right)}-\frac{\lambda}{2}\right)\\
&=\frac{1}{2}+\frac{1}{2}\,\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}
\end{aligned} \tag{3.24}$$

where $\xi = -\frac{a}{2\sqrt{\lambda^2 - 4\mu}} x + \frac{a^2 + 4b}{4\sqrt{\lambda^2 - 4\mu}} \frac{t^\alpha}{\Gamma(1+\alpha)}$

So,

$$u_{1,1}(x,t) = \frac{1}{2} + \frac{1}{2} \frac{A_1 \sinh\left(-\frac{a}{4}x + \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + A_2 \cosh\left(-\frac{a}{4}x + \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)}{A_2 \sinh\left(-\frac{a}{4}x + \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + A_1 \cosh\left(-\frac{a}{4}x + \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{3.25}$$

In particular, if $a = 4, b = 2, \alpha = 1, A_1 \neq 0, A_2 = 0$, then (3.25) becomes

$$u_{1,1}(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(3t - x), \tag{3.26}$$

or,

$$u_{1,2}(x,t) = U_{1,2}(\xi) = \frac{1}{2} - \frac{1}{2} \frac{A_1 \sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right) + A_2 \cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2 \sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right) + A_1 \cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)} \tag{3.27}$$

where $\xi = \frac{a}{2\sqrt{\lambda^2 - 4\mu}} x - \frac{a^2 + 4b}{4\sqrt{\lambda^2 - 4\mu}} \frac{t^\alpha}{\Gamma(1+\alpha)}$.

So,

$$u_{1,2}(x,t) = \frac{1}{2} - \frac{1}{2} \frac{A_1 \sinh\left(\frac{a}{4}x - \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + A_2 \cosh\left(\frac{a}{4}x - \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)}{A_2 \sinh\left(\frac{a}{4}x - \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + A_1 \cosh\left(\frac{a}{4}x - \frac{a^2+4b}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)}. \tag{3.28}$$

Also, if $a = 4, b = 2, \alpha = 0.75, A_1 = 0, A_2 \neq 0$, then (3.28) becomes

$$u_{1,2}(x,t) = \frac{1}{2} - \frac{1}{2}\coth\left(x - 3\frac{t^{0.75}}{\Gamma(1.75)}\right) \tag{3.29}$$

- **Case 2**: when $\lambda^2 - 4\mu < 0$, after simplification, then

$$u_{2,1}(x,t) = \frac{1}{2} + \frac{i}{2} \frac{A_2 \cos\left(-\frac{ia}{4}x + \frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) - A_1 \sin\left(-\frac{ia}{4}x + \frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)}{A_2 \sin\left(-\frac{ia}{4}x + \frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) + A_1 \cos\left(-\frac{ia}{4}x + \frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)} \tag{3.30}$$

and

$$u_{2,2}(x,t)=\frac{1}{2}-\frac{i}{2}\frac{A_2\cos\left(\frac{ia}{4}x-\frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)-A_1\sin\left(\frac{ia}{4}x-\frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)}{A_2\sin\left(\frac{ia}{4}x-\frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)+A_1\cos\left(\frac{ia}{4}x-\frac{i(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right)}. \tag{3.31}$$

If $A_1=0, A_2\neq 0$, with the fact that ($\sin(i\theta)=i\sinh(\theta), \cos(i\theta)=\cosh(\theta)$), then

$$u_{2,1}(x,t)=\frac{1}{2}+\frac{1}{2}\coth\left(-\frac{a}{4}x+\frac{(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \tag{3.32}$$

and, if $A_2=0, A_1\neq 0$, then

$$u_{2,2}(x,t)=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{a}{4}x-\frac{(a^2+4b)}{8}\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \tag{3.33}$$

