ORIGINAL PAPER

MIN MATRICES WITH HYPER LUCAS NUMBERS

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Abstract. *In this paper, we examine Min matrix* $L=\left[L_{k+\min(i,j)-1}^{(r)}\right]_{i,j=1}^{n}$, *where* $L_n^{(r)}$ *denotes the nth hyper-Lucas number of order r. We mainly focus on characteristic polynomial of L. Also, we compute determinants, inverses of L and its Hadamard inverse. Moreover, we give a numerical example to illustrate our results.*
Keywords: *Max and Min matrix; Hyper-Lucas number; determinant; inverse.*

1. INTRODUCTION

There are many special matrices in mathematics. Two of them are Max and Min matrices. The elements of Max and Min matrices are defined by means of maximum and minimum concepts. The fundamental examples of Max and Min matrices are

$$A=\left[\min(i,j)\right]_{i,j=1}^{n}=\begin{bmatrix}1&1&1&\dots&1\\1&2&2&\dots&2\\1&2&3&\dots&3\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&2&3&\dots&n\end{bmatrix}\quad\text{and}\quad B=\left[\max(i,j)\right]_{i,j=1}^{n}=\begin{bmatrix}1&2&3&\dots&n\\2&2&3&\dots&n\\3&3&3&\dots&n\\\vdots&\vdots&\vdots&\ddots&\vdots\\n&n&n&\dots&n\end{bmatrix}.$$

There are interesting relationships Fibonacci numbers and Max and Min matrices [1, 2]. Also, determinants and inverses of these matrices can be computed easily [3-7]. For the Min and Max matrices

$$(T)_{\min}=\left[\min\left(z_i,z_j\right)\right]_{i,j=1}^{n}=\begin{bmatrix}z_1&z_1&z_1&\dots&z_1\\z_1&z_2&z_2&\dots&z_2\\z_1&z_2&z_3&\dots&z_3\\\vdots&\vdots&\vdots&\ddots&\vdots\\z_1&z_2&z_3&\dots&z_n\end{bmatrix}\text{ and }(T)_{\max}=\begin{bmatrix}z_1&z_2&z_3&\dots&z_n\\z_2&z_2&z_3&\dots&z_n\\z_3&z_3&z_3&\dots&z_n\\\vdots&\vdots&\vdots&\ddots&\vdots\\z_n&z_n&z_n&\dots&z_n\end{bmatrix},$$

by means of Mattila and Haukkanen [6], we know that for $z_n\neq 0$ the *(i,j)*th entry of inverse matrix $(T)_{\max}$ is

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$$\begin{cases} 0, & |i-j|>1, \\ \dfrac{1}{z_1-z_2}, & i=j=1, \\ \dfrac{1}{z_{i-1}-z_i}+\dfrac{1}{z_i-z_{i+1}}, & 1<i=j<n, \\ \dfrac{1}{|z_i-z_j|}, & |i-j|=1, \\ \dfrac{1}{z_{n-1}-z_n}+\dfrac{1}{z_n}, & i=j=n \end{cases} \tag{1}$$

and for $z_1 \neq 0$ the *(i,j)*th entry of inverse matrix $(T)_{\min}$ is

$$\begin{cases} 0, & |i-j|>1, \\ \dfrac{z_2}{z_1(z_2-z_1)}, & i=j=1, \\ \dfrac{1}{z_i-z_{i-1}}+\dfrac{1}{z_{i+1}-z_i}, & 1<i=j<n, \\ \dfrac{1}{|z_i-z_j|}, & |i-j|=1, \\ \dfrac{1}{z_n-z_{n-1}}, & i=j=n. \end{cases} \tag{2}$$

In this study, we mainly focus on characteristic polynomial of matrix

$$L=\left[L_{k+\min(i,j)-1}^{(r)}\right]_{i,j=1}^{n}=\begin{bmatrix} L_k^{(r)} & L_k^{(r)} & L_k^{(r)} & \cdots & L_k^{(r)} \\ L_k^{(r)} & L_{k+1}^{(r)} & L_{k+1}^{(r)} & \cdots & L_{k+1}^{(r)} \\ L_k^{(r)} & L_{k+1}^{(r)} & L_{k+2}^{(r)} & \cdots & L_{k+2}^{(r)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_k^{(r)} & L_{k+1}^{(r)} & L_{k+2}^{(r)} & \cdots & L_{k+n-1}^{(r)} \end{bmatrix}, \tag{3}$$

where $L_n^{(r)}$ denotes the *n*th hyper-Lucas number of order *r*. Moreover, we compute determinant, inverse and Hadamard inverse of this matrix.

The numbers 2, 1, 3, 4, 7, 11, 18, ... are called as Lucas numbers. The Lucas numbers have the recurrence relation $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$ [8]. There are many generalizations of Lucas numbers. One of them is hyper-Lucas numbers. *n*th hyper-Lucas number of order *r* is defined by the rule $L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)}$ under the conditions $L_n^{(0)} = L_n$,

$L_0^{(r)} = 2$ and $L_1^{(r)} = 2r+1$ [9]. Also, the sequence of hyper-Lucas numbers has the recurrence relation $L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}$ [9].

2. MAIN RESULTS

Theorem 2.1. The characteristic polynomial, $p(n,\lambda)$, of $n\times n$ matrix L in (3) holds

$$p(n,\lambda) = \left(2\lambda - L_{k+n-1}^{(r-1)}\right) p(n-1,\lambda) - \lambda^2 p(n-2,\lambda) \tag{4}$$

where $p(0,\lambda)=1$, $p(1,\lambda)=\lambda - L_k^{(r)}$ and $n\geq 2$.

Proof: Since $p(n,\lambda)=\det(\lambda I - L)$, we write

$$p(n,\lambda) = \det\begin{bmatrix} \lambda - L_k^{(r)} & -L_k^{(r)} & -L_k^{(r)} & \cdots & -L_k^{(r)} \\ -L_k^{(r)} & \lambda - L_{k+1}^{(r)} & -L_{k+1}^{(r)} & \cdots & -L_{k+1}^{(r)} \\ -L_k^{(r)} & -L_{k+1}^{(r)} & \lambda - L_{k+2}^{(r)} & \cdots & -L_{k+2}^{(r)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -L_k^{(r)} & -L_{k+1}^{(r)} & -L_{k+2}^{(r)} & \cdots & \lambda - L_{k+n-1}^{(r)} \end{bmatrix}.$$

By using elementary row and column operations, we have

$$p(n,\lambda) = \det\begin{bmatrix} \lambda - L_k^{(r)} & -L_k^{(r)} & -L_k^{(r)} & \cdots & -L_k^{(r)} & -L_k^{(r)} \\ -\lambda & \lambda - L_{k+1}^{(r-1)} & -L_{k+1}^{(r-1)} & \cdots & -L_{k+1}^{(r-1)} & -L_{k+1}^{(r-1)} \\ 0 & -\lambda & \lambda - L_{k+2}^{(r-1)} & \cdots & -L_{k+2}^{(r-1)} & -L_{k+2}^{(r-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda - L_{k+n-2}^{(r-1)} & -L_{k+n-2}^{(r-1)} \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda - L_{k+n-1}^{(r-1)} \end{bmatrix}$$

$$= \det\begin{bmatrix} \lambda - L_k^{(r)} & -\lambda & 0 & \cdots & 0 & 0 \\ -\lambda & 2\lambda - L_{k+1}^{(r-1)} & -\lambda & \cdots & 0 & 0 \\ 0 & -\lambda & 2\lambda - L_{k+2}^{(r-1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\lambda - L_{k+n-2}^{(r-1)} & -\lambda \\ 0 & 0 & 0 & \cdots & -\lambda & 2\lambda - L_{k+n-1}^{(r-1)} \end{bmatrix},$$

respectively. Then, by computating the above last determinant by the last row, we have

$$p(n,\lambda) = \left(2\lambda - L_{k+n-1}^{(r-1)}\right) p(n-1,\lambda) - \lambda^2 p(n-2,\lambda),$$

for $n \geq 2$, under the conditions $p(0,\lambda)=1$ and $p(1,\lambda)=\lambda - L_k^{(r)}$. □

Now, we have the sequence $p(0,\lambda)=1$, $p(1,\lambda)$, $p(1,\lambda)$, … , $p(n,\lambda)$.

Corollary 2.2. The determinant of $n \times n$ matrix L in (3) is

$$\det(L) = L_k^{(r)} \prod_{i=1}^{n-1} L_{k+i}^{(r-1)} . \tag{5}$$

Proof: Let $p(n,\lambda)$ be the characteristic polynomial of $n \times n$ matrix L. Then,

$$\det(L) = (-1)^n \, p(n,0).$$

By (4), we have

$$\begin{aligned} p(n,0) &= -L_{k+n-1}^{(r-1)} p(n-1,0) = -L_{k+n-1}^{(r-1)} \left[-L_{k+n-2}^{(r-1)} p(n-2,0) \right] \\ &= \cdots = (-1)^n L_{k+n-1}^{(r-1)} L_{k+n-2}^{(r-1)} L_{k+n-3}^{(r-1)} \cdots L_{k+1}^{(r-1)} L_k^{(r)} . \end{aligned}$$

Thus,

$$\det(L) = L_k^{(r)} \prod_{i=1}^{n-1} L_{k+i}^{(r-1)} .$$ □

Corollary 2.3. If $\lambda = 0$, then $p(i,\lambda)$ does not vanish for $1 \leq i \leq n$.

Proof: The relation (4) yields

$$p(i,0) = -L_{k+i-1}^{(r-1)} p(i-1,0) = \cdots = (-1)^i L_{k+i-1}^{(r-1)} L_{k+i-2}^{(r-1)} L_{k+i-3}^{(r-1)} \cdots L_{k+1}^{(r-1)} L_k^{(r)} .$$

Since $L_k^{(r)} \neq 0$ and $L_s^{(r-1)} \neq 0$ for $s \geq 0$, $p(i,0) \neq 0$ for $1 \leq i \leq n$. □

Corollary 2.4. Two consecutive terms $p(i,\lambda)$, $p(i+1,\lambda)$ can not have a common zero, for $1 \leq i \leq n-1$.

Proof: If $p(i+1,\lambda_0) = p(i,\lambda_0) = 0$ for $1 \leq i \leq n-1$, then from Corollary 2.3 $\lambda_0 \neq 0$ and the equation (4) holds

$$p(i-1,\lambda_0) = p(i-2,\lambda_0) = \cdots = p(0,\lambda_0) = 0 .$$

But it is impossible since $p(0,\lambda_0) = 1$ and $\lambda_0 \neq 0$. □

Theorem 2.5. The inverse of $n\times n$ matrix L in (3) is

$$L^{-1}=\begin{bmatrix} \frac{1}{L_k^{(r)}}+\frac{1}{L_{k+1}^{(r-1)}} & -\frac{1}{L_{k+1}^{(r-1)}} & 0 & \cdots & 0 & 0 \\ -\frac{1}{L_{k+1}^{(r-1)}} & \frac{1}{L_{k+1}^{(r-1)}}+\frac{1}{L_{k+2}^{(r-1)}} & -\frac{1}{L_{k+2}^{(r-1)}} & \cdots & 0 & 0 \\ 0 & -\frac{1}{L_{k+2}^{(r-1)}} & \frac{1}{L_{k+2}^{(r-1)}}+\frac{1}{L_{k+3}^{(r-1)}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_{k+n-2}^{(r-1)}}+\frac{1}{L_{k+n-1}^{(r-1)}} & -\frac{1}{L_{k+n-1}^{(r-1)}} \\ 0 & 0 & 0 & \cdots & -\frac{1}{L_{k+n-1}^{(r-1)}} & \frac{1}{L_{k+n-1}^{(r-1)}} \end{bmatrix}.$$

Proof: For $z_i=L_{k+i-1}^{(r)}$, the equation (2) and the relation $L_n^{(r)}=L_{n-1}^{(r)}+L_n^{(r-1)}$ immediately yield desired result. □

Theorem 2.6. The determinant of Hadamard inverse of $n\times n$ matrix L in (3) is

$$\det\left(L^{\circ-1}\right)=\frac{(-1)^{n-1}}{L_k^{(r)}}\prod_{i=1}^{n-1}\frac{L_{k+i}^{(r-1)}}{L_{k+i}^{(r)}L_{k+i-1}^{(r)}}.$$

Proof: By using elementary row operations on

$$\det\left(L^{\circ-1}\right)=\det\begin{bmatrix} \frac{1}{L_k^{(r)}} & \frac{1}{L_k^{(r)}} & \frac{1}{L_k^{(r)}} & \cdots & \frac{1}{L_k^{(r)}} \\ \frac{1}{L_k^{(r)}} & \frac{1}{L_{k+1}^{(r)}} & \frac{1}{L_{k+1}^{(r)}} & \cdots & \frac{1}{L_{k+1}^{(r)}} \\ \frac{1}{L_k^{(r)}} & \frac{1}{L_{k+1}^{(r)}} & \frac{1}{L_{k+2}^{(r)}} & \cdots & \frac{1}{L_{k+2}^{(r)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_k^{(r)}} & \frac{1}{L_{k+1}^{(r)}} & \frac{1}{L_{k+2}^{(r)}} & \cdots & \frac{1}{L_{k+n-1}^{(r)}} \end{bmatrix},$$

we have upper triangular form:

$$\det\left(L^{\circ -1}\right)=\det\begin{bmatrix} \frac{1}{L_k^{(r)}} & \frac{1}{L_k^{(r)}} & \frac{1}{L_k^{(r)}} & \cdots & \frac{1}{L_k^{(r)}} \\ 0 & \frac{1}{L_{k+1}^{(r)}}-\frac{1}{L_k^{(r)}} & \frac{1}{L_{k+1}^{(r)}}-\frac{1}{L_k^{(r)}} & \cdots & \frac{1}{L_{k+1}^{(r)}}-\frac{1}{L_k^{(r)}} \\ 0 & 0 & \frac{1}{L_{k+2}^{(r)}}-\frac{1}{L_{k+1}^{(r)}} & \cdots & \frac{1}{L_{k+2}^{(r)}}-\frac{1}{L_{k+1}^{(r)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_{k+n-1}^{(r)}}-\frac{1}{L_{k+n-2}^{(r)}} \end{bmatrix}.$$

Thus,

$$\begin{aligned}\det\left(L^{\circ -1}\right)&=\frac{1}{L_k^{(r)}}\left(\frac{1}{L_{k+1}^{(r)}}-\frac{1}{L_k^{(r)}}\right)\left(\frac{1}{L_{k+2}^{(r)}}-\frac{1}{L_{k+1}^{(r)}}\right)\cdots\left(\frac{1}{L_{k+n-1}^{(r)}}-\frac{1}{L_{k+n-2}^{(r)}}\right)\\ &=\frac{1}{L_k^{(r)}}\frac{L_k^{(r)}-L_{k+1}^{(r)}}{L_{k+1}^{(r)}L_k^{(r)}}\frac{L_{k+1}^{(r)}-L_{k+2}^{(r)}}{L_{k+2}^{(r)}L_{k+1}^{(r)}}\cdots\frac{L_{k+n-2}^{(r)}-L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r)}L_{k+n-2}^{(r)}}\\ &=\frac{1}{L_k^{(r)}}\frac{-L_{k+1}^{(r-1)}}{L_{k+1}^{(r)}L_k^{(r)}}\frac{-L_{k+2}^{(r-1)}}{L_{k+2}^{(r)}L_{k+1}^{(r)}}\cdots\frac{-L_{k+n-1}^{(r-1)}}{L_{k+n-1}^{(r)}L_{k+n-2}^{(r)}}\\ &=\frac{(-1)^{n-1}}{L_k^{(r)}}\prod_{i=1}^{n-1}\frac{L_{k+i}^{(r-1)}}{L_{k+i}^{(r)}L_{k+i-1}^{(r)}}.\end{aligned}$$

□

Theorem 2.7. The inverse of Hadamard inverse of $n\times n$ matrix L in (3) is

$$\left(L^{\circ -1}\right)^{-1}=\begin{bmatrix} L_k^{(r)}-\frac{L_k^{(r)}L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} & \frac{L_k^{(r)}L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} & 0 & \cdots & 0 & 0 \\ \frac{L_k^{(r)}L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} & -\frac{L_k^{(r)}L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}}-\frac{L_{k+1}^{(r)}L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} & \frac{L_{k+1}^{(r)}L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} & \cdots & 0 & 0 \\ 0 & \frac{L_{k+1}^{(r)}L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} & -\frac{L_{k+1}^{(r)}L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}}-\frac{L_{k+2}^{(r)}L_{k+3}^{(r)}}{L_{k+3}^{(r-1)}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{L_{k+n-3}^{(r)}L_{k+n-2}^{(r)}}{L_{k+n-2}^{(r-1)}}-\frac{L_{k+n-2}^{(r)}L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} & \frac{L_{k+n-2}^{(r)}L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} \\ 0 & 0 & 0 & \cdots & \frac{L_{k+n-2}^{(r)}L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} & -\frac{L_{k+n-2}^{(r)}L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} \end{bmatrix}.$$

Proof: Consider the matrix

$$L_1 = \begin{bmatrix} \frac{1}{L_{k+n-1}^{(r)}} & \frac{1}{L_{k+n-2}^{(r)}} & \cdots & \frac{1}{L_{k}^{(r)}} \\ \frac{1}{L_{k+n-2}^{(r)}} & \frac{1}{L_{k+n-2}^{(r)}} & \cdots & \frac{1}{L_{k}^{(r)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{k}^{(r)}} & \frac{1}{L_{k}^{(r)}} & \cdots & \frac{1}{L_{k}^{(r)}} \end{bmatrix}.$$

Then

$$L^{\circ-1} = \begin{bmatrix} \frac{1}{L_{k}^{(r)}} & \frac{1}{L_{k}^{(r)}} & \cdots & \frac{1}{L_{k}^{(r)}} \\ \frac{1}{L_{k}^{(r)}} & \frac{1}{L_{k+1}^{(r)}} & \cdots & \frac{1}{L_{k+1}^{(r)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{k}^{(r)}} & \frac{1}{L_{k+1}^{(r)}} & \cdots & \frac{1}{L_{k+n-1}^{(r)}} \end{bmatrix} = I_1 L_1 I_1$$

where

$$I_1 = I_1^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix L_1 has the Max matrix form. By considering Equation (1) for inverse of L_1, we have

$$\left(L^{\circ-1}\right)^{-1} = I_1^{-1} L_1^{-1} I_1^{-1} = I_1 L_1^{-1} I_1$$

$$= \begin{bmatrix} L_k^{(r)} - \frac{L_k^{(r)} L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} & \frac{L_k^{(r)} L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} & 0 & \cdots & 0 & 0 \\ \frac{L_k^{(r)} L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} & -\frac{L_k^{(r)} L_{k+1}^{(r)}}{L_{k+1}^{(r-1)}} - \frac{L_{k+1}^{(r)} L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} & \frac{L_{k+1}^{(r)} L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} & \cdots & 0 & 0 \\ 0 & \frac{L_{k+1}^{(r)} L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} & -\frac{L_{k+1}^{(r)} L_{k+2}^{(r)}}{L_{k+2}^{(r-1)}} - \frac{L_{k+2}^{(r)} L_{k+3}^{(r)}}{L_{k+3}^{(r-1)}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{L_{k+n-3}^{(r)} L_{k+n-2}^{(r)}}{L_{k+n-2}^{(r-1)}} - \frac{L_{k+n-2}^{(r)} L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} & \frac{L_{k+n-2}^{(r)} L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} \\ 0 & 0 & 0 & \cdots & \frac{L_{k+n-2}^{(r)} L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} & -\frac{L_{k+n-2}^{(r)} L_{k+n-1}^{(r)}}{L_{k+n-1}^{(r-1)}} \end{bmatrix}.$$

□

Now, we give a table related to hyper-Lucas numbers and an example to illustrate our results.

Table 1. Some hyper-Lucas numbers

$L_n^{(r)}$ \ r	0	1	2	3	4	5	6	7	8
$L_0^{(r)}$	2	2	2	2	2	2	2	2	2
$L_1^{(r)}$	1	3	5	7	9	11	13	15	17
$L_2^{(r)}$	3	6	11	18	27	38	51	66	83
$L_3^{(r)}$	4	10	21	39	66	104	155	221	304
$L_4^{(r)}$	7	17	38	77	143	247	402	623	927
$L_5^{(r)}$	11	28	66	143	286	533	935	1558	2485
$L_6^{(r)}$	18	46	112	255	541	1074	2009	3567	6052
$L_7^{(r)}$	29	75	187	442	983	2057	4066	7633	13685
$L_8^{(r)}$	47	122	309	751	1734	3791	7857	15490	29175

Example 2.8. For $n=5,\ r=2,\ k=3$, the matrix L is

$$L=\begin{bmatrix} 21 & 21 & 21 & 21 & 21 \\ 21 & 38 & 38 & 38 & 38 \\ 21 & 38 & 66 & 66 & 66 \\ 21 & 38 & 66 & 112 & 112 \\ 21 & 38 & 66 & 112 & 187 \end{bmatrix}.$$

Then

$$\det(L)=L_3^{(2)}\prod_{i=1}^{4}L_{3+i}^{(1)}=L_3^{(2)}L_4^{(1)}L_5^{(1)}L_6^{(1)}L_7^{(1)}=21.17.28.46.75$$

$$=34486200,$$

$$L^{-1}=\begin{bmatrix} \frac{1}{L_3^{(2)}}+\frac{1}{L_4^{(2)}} & -\frac{1}{L_4^{(1)}} & 0 & 0 & 0 \\ -\frac{1}{L_4^{(1)}} & \frac{1}{L_4^{(1)}}+\frac{1}{L_5^{(1)}} & -\frac{1}{L_5^{(1)}} & 0 & 0 \\ 0 & -\frac{1}{L_5^{(1)}} & \frac{1}{L_5^{(1)}}+\frac{1}{L_6^{(1)}} & -\frac{1}{L_6^{(1)}} & 0 \\ 0 & 0 & -\frac{1}{L_6^{(1)}} & \frac{1}{L_6^{(1)}}+\frac{1}{L_7^{(1)}} & -\frac{1}{L_7^{(1)}} \\ 0 & 0 & 0 & -\frac{1}{L_7^{(1)}} & \frac{1}{L_7^{(1)}} \end{bmatrix}=\begin{bmatrix} \frac{38}{357} & -\frac{1}{17} & 0 & 0 & 0 \\ -\frac{1}{17} & \frac{45}{476} & -\frac{1}{28} & 0 & 0 \\ 0 & -\frac{1}{28} & \frac{74}{1288} & -\frac{1}{46} & 0 \\ 0 & 0 & -\frac{1}{46} & \frac{121}{3450} & -\frac{1}{75} \\ 0 & 0 & 0 & -\frac{1}{75} & \frac{1}{75} \end{bmatrix},$$

$$\det\left(L^{\circ-1}\right)=\frac{(-1)^4}{L_3^{(2)}}\prod_{i=1}^{4}\frac{L_{3+i}^{(1)}}{L_{3+i}^{(2)}L_{2+i}^{(2)}}=\frac{\left(-L_4^{(1)}\right)\left(-L_5^{(1)}\right)\left(-L_6^{(1)}\right)\left(-L_7^{(1)}\right)}{\left(L_3^{(2)}\right)^2\left(L_4^{(2)}\right)^2\left(L_5^{(2)}\right)^2\left(L_6^{(2)}\right)^2 L_7^{(2)}}$$

$$=\frac{17.28.46.75}{(21)^2(38)^2.(66)^2.(112)^2 187}=\frac{9775}{38731295522304},$$

$$\left(L^{\circ-1}\right)^{-1}=\begin{bmatrix} L_3^{(2)}-\frac{L_3^{(2)}L_4^{(2)}}{L_4^{(1)}} & \frac{L_3^{(2)}L_4^{(2)}}{L_4^{(1)}} & 0 & 0 & 0\\ \frac{L_3^{(2)}L_4^{(2)}}{L_4^{(1)}} & -\frac{L_3^{(2)}L_4^{(2)}}{L_4^{(1)}}-\frac{L_4^{(2)}L_5^{(2)}}{L_5^{(1)}} & \frac{L_4^{(2)}L_5^{(2)}}{L_5^{(1)}} & 0 & 0\\ 0 & \frac{L_4^{(2)}L_5^{(2)}}{L_5^{(1)}} & -\frac{L_4^{(2)}L_5^{(2)}}{L_5^{(1)}}-\frac{L_5^{(2)}L_6^{(2)}}{L_6^{(1)}} & \frac{L_5^{(2)}L_6^{(2)}}{L_6^{(1)}} & 0\\ 0 & 0 & \frac{L_5^{(2)}L_6^{(2)}}{L_6^{(1)}} & -\frac{L_5^{(2)}L_6^{(2)}}{L_6^{(1)}}-\frac{L_6^{(2)}L_7^{(2)}}{L_7^{(1)}} & \frac{L_6^{(2)}L_7^{(2)}}{L_7^{(1)}}\\ 0 & 0 & 0 & \frac{L_6^{(2)}L_7^{(2)}}{L_7^{(1)}} & -\frac{L_6^{(2)}L_7^{(2)}}{L_7^{(1)}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{-441}{17} & \frac{798}{17} & 0 & 0 & 0\\ \frac{798}{17} & -\frac{64980}{476} & \frac{2508}{28} & 0 & 0\\ 0 & \frac{2508}{28} & -\frac{322344}{1288} & \frac{7392}{46} & 0\\ 0 & 0 & \frac{7392}{46} & -\frac{1517824}{3450} & \frac{20944}{75}\\ 0 & 0 & 0 & \frac{20944}{75} & -\frac{20944}{75} \end{bmatrix}.$$

Also,

$$p(0,\lambda)=1,\quad p(1,\lambda)=|\lambda-21|=\lambda-21,$$

$$p(2,\lambda)=\begin{vmatrix}\lambda-21 & -21\\ -21 & \lambda-38\end{vmatrix}=\lambda^2-59\lambda+357,$$

$$p(3,\lambda)=\begin{vmatrix}\lambda-21 & -21 & -21\\ -21 & \lambda-38 & -38\\ -21 & -38 & \lambda-66\end{vmatrix}=\lambda^3-125\lambda^2+2366\lambda-9996,$$

$$p(4,\lambda)=\begin{vmatrix}\lambda-21 & -21 & -21 & -21\\ -21 & \lambda-38 & -38 & -38\\ -21 & -38 & \lambda-66 & -66\\ -21 & -38 & -66 & \lambda-112\end{vmatrix}=\lambda^4-237\lambda^3+10125\lambda^2-128828\lambda+459816,$$

$$p(5,\lambda)=\begin{vmatrix} \lambda-21 & -21 & -21 & -21 & -21 \\ -21 & \lambda-38 & -38 & -38 & -38 \\ -21 & -38 & \lambda-66 & -66 & -66 \\ -21 & -38 & -66 & \lambda-112 & -112 \\ -21 & -38 & -66 & -112 & \lambda-187 \end{vmatrix}$$
$$=\lambda^5-424\lambda^4+35659\lambda^3-1007035\lambda^2+10581732\lambda-34486200.$$

Thus,

$$p(1,\lambda)=\lambda-21,\quad p(2,\lambda)=(2\lambda-17)(\lambda-21)-\lambda^2=\lambda^2-59\lambda+357,$$
$$p(3,\lambda)=(2\lambda-28)\left(\lambda^2-59\lambda+357\right)-\lambda^2(\lambda-21)=\lambda^3-125\lambda^2+2366\lambda-9996,$$
$$p(4,\lambda)=(2\lambda-46)\left(\lambda^3-125\lambda^2+2366\lambda-9996\right)-\lambda^2\left(\lambda^2-59\lambda+357\right)$$
$$=\lambda^4-237\lambda^3+10125\lambda^2-128828\lambda+459816,$$
$$p(5,\lambda)=(2\lambda-75)\left(\lambda^4-237\lambda^3+10125\lambda^2-128828\lambda+459816\right)$$
$$-\lambda^2\left(\lambda^3-125\lambda^2+2366\lambda-9996\right)$$
$$=\lambda^5-424\lambda^4+35659\lambda^3-1007035\lambda^2+10581732\lambda-34486200.$$

3. CONCLUSION

In the present paper, we compute characteristic polynomial, determinant and inverse of the Min matrix $L=\left[L^{(r)}_{k+\min(i,j)-1}\right]^{n}_{i,j=1}$, where $L^{(r)}_n$ denotes the nth hyper-Lucas number of order r. We show that the characteristic polynomial of L has a recurrence relation and we obtain determinant of L by using that recurrence relation. Also, we compute determinant, inverse of Hadamard inverse of L.

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