ORIGINAL PAPER

A STUDY ON THE HARMONIC EVOLUTE SURFACES OF QUASI BINORMAL SURFACES

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Abstract.*In this paper, harmonic evolute surface of quasi binormal surface associated with quasi frame is studied. After constructing quasi binormal surface, we determine harmonic evolute surface of quasi binormal surface by using first and second fundamental forms. We then obtain some new results about these new surfaces and give some applications for them.*

Keywords: *Binormal spherical image; Bonnet Surfaces; curvatures; fundamental forms.*

1. INTRODUCTION

The ruled surface generated by the motion of a straight line along a curve is one of the most atractive surfaces to study since the application of surfaces to physics and engineering is countless and these surfaces are the easiest of all surfaces to parametrize. After it was initially discovered by Gaspard Mongea, many researchers have studied ruled surfaces [1-4]. Besides ruled surfaces, harmonic evolute surfaces of various surfaces have been also studied [5-8]. In addition to these studies, we work on the harmonic evolute surfaces of the ruled surfaces generated by quasi binormal vector [9].

This paper is consisting of three sections. After giving basic notation about Frenet frame and quasi frame in 3-dimensional Euclidean space, we work on the ruled surfaces generated by quasi binormal vector. Then we calculated mean curvature of this quasi binormal surface by using first and second fundamental forms in the second section. In the following section, we define harmonic evolute surface of quasi binormal surface associated with quasi frame. We finally construct the qausi binormal surface and the harmonic evolute surface of this quasi binormal surface.

2. MATERIALS AND METHODS

By way of design and style, this model is kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Frenet-Serret frame is established by way of subsequent equations for a presented framework [10],

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$$\begin{bmatrix} \nabla_{\mathbf{t}}\mathbf{t} \\ \nabla_{\mathbf{t}}\mathbf{n} \\ \nabla_{\mathbf{t}}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{1.1}$$

where $\kappa = \|\mathbf{t}\|$ and τ are the curvature and torsion of γ, respectively.

After Bishop in 1975 showed that there is more than one way to frame a curve in [11], Yilmaz and Turgut introduced second type of Bishop frame in [12]. Besides them, Dede et. al. defined quasi frame in [13]. The quasi frame of a regular curve γ is given by

$$\mathbf{t_q} = \mathbf{t}, \mathbf{n_q} = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b_q} = \mathbf{t_q} \wedge \mathbf{n_q}, \tag{1.2}$$

where $\mathbf{k}$ is the projection vector.

For simplicity, we have chosen the projection vector $\mathbf{k} = (0,0,1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel, the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0,1,0)$ or $\mathbf{k} = (1,0,0)$.

If the angle between the quasi normal vector $\mathbf{n_q}$ and the normal vector $\mathbf{n}$ is chosen as ψ, then the following relation is obtained between the quasi and FS frame.

$$\begin{aligned} \mathbf{t_q} &= \mathbf{t}, \\ \mathbf{n_q} &= \cos\psi\mathbf{n} + \sin\psi\mathbf{b}, \\ \mathbf{b_q} &= -\sin\psi\mathbf{n} + \cos\psi\mathbf{b}. \end{aligned} \tag{1.3}$$

Therefore, by using the above equations the variation of parallel adapted quasi frame is obtained by

$$\begin{bmatrix} \nabla_{\mathbf{t_q}}\mathbf{t_q} \\ \nabla_{\mathbf{t_q}}\mathbf{n_q} \\ \nabla_{\mathbf{t_q}}\mathbf{b_q} \end{bmatrix} = \begin{bmatrix} 0 & \grave{u}_1 & \grave{u}_2 \\ -\grave{u}_1 & 0 & \grave{u}_3 \\ -\grave{u}_2 & -\grave{u}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t_q} \\ \mathbf{n_q} \\ \mathbf{b_q} \end{bmatrix},$$

where

$$\grave{u}_1 = \left\langle \nabla_{\mathbf{t_q}}\mathbf{t_q}, \mathbf{n_q} \right\rangle, \quad \grave{u}_2 = \left\langle \nabla_{\mathbf{t_q}}\mathbf{t_q}, \mathbf{b_q} \right\rangle, \quad \grave{u}_3 = \left\langle \nabla_{\mathbf{t_q}}\mathbf{n_q}, \mathbf{b_q} \right\rangle,$$

and the vector products of the quasi vectors are given by

$$\mathbf{t_q} \times \mathbf{n_q} = \mathbf{b_q}, \mathbf{n_q} \times \mathbf{b_q} = \mathbf{t_q}, \mathbf{b_q} \times \mathbf{t_q} = \mathbf{n_q}.$$

Let n be the standard unit normal vector field on a surface ϕ defined by

$$n = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|},$$

where $\phi_s = \partial\phi/\partial s, \phi_t = \partial\phi/\partial t$. Then, the first fundamental form I and the second fundamental form II of a surface ϕ are defined by

$$I = Eds^2 + 2Fdsdt + Gdt^2,$$
$$II = eds^2 + 2fdsdt + gdt^2,$$

where

$$E = \langle \phi_s, \phi_s \rangle, F = \langle \phi_s, \phi_t \rangle, G = \langle \phi_t, \phi_t \rangle,$$
$$e = \langle \phi_{ss}, n \rangle, f = \langle \phi_{st}, n \rangle, g = \langle \phi_{tt}, n \rangle$$

respectively, [14-15].

The mean curvature H is given by

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}.$$

Theorem 2.1. The surface is minimal if and only if it has vanishing mean curvature [10-14].

3. HARMONIC EVOLUTE SURFACES OF QUASI BINORMAL SURFACES

In this section, we aim to explore harmonic evolute surface of quasi binormal surface associated with quasi frame when the mean curvature does not vanish.

Firstly, we construct quasi binormal surface of a quasi curve as

$$\phi^{\mathbf{b_q}}(s,t) = \alpha + t\mathbf{b_q}.$$

Definition 3.1. If $E=G$, $F=0$, $f=c\neq 0$ $(c = \text{const.})$ *are satisfied then the surface is called A-net on a surface, [10].*

Theorem 3.2. *A surface is a Bonnet surface if and only if surface has an A-net, [10].*

Theorem 3.3. *Let* $\phi^{\mathbf{b_q}}$ *is a quasi binormal surface of a quasi curve in space.* $\phi^{\mathbf{b_q}}$ *is minimal iff*

$$t^2\text{ù}_3(\text{ù}_1\text{ù}_3 + (\text{ù}_2)_s) + (1 - t\text{ù}_1)(\text{ù}_1(1 - t\text{ù}_2) - t(\text{ù}_3)_s = 0.$$

Proof: From the definition of quasi binormal surface, we have

$$\phi_s^{\mathbf{b_q}} = (1 - t\text{ù}_2)\mathbf{t_q} - t\text{ù}_3\mathbf{n_q},$$
$$\phi_t^{\mathbf{b_q}} = \mathbf{b_q}.$$

By using this field, the coefficients of the first fundamental form are given

$$E = (1 - t\text{ù}_2)^2 + t^2\text{ù}_3^2,$$
$$F = 0, G = 1.$$

The second partial derivatives of $\phi^{\mathbf{b_q}}$ are expressed as follows:

$$\begin{aligned}
\phi_{ss}^{\mathbf{b_q}} &= -t((\grave{u}_2)_s - \grave{u}_1\grave{u}_3)\mathbf{t_q} + (\grave{u}_1(1-t\grave{u}_2) - t(\grave{u}_3)_s)\mathbf{n_q} + (\grave{u}_2(1-t\grave{u}_2) - t\grave{u}_3^2)\mathbf{b_q},\\
\phi_{ts}^{\mathbf{b_q}} &= -\grave{u}_2\mathbf{t_q} - \grave{u}_3\mathbf{n_q},\\
\phi_{tt}^{\mathbf{b_q}} &= 0.
\end{aligned}$$

So, an algebraic calculus shows that

$$\phi_s^{\mathbf{b_q}} \times \phi_t^{\mathbf{b_q}} = -t\grave{u}_3\mathbf{t_q} - (1-t\grave{u}_2)\mathbf{n_q}.$$

Moreover, by the definition of the unit normal vector, we have

$$n^{\mathbf{b_q}} = \frac{-t\grave{u}_3}{\sqrt{t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2}}\mathbf{t_q} - \frac{(1-t\grave{u}_1)}{\sqrt{t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2}}\mathbf{n_q}.$$

Therefore, the coefficients of the second fundamental form are given

$$\begin{aligned}
e &= \frac{1}{\sqrt{t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2}}(t^2\grave{u}_3(\grave{u}_1\grave{u}_3 + (\grave{u}_2)_s) + (1-t\grave{u}_1)(\grave{u}_1(1-t\grave{u}_2) - t(\grave{u}_3)_s),\\
f &= \frac{\grave{u}_3}{\sqrt{t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2}},\\
g &= 0.
\end{aligned}$$

The mean curvature of $\phi^{\mathbf{b_q}}$ is presented

$$H^{\mathbf{b_q}} = \frac{1}{2(t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2)^{\frac{3}{2}}}(t^2\grave{u}_3(\grave{u}_1\grave{u}_3 + (\grave{u}_2)_s) + (1-t\grave{u}_1)(\grave{u}_1(1-t\grave{u}_2) - t(\grave{u}_3)_s).$$

This completes the proof.

Theorem 3.4. *Let $\phi^{\mathbf{b_q}}$ be a quasi binormal surface of a quasi curve in space. $\phi^{\mathbf{b_q}}$ is not a Bonnet surface.*

Assume that, $\phi^{\mathbf{b_q}}$ is not minimal. Then, a harmonic evolute surface of the quasi binormal surface is given by

$$\phi^h(s,t) = \phi^{\mathbf{b_q}}(s,t) + \frac{1}{H^{\mathbf{b_q}}}n^{\mathbf{b_q}}.$$

Theorem 3.5. *A harmonic evolute surface of $\phi^{\mathbf{b_q}}$ is given by*

$$\phi^h(s,t) = \alpha + t\mathbf{b_q} - \frac{t\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2}}\mathbf{t_q} - \frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2 + (1-t\grave{u}_2)^2}}\mathbf{n_q}.$$

Theorem 3.6. *A harmonic evolute surface of $\phi^{\mathbf{b_q}}$ is a Bonnet surface if and only if*

$$(1-tx_2-ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})-(\frac{tù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)^2$$

$$+(tù_3+\frac{tù_1ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}+(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)^2$$

$$+(ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})-\frac{tù_2ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})^2$$

$$=1+(\frac{ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})^2+(\frac{ù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})^2,$$

$$(1-tx_2-ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$-(\frac{tù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)(\frac{ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$-(tù_3+\frac{tù_1ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}$$

$$+(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)(\frac{ù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$+(ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})-\frac{tù_2ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})=0,$$

and

$$f^{\phi^h}=\frac{1}{\pi}[(ù_2+(\frac{-ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s+(\frac{ù_1^2}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}))$$

$$(tù_3+\frac{tù_1ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}+(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s$$

$$-(ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})-\frac{tù_2ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$(\frac{ù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}))+((ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$-\frac{tù_2ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})\frac{ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}$$

$$-(1-tx_2-ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})-(\frac{tù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)$$

$$(-(x_3+\frac{-x_1ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}+(\frac{-ù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s))$$

$$+(\frac{ù_3(x_1+x_2)}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})(-(1-tx_2+-ù_1(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$-(\frac{tù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)(\frac{tù_2ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})$$

$$+(\frac{ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})((tù_3+\frac{tù_1ù_3}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}}$$

$$+(\frac{1-tù_1}{H^{\mathbf{bq}}\sqrt{t^2ù_3^2+(1-tù_2)^2}})_s)))=\rho,$$

where ρ is constant.

Proof: