ORIGINAL PAPER

HERMITE POLYNOMIAL APPROACH FOR SOLVING SINGULAR PERTURBATED DELAY DIFFERENTIAL EQUATIONS

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Manuscript received: 05.04.2020; Accepted paper: 06.10.2020;
Published online: 30.12.2020.

Abstract. *In this study, a collocation approach based on the Hermite polyomials is applied to solve the singularly perturbated delay differential eqautions by boundary conditions. By means of the matix relations of the Hermite polynomials and the derivatives of them, main problem is reduced to a matrix equation. And then, collocation points are placed in equation of the matrix. Hence, the singular perturbed problem is transformed into an algebraic system of linear equations. This system is solved and thus the coefficients of the assumed approximate solution are determined. Numerical applications are made for various values of N.*

Keywords: *Boundary value problems; Singular perturbated delay differential equations; Hermite polynomials; collocation sechme; collocation points.*

1. INTRODUCTION

Singularly-perturbed delay differential equations contain a small parameter ε. These problems have many improtant in characterize of mathematical models in many fields of engineering and science. For example, fluid mechanics, plasma dynamics, hydrodynamics, elasticity, fluid dynamics, aero dynamics.

The obtaining of the exact solutions of these problems are difficult. Therefore, numerical techniques for the solutions of them are needed.

In last years, the mentioned these problems have been numerically studied by some researchers [1-18].

The aim of this paper is to find numerical solutions of the singularly-perturbed delay differential equation

$$L[y(t)] = \varepsilon y''(t) + s(t)y'(x-\delta) + f(t)y(t) = g(t), \quad 0 \le t \le b \tag{1}$$

under the boundary conditions

$$y(0) = \alpha \text{ and } y(b) = \beta \tag{2}$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$), δ is a small shifting parameter $0 < \delta \ll 1$, α and β are known constants, $y(t)$ is an unknown function, and $s(t)$ and $f(t)$ are the known functions on interval $0 \le t \le b < \infty$.

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In this paper, we present a numerical approach based on the Hermite polynomial approximation for solving the problem (1)-(2). After applying our method, we will gain the approximate solution in form of the truncated Hermite series as

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n H_n(t)\,,\ 0 \le t \le b < \infty\,. \tag{3}$$

In here, $H_n(t)$, $(n = 0,1,2,...)$ show the Hermite polynomials expressed by

$$H_n(t) = \sum_{k=0}^{\left[\left[\frac{n}{2}\right]\right]} \frac{(-1)^k (n)!}{k!(n-2k)!} (2t)^{n-2k}\ , \tag{4}$$

N is selected any positive integer, a_n, $n = 0,1,2,...,N$ shows the unknown coefficients.

The Hermite polynomials [20-22] has been used for solving pantograph type delay differential equations and integro-differential equations before. In this article, we introduce collocation method based on the Hermite polyomials for solving the singularly-perturbed delay differential equations. In defined of the method, we consider the collocation points defined by

$$t_i = \frac{b}{N-2} i\ ,\ i = 0,1,\ldots,N-2,\ \ 0 \le t \le b, \tag{5}$$

2. METHOD OF SOLUTIONS

2.1. MAIN MATRIX FORMS

Firstly, let us expressd the Hermite polynomials (4) in the matrix form as

$$\mathbf{H}(t) = \mathbf{X}(t)\mathbf{D}^T \tag{6}$$

where

$$\mathbf{H}(t) = \begin{bmatrix} H_0(t) & H_1(t) & \cdots & H_N(t) \end{bmatrix},\ \ \mathbf{T}(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix}$$

and in case odd values of N :

$$\mathbf{D}=\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{(\frac{N-5}{2})}\frac{2^0}{0!}\frac{(N-1)!}{(\frac{N-1}{2})!} & 0 & \cdots & 0 & 2^{N-1} \\ 0 & (-1)^{(\frac{N-1}{2})}\frac{2^1}{1!}\frac{N!}{(\frac{N-1}{2})!} & \cdots & 0 & 2^N \end{bmatrix},$$

in case even values of N :

$$\mathbf{D}=\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^{(\frac{N-2}{2})}\frac{2^1}{1!}\frac{(N-1)!}{(\frac{N-2}{2})!} & \cdots & 0 & 2^{N-1} \\ (-1)^{(\frac{N-4}{2})}\frac{2^0}{0!}\frac{N!}{(\frac{N}{2})!} & 0 & \cdots & 0 & 2^N \end{bmatrix}.$$

By using (6), the matrix form of derivative of the Hermite polynomials is obtained as

$$\mathbf{H}^{(1)}(t)=\mathbf{T}(t)\mathbf{B}\mathbf{D}^T \tag{7}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In here, $\mathbf{T}^{(1)}(t)=\mathbf{T}(t)\mathbf{B}$.

Similarly, second derivative form of the Hermite polynomials is found as

$$\mathbf{H}^{(2)}(t)=\mathbf{T}(t)\mathbf{B}^{2}\mathbf{D}^{T}. \tag{8}$$

Now, we write the solution form (3) in the following matrix form

$$y_N(t)=\mathbf{H}(t)\mathbf{A}$$

where

$$\mathbf{H}(t)=\left[H_0(t)\ \ H_1(t)\ \ ...\ \ H_N(t)\right] \text{ and } \mathbf{A}=\left[a_0\ \ a_1\ \ ...\ \ a_N\right]^T.$$

By pacing Eq. (6) into in here, we have the matrix form

$$y_N(t)=\mathbf{T}(t)\mathbf{D}^{T}\mathbf{A} \tag{9}$$

By using the matrix forms (7)-(8), first two derivatives of the solutions form (3) can be expressed in the matrix form, repectively, as

$$y_N^{(1)}(t)=\mathbf{T}(t)\mathbf{B}\mathbf{D}^{T}\mathbf{A} \tag{10}$$

and

$$y_N^{(2)}(t)=\mathbf{T}(t)\mathbf{B}^{2}\mathbf{D}^{T}\mathbf{A}. \tag{11}$$

When we write $t\to t-\delta$ in Eq. (10), we have

$$y_N^{(1)}(t-\delta)=\mathbf{T}(t-\delta)\mathbf{B}\mathbf{D}^{T}\mathbf{A}.$$

The relation between $\mathbf{T}(t-\delta)$ and $\mathbf{T}(t)$ is

$$\mathbf{T}(t-\delta)=\mathbf{T}(t)\mathbf{C}_{\delta},$$

where

$$\mathbf{C}_{\delta}=\begin{bmatrix} \binom{0}{0}(-\delta)^0 & \binom{1}{0}^0(-\delta)^1 & \cdots & \binom{N}{0}^0(-\delta)^N \\ 0 & \binom{1}{1}^1(-\delta)^0 & \cdots & \binom{N}{1}(-\delta)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{N}{N}(-\delta)^0 \end{bmatrix}.$$

Thus, Eq. (11) becomes as follows:

$$y_N^{(1)}(t-\delta)=\mathbf{T}(t)\mathbf{C}_{\delta}\mathbf{B}\mathbf{D}^{T}\mathbf{A}. \tag{12}$$

2.2. THE SOLUTION OF THE PROBLEM

By aid of the matrix forms in the previous Section and the collocation points, the problem will be reduced to a system of algebraic equations. For this aim, we put the matrix forms (9), (11) and (12) into Eq. (1) and so, we have the matrix equation

$$\varepsilon \mathbf{T}(t)\mathbf{B}^2\mathbf{D}^T\mathbf{A} + s(t)\mathbf{T}(t)\mathbf{C}_\delta \mathbf{B}\mathbf{D}^T\mathbf{A} + f(t)\mathbf{T}(t)\mathbf{D}^T\mathbf{A} = g(t)\,. \tag{13}$$

Now, we put the collocation points (5) into Eq. (13) and thus, we obtain the system

$$\varepsilon \mathbf{T}(t_i)\mathbf{B}^2\mathbf{D}^T\mathbf{A} + s(t_i)\mathbf{T}(t_i)\mathbf{C}_\delta \mathbf{B}\mathbf{D}^T\mathbf{A} + f(t_i)\mathbf{T}(t_i)\mathbf{D}^T\mathbf{A} = g(t_i)\,, \quad i = 0,1,\ldots,N-2\,.$$

We can briefly expres this system as

$$\left\{\left[\varepsilon \mathbf{T}\mathbf{B}^2 + \mathbf{S}\mathbf{X}\mathbf{C}_\delta \mathbf{B} + \mathbf{F}\mathbf{T}\right]\mathbf{D}^T\right\}\mathbf{A} = \mathbf{G}\,. \tag{14}$$

Here,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}_{(N-1)\times(N-1)}, \ \mathbf{S} = \begin{bmatrix} s(t_0) & 0 & 0 & 0 \\ 0 & s(t_1) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & s(t_{N-2}) \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} f(t_0) & 0 & 0 & 0 \\ 0 & f(t_1) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f(t_{N-2}) \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}(t_0) \\ \mathbf{T}(t_1) \\ \vdots \\ \mathbf{T}(t_{N-2}) \end{bmatrix}, \ \mathbf{T}(t_i) = \begin{bmatrix} 1 & t_i & t_i^2 & \cdots & t_i^N \end{bmatrix} \text{ for } i = 0,1,\ldots,N-2, \text{ and } \mathbf{G} = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_{N-2}) \end{bmatrix}.$$

This system is a system of $(N-1)$ algebraic equations and the $(N+1)$ unknown coefficients $a_0, a_1, \ldots, a_N$. Briefly, let us write the system (14) as

$$\mathbf{W}\mathbf{A} = \mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}] \tag{15}$$

where

$$\mathbf{W} = \left[w_{ij}\right] = \left\{\varepsilon \mathbf{T}\mathbf{B}^2 + \mathbf{S}\mathbf{T}\mathbf{C}_\delta \mathbf{B} + \mathbf{F}\mathbf{T}\right\}\mathbf{D}^T, \quad j = 0,1,\ldots,N \text{ and } i = 0,1,\ldots,N-2\,.$$

To obtain the matrix transforms of the conditions (2), we put $t \to 0$ and $t \to b$ in the relation (9) and then we have

$$y_N(0) = \mathbf{T}(0)\mathbf{D}^T\mathbf{A} = \alpha \text{ and } y_N(b) = \mathbf{T}(b)\mathbf{D}^T\mathbf{A} = \beta\,.$$

Hence, the mtrix forms of the conditions (2) become, respectively,

$$\mathbf{U}_0\mathbf{A} = \alpha \text{ and } \mathbf{U}_1\mathbf{A} = \beta$$

where

$$\mathbf{U}_0 = \mathbf{T}(0)\mathbf{D}^T = \begin{bmatrix} k_{1\,0} & k_{1\,1} & k_{1\,2} & \dots & k_{1\,N} \end{bmatrix}$$

and

$$\mathbf{U}_1 = \mathbf{T}(b)\mathbf{D}^T = \begin{bmatrix} k_{2\,0} & k_{2\,1} & k_{2\,2} & \dots & k_{2\,N} \end{bmatrix}.$$

Now, we can write the augmented matrix forms of the conditions (2) as

$$\left[\mathbf{U}_0;\alpha\right] \text{ and } \left[\mathbf{U}_1;\beta\right]. \tag{16}$$

From the equations (15) and (16), we gain the system

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}}. \tag{17}$$

This system corresponds to a system of $(N+1)$ linear algebraic equations with unknown coefficients $a_0, a_1, \dots, a_N$.

For simplicity, the two row matrices in (16) are added in the augmented matrix in (15), the augmented matrix (17) are obtained as follows:

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = \begin{bmatrix} w_{0\,0} & w_{0\,1} & w_{0\,2} & \dots & w_{0\,N} & ; & g(x_0) \\ w_{1\,0} & w_{1\,1} & w_{1\,2} & \dots & w_{1\,N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-2\,0} & w_{N-2\,1} & w_{N-2\,2} & \dots & w_{N-2\,N} & \vdots & g(x_{N-2}) \\ k_{1\,0} & k_{1\,1} & k_{1\,2} & \dots & k_{1\,N} & ; & \alpha \\ k_{2\,0} & k_{2\,1} & k_{2\,2} & \dots & k_{2\,N} & ; & \beta \end{bmatrix}. \tag{18}$$

If $\operatorname{rank} \tilde{\mathbf{W}} = \operatorname{rank}\,[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] = N+1$, then the unknown coefficient matrix is computed by

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}.$$

As a result, we substitute the determined coefficients $a_0, a_1, \dots, a_N$ into Eq. (3) and thus we have the Hermite approximate solution

$$y_{\varepsilon,\delta,N}(t) = \sum_{n=0}^{N} a_n H_n(t). \tag{19}$$

In here, we note that if $\operatorname{rank} \tilde{\mathbf{W}} = \operatorname{rank}\,[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] < N+1$, a particular solution may be found. Otherwise if $\operatorname{rank} \tilde{\mathbf{W}} \neq \operatorname{rank}\,[\tilde{\mathbf{W}};\tilde{\mathbf{G}}] < N+1$, then a solution is no available.

3. RESIDUAL FUNCTION TO ESTIMATE ERRORS

In this part, we use the residual function of the approximate solution to test the reliability of the method. When the approximate solution (19) is putted in Eq. (1), the residual function of it becomes as follows:

$$R_{\varepsilon,\delta,N}(t) = \varepsilon y^{(2)}_{\varepsilon,\delta,N}(t) + s(t) y^{(1)}_{\varepsilon,\delta,N}(t-\delta) + f(t) y_{\varepsilon,\delta,N}(t) - g(t)\,. \tag{20}$$

If this residual function approachs to zero as value of N increases, then it is said that the more accurate the solution is obtained as value of N increases.

Also, we will compute the errors by using the point-wise error function defined by

$$R_{\varepsilon,\delta,N}(x) = \left| y_{\varepsilon,\delta,N}(x) - y_{\varepsilon,\delta,2N}(x) \right|.$$

4. NUMERICAL APPLICATIONS

In this section, we make applications of our method on example. We calculated all numerical computations by means of a code written in Matlab.

Example [23] We apply our method to solve the singularly-perturbed delay differential equation

$$\varepsilon y''(t) + (1+t) y'(t-\delta) - e^{-t} y(t) = 1, \quad 0 \le t \le 1 \tag{21}$$

with the boundary conditions

$$y(0) = 0 \text{ and } y(1) = 1\,.$$

The fundamental matrix equation our problem is

$$\left\{ \left[\varepsilon \mathbf{T}\mathbf{B}^2 + \mathbf{P}\mathbf{T}\mathbf{C}_\delta \mathbf{B} + \mathbf{R}\mathbf{T} \right] \mathbf{D}^T \right\} \mathbf{A} = \mathbf{G}\,.$$

In Fig. 1 (a), (c) and (e), we give comparisons of our approximate solutions $y_{\varepsilon,\delta,N}(t)$ and $y_{\varepsilon,\delta,2N}(t)$ for $\delta = 2^{-2}$, $\varepsilon = 0.5$ and $(N,2N) = (3,6),(5,10),(7,14)$. Also, we compare residual errors $R_{\varepsilon,\delta,N}(t)$ and $R_{\varepsilon,\delta,2N}(t)$ for for $\delta = 2^{-2}$, $\varepsilon = 0.5$ and $(N,2N) = (3,6),(5,10),(7,14)$ in Fig. 1(b), (d) and (f). Simlarly, Fig. 2 (a), (b) show graphics of solution and residual error functions for $\delta = 2^{-2}$, $\varepsilon = 0.01$ and $(N,2N) = (7,14)$. Lastly, we display graphics of solution and residual error functions for $\delta = 2^{-2}$, $\varepsilon = 0.001$ and $(N,2N) = (7,14)$ in Fig. 3 (a), (b).

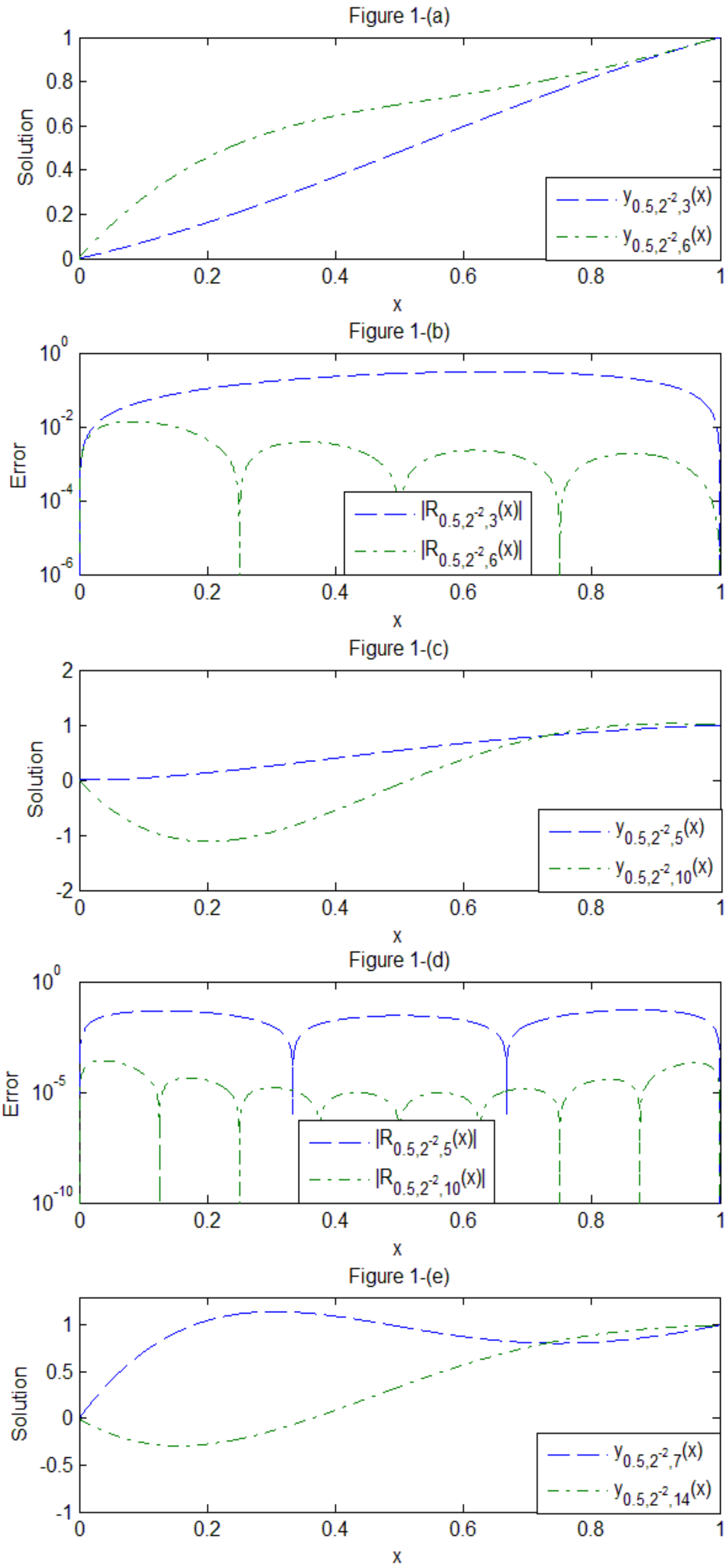
Figure 1-(a)
Solution
x
$y_{0.5,2^{-2},3}(x)$
$y_{0.5,2^{-2},6}(x)$
Figure 1-(b)
Error
x
$|R_{0.5,2^{-2},3}(x)|$
$|R_{0.5,2^{-2},6}(x)|$
Figure 1-(c)
Solution
x
$y_{0.5,2^{-2},5}(x)$
$y_{0.5,2^{-2},10}(x)$
Figure 1-(d)
Error
x
$|R_{0.5,2^{-2},5}(x)|$
$|R_{0.5,2^{-2},10}(x)|$
Figure 1-(e)
Solution
x
$y_{0.5,2^{-2},7}(x)$
$y_{0.5,2^{-2},14}(x)$

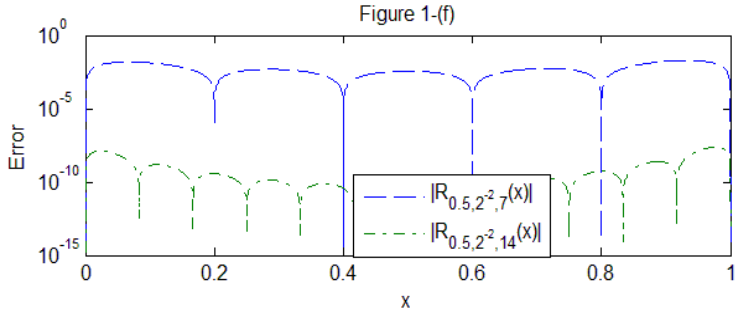

Figure 1. Comparision of the approximate solutions and the residual errors for $\delta = 2^{-2}$, $\varepsilon = 0.5$ and $(N,2N) = (3,6),(5,10),(7,14)$.

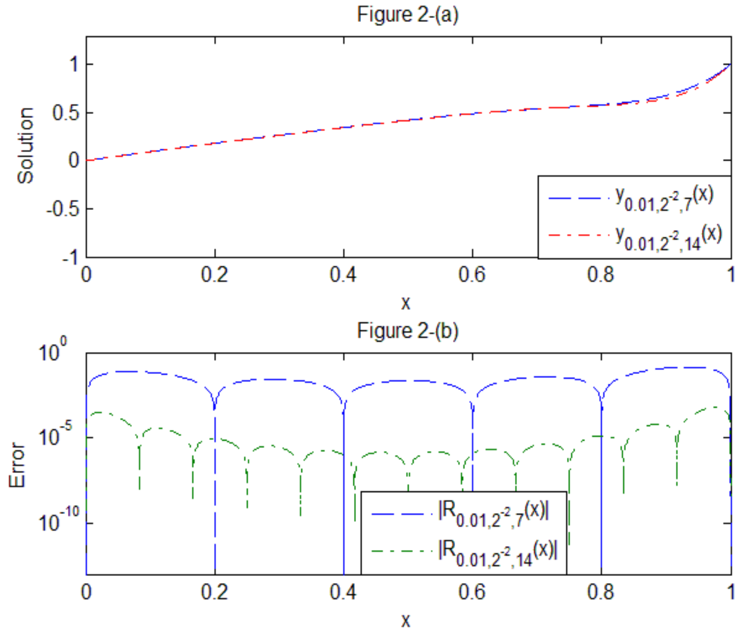

Figure 2. Comparision of the approximate solutions and the residual errors for $\delta = 2^{-2}$, $\varepsilon = 0.01$ and $(N,2N) = (7,14)$.

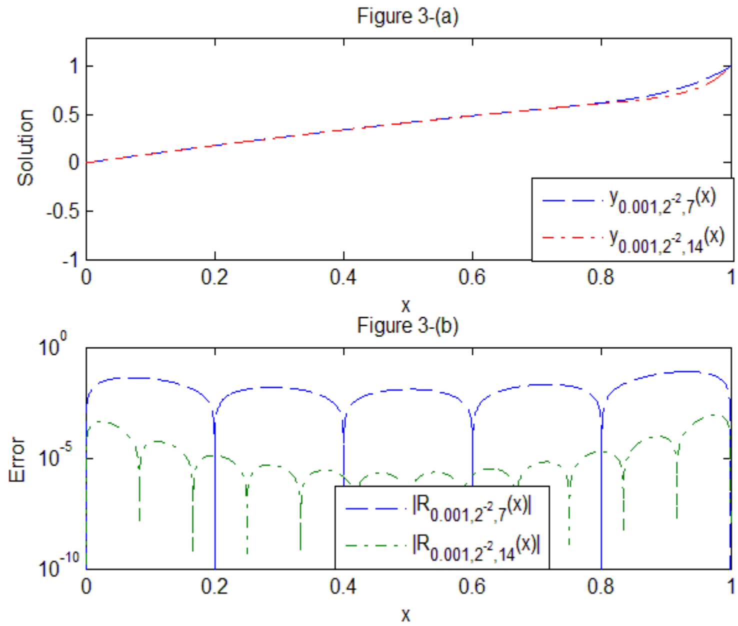

Figure 3. Comparision of the approximate solutions and the residual errors for $\delta = 2^{-2}$, $\varepsilon = 0.001$ and $(N,2N) = (7,14)$.

5. CONCLUSIONS

In this article, a Hermite approximation is developed for solving the singular-perturbated delay differential equations under the boundary conditions. Singular perturbed delay differential equations have not the exact solutons generally. In example, we applied the method to a problem which has not the exact solution. In Fig. 1, we computed the approximate solutions and residual errors $\delta = 2^{-2}$, $\varepsilon = 0.5$ and $(N, 2N) = (3,6),(5,10),(7,14)$.

Fig. 2 (a)-(b) show the approximate solution and residual error function for $\delta = 2^{-2}$, $\varepsilon = 0.01$ and $(N, 2N) = (7,14)$. Similarly, we plotted the graphics of solution and residual error functions for $\delta = 2^{-2}$, $\varepsilon = 0.001$ and $(N, 2N) = (7,14)$. It is seen from graphics that the errors low while N values increase. However, it is observed from figures that the errors increase as in other methods in literature while value of ε is decrease. In the future the method can be developed for singular partial differential equations.

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