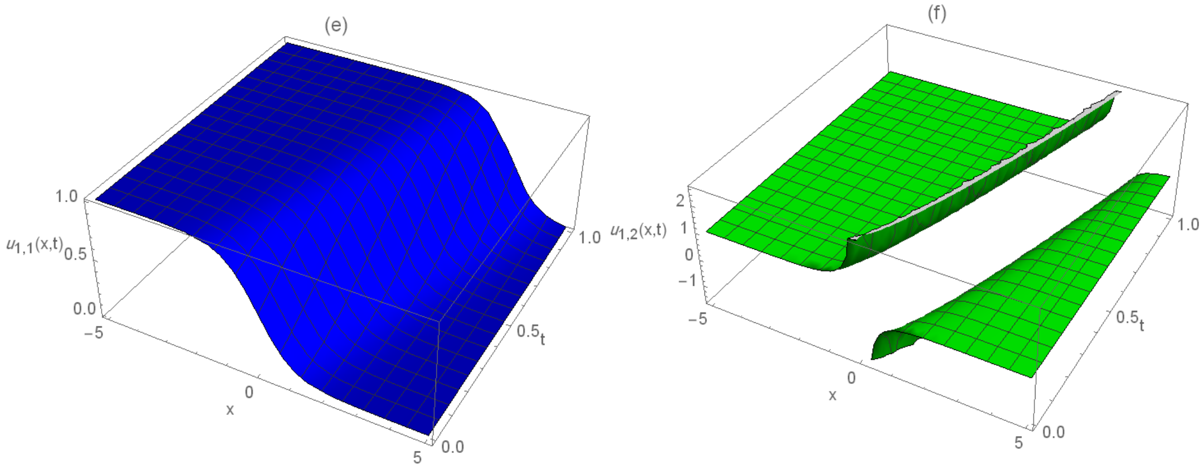

Figure 4. The 3D surfaces of the exact solutions of Eq. (3.18), (e) given by (3.26) and (f) (3.29), for $(x,t)\in[-5,5]\times[0,1]$.

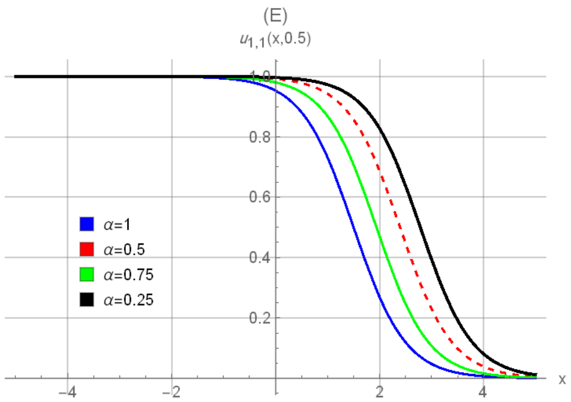

Figure 5. The 2D surfaces of the exact solutions of Eq. (3.18), (e) given by (3.26), for $x\in[-5,5], t=0.5$ and $\alpha=1, 0.75, 0.5, 0.25$.

3.3. EXACT SOLUTIONS OF TIME-FRACTIONAL GENERALIZED SHALLOW WATER WAVE EQUATION

Let's consider the (TFGSWW) equation of the form

$$\frac{\partial^\nu}{\partial t^\nu}\Big(\frac{\partial^3 u(x,t)}{\partial x^3}\Big)+\alpha\frac{\partial u(x,t)}{\partial x}\frac{\partial^\nu}{\partial t^\nu}\Big(\frac{\partial u(x,t)}{\partial x}\Big)+\beta\frac{\partial^2 u(x,t)}{\partial x^2}\frac{\partial^\nu u(x,t)}{\partial t^\nu} \qquad (3.34)$$
$$-\frac{\partial^\nu}{\partial t^\nu}\Big(\frac{\partial u(x,t)}{\partial x}\Big)-\frac{\partial^2 u(x,t)}{\partial x^2}=0.$$

The fractional complex transfomation $u(x,t)=U(\xi),\xi=kx+\omega\frac{t^\nu}{\Gamma(1+\nu)}$, transform the Eq. (3.34) to the following ordinary differential equation:

$$k^3\omega U^{(4)}-\left(k^2+k\omega\right)U''+U'U''\left(\alpha k^2\omega+\beta k^2\omega\right)=0. \qquad (3.35)$$

Integrating Eq. (3.35) with respect to ξ once, yields

$$k^3\omega U^{(3)}-\left(k^2+k\omega\right)U'+\frac{1}{2}\left(\alpha k^2\omega+\beta k^2\omega\right)\left(U'\right)^2=0. \qquad (3.36)$$

By the same procedure as illustrated below, the value of N can be determined by balancing $U^{(3)}$ and U'U" in Eq. (3.36). So $N=1$. Let's suppose that the solutions of Eq. (3.36) is of the form

$$U(\xi)=a_0+a_1\Big(\frac{G'}{G}\Big),\ a_1\neq 0,$$

(3.37)

where $G=G(\xi)$ satisfies (2.5). Substituting Eqs. (3.37) into Eq. (3.36), collecting the co-efficients of $\Big(\frac{G'}{G}\Big)^i,(i=0,1,2,\ldots)$ and set it to zero, yields a set of algebraic equations for a_0,a_1,k,ω. These systems are

$$\begin{aligned}
&-a_1\lambda^2k^3\mu\omega-2a_1k^3\mu^2\omega+\frac{1}{2}\alpha a_1^2k^2\mu^2\omega+\frac{1}{2}a_1^2\beta k^2\mu^2\omega+a_1k^2\mu+a_1k\omega\mu=0,\\
&a_1\left(-\lambda^3\right)k^3\omega-8a_1\lambda k^3\mu\omega+\alpha a_1^2\lambda k^2\mu\omega+a_1^2\beta\lambda k^2\mu\omega+a_1+k^2\lambda+a_1\lambda k\omega=0,\\
&-7a_1\lambda^2k^3\omega-8a_1k^3\mu\omega+\frac{1}{2}\alpha a_1^2\lambda^2k^2\omega+\alpha a_1^2k^2\mu\omega+\frac{1}{2}a_1^2\beta\lambda^2k^2\omega+a_1^2\beta k^2\mu\omega\\
&+a_1+k^2+a_1k\omega=0,\\
&-12a_1\lambda k^3\omega+\alpha a_1^2\lambda k^2\omega+a_1^2\beta\lambda k^2\omega=0,\\
&-6a_1k^3\omega+\frac{1}{2}\alpha a_1^2k^2\omega+\frac{1}{2}a_1^2\beta k^2\omega=0.
\end{aligned} \qquad (3.38)$$

The roots of Eqs. (3.38) are obtained by the aid of *Mathematica* as

$$\left\{a_0 = a_0, a_1 \to \frac{12k}{\alpha+\beta}, k = k, \omega \to \frac{k}{k^2\left(\lambda^2-4\mu\right)-1}\right\}, \tag{3.39}$$

where $\frac{k}{k^2\left(\lambda^2-4\mu\right)-1} \neq 0$ and $\alpha+\beta \neq 0$.

Now substituting (3.39) and using Eqs. (2.6) into (3.37), three kinds of traveling wave solutions of Eq. (3.34) are obtained as follows:

- **Case 1**: when $\lambda^2-4\mu>0$, then, hyperbolic function solutions are obtained:

$$u_1(x,t) = a_0 - \frac{6k\lambda}{\alpha+\beta} + \frac{6k\sqrt{\lambda^2-4\mu}}{\alpha+\beta}\left(\frac{A_1\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_2\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}{A_2\sinh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)+A_1\cosh\left(\frac{1}{2}\xi\sqrt{\lambda^2-4\mu}\right)}\right) \tag{3.40}$$

where $\xi = kx + \frac{k}{k^2\left(\lambda^2-4\mu\right)-1}\frac{t^{\nu}}{\Gamma(1+\nu)}$.

In particular, if $a_0 = 0, \lambda = -1, \mu = 0, k = 2, \alpha = -2, \beta = -1, \nu = 0.75, A_1 \neq 0, A_2 = 0$, then

$$u_1(x,t) = -4\tanh\left(\frac{2t^{0.75}}{3\Gamma(1.75)} + 2x\right) - 4. \tag{3.41}$$

- **Case 2**: when $\lambda^2-4\mu<0$, then, trigonometric function solutions are obtained:

$$u_2(x,t) = a_0 - \frac{6k\lambda}{\alpha+\beta} + \frac{6k\sqrt{4\mu-\lambda^2}}{\alpha+\beta}\left(\frac{A_2\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)-A_1\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}{A_2\sin\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)+A_1\cos\left(\frac{1}{2}\xi\sqrt{4\mu-\lambda^2}\right)}\right) \tag{3.42}$$

where $\xi = kx + \frac{k}{k^2\left(\lambda^2-4\mu\right)-1}\frac{t^{\nu}}{\Gamma(1+\nu)}$.

If $a_0 = 0, \lambda = 0, \mu = \frac{1}{4}, k = 2, \alpha = 2, \beta = 1, \nu = 0.5, A_1 \neq 0, A_2 = 0$, then

$$u_2(x,t) = -4\tan\left(\frac{2t^{0.5}}{3\Gamma(1.5)} + 2x\right). \tag{3.43}$$

- **Case 3**: when $\lambda^2 - 4\mu = 0$, then rational function solutions are obtained:

$$u_3(x,t) = a_0 - \frac{6k\lambda}{\alpha+\beta} + \frac{12k}{\alpha+\beta}\left(\frac{A_2}{A_1 + A_2\left(kx - k\,\frac{t^{\nu}}{\Gamma(1+\nu)}\right)}\right). \tag{3.44}$$

If $a_0 = 4, \lambda = 1, \mu = \frac{1}{4}, k = 1, \alpha = 2, \beta = 1, \nu = 0.25, A_1 = A_2 = 1$, then

$$u_3(x,t) = \frac{4}{\left(x - \frac{t^{0.25}}{\Gamma(1.25)}\right) + 1} + 2. \tag{3.45}$$

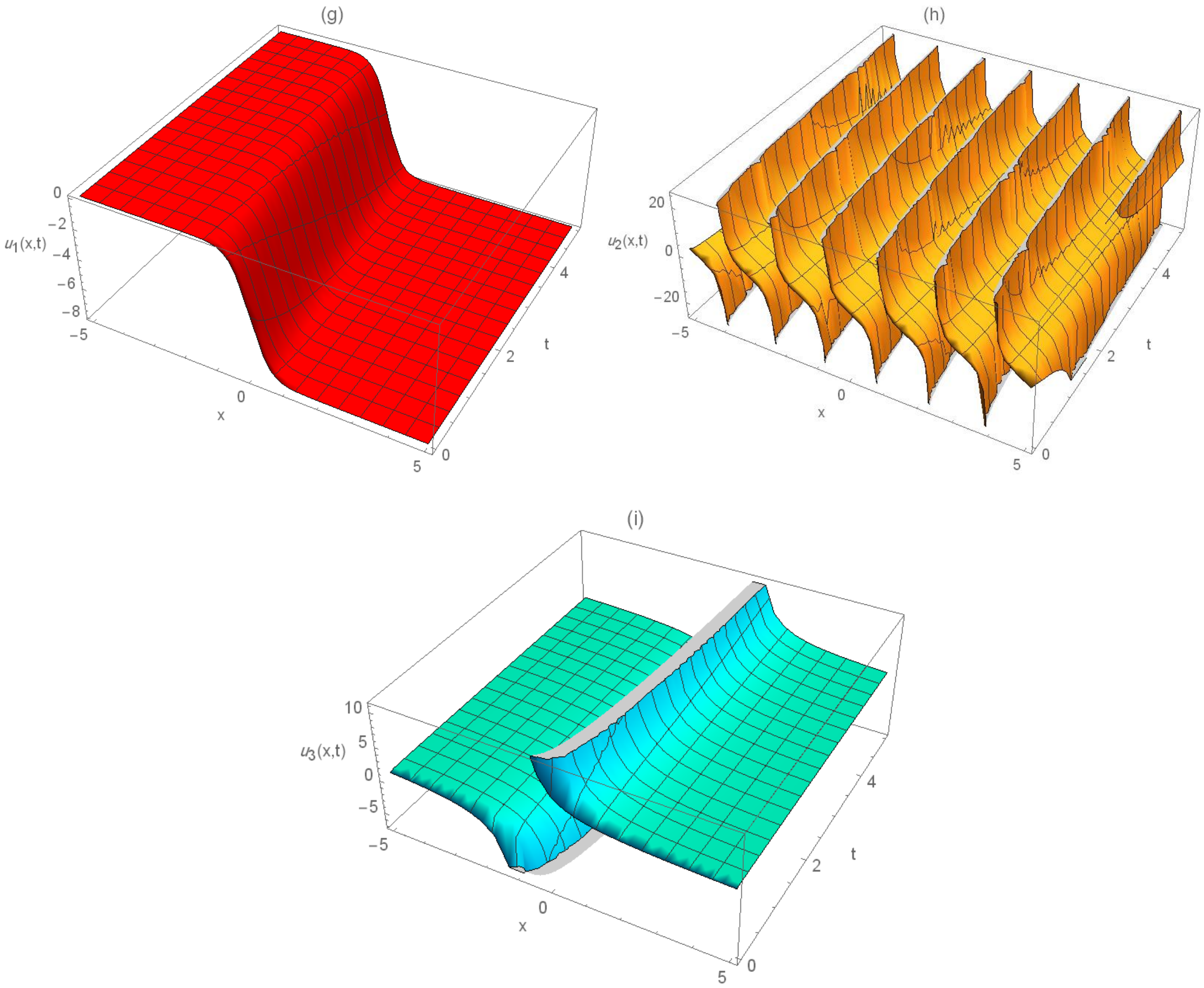

Figure 6. The 3D surfaces of the exact solutions of Eq. (3.34), (g) given by (3.41), (h). (3.43), and (i) (3.45) , for $(x,t) \in [-5,5] \times [0,5]$.

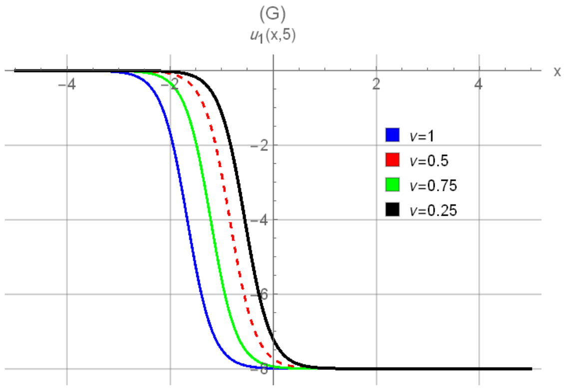

Figure 7. The 2D surfaces of the exact solutions of Eq. (3.34), (g) given by (3.41), for $x \in [-5,5], t = 5$ **and** $\nu = 1, 0.75, 0.5, 0.25$.

4. CONCLUSION

In this work the solutions of three important nonlinear time-fractional evolution equations, TFKKE, TFGBFE, and TFGSWW were found. Through using the $(\frac{G'}{G})$-expansion method, three types exact solutions (hyperbolic, trigonometric, and rational solutions) are derived . The availability of computer systems like Mathematica facilitates the tedious algebraic calculations and plots of surfaces of solutions. The proposed method in this paper is also a standard, direct and computerizable method, which allows us to do complicated and tedious algebraic calculation.

REFERENCES

[1] Alquran, M., Qawasmeh, A., *Journal of Applied Analysis and Computation*, **4**, 221, 2014.
[2] Bangqing, L., Meiping, X., Yulan, M., *Applied Mechanics and Materials***, 20-23,** 1516, 2010.
[3] Bekir, A., *Physics Letters A*, **372,** 3400, 2008.
[4] Borhanifar, A., Abazari, R., *Applied Mathematics*, **3**, 158, 2012.
[5] Malik, A., Chand, F., Kumar, H. Mishra, S.C., *PRAMANA -journal of physics*, **78**, 513, 2012.
[6] Naher, H., Abdullah, F.A., *Applied Mathematics*, **3**, 1144, 2012.
[7] Parkes, E.J., *Applied Mathematics and Computation*, **217**, 1759; 2010.
[8] Shahoot, A.M., Alurrfi, K.A.E., Elmrid, M.O.M., Almsir, A.M., Arwiniya, A.M.H., *Journal of Taibah University for Science*, **13**, 63, 2019.
[9] Shi, Y., Li, X., Zhang, B.G., *Advances in Mathematical Physics*, **2018**, 8583418, 2018.
[10] Wang, M., Li, X., Zhang, J., *Physics Letters A*, **372,** 417, 2008.
[11] Salas, A.H., *Advanced* Studies *in Theoretical Physics*, **6**, 879, 2012.
[12] Wazwaz, A.M., *Applied Mathematics and Computation*, **182**, 283, 2006.
[13] Fordy, A. Gibbons, J., *Physics Letters*, Volume **75A**, 325, 1980.
[14] Hereman, W., Nuseir, A., *Mathematics and Computers in Simulation*, **43**, 13, 1997.
[15] Wang, P., Xiao, S.H., *Physica Scripta*, **93**(10), 105201, 2018.

[16] Wang, P., *Modern Physics Letters B*, **31**, 1750057, 2017.
[17] Reyes, E.G., *Journal of Mathematical Physics*, **46**, 073507, 2005.
[18] Inc, M., *Applied Mathematics and Computation*, **172,** 72, 2006.
[19] Parker, A., *Physica D*, **137**, 34, 2000.
[20] Charalambous, K., Halder, A.K., Leach, G.L., *AIP Conference Proceedings **2153***, 020006, 2019.
[21] Hu, X.B., Wang, D.L., Qian, X. M., *Physics Letters A*, **262,** 409, 1999.
[22] Djilali, M., Hakem, A., Benali, A., *International Journal of Analysis and Applications,* **18,** 493, 2020.
[23] Fuchssteiner, B., Oevel, W., *Journal of Mathematical Physics*, **23**, 358, 1982.
[24] Fuchssteiner, B., Oevel, W., Wiwianka, W., *Computer Physics Communications*, **44**, 47, 1987.
[25] Behzadi, S. S., *Mathematical and Computational Applications*, **16,** 443, 2011.
[26] Chandraker, V.A., Awasthi, A., Jayaraj, S., *Procedia Technology,* **25,** 1217, 2016.
[27] Chen, B.K., Min, S.Q., Wang, B.H., *Communications in Theoretical Physics,* **52**, 443, 2010.
[28] Gupta, A.K., Ray, S.S., *International Journal of Mathematics and Mathematical Sciences*, **2014**, 682910, 2014.
[29] Lu, J., Yu-Cui, G., Shu-Jiang, X., *Chinese Physics,* **16**, 2514, 2007.
[30] Zhang, X.L., Wang, J., Zhang, H.Q., *Communications in Theoretical Physics*. **46,** 779, 2006.
[31] Wazwaz, A.M., *Applied Mathematics and Computation*, **169**, 321, 2005.
[32] Xinyi, W., *Chinese Physics. Letters,* **7**, 145, 1990.
[33] Wazwaz, A.M., *Applied Mathematics and Computation*, **195**, 754, 2008.
[34] Clarkson, P.A., Mansfield, E.L., *Nonlinearity,* **7,** 975, 1994.
[35] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H., *Studies in Applied Math*ematics, **53,** 249, 1974.
[36] Hirota, R., Satsuma, J., *Journal of Physical Society of Japan*. **40**, 611, 1976.
[37] Wazwaz, A.M., *Applied Mathematics and Computation,* **202,** 275, 2008.
[38] Elwakil, S.A., El-Labany, S.K., Zahran, M.A., Sabry, R., Chaos, *Solitons and Fractals*, **17** 121, 2003.
[39] Mansfield, E.L., Clarkson, P.A., *Mathematics and Computers in Simulation,* **43,** 39, 1997.
[40] Kilbas, A.A., Srivastava, H.M. Trujillo, J.J., *Theory and Applications of Fractional Differential Equations,* Elsevier B.V., Amsterdam, Netherlands, 2006.
[41] Oldham, K.B., Spanier, J., *The Fractional Calculus,* Academic Press, New York, 1974.
[42] Podlubny, I., *Fractional Differential Equations,* Academic Press, New York, 1999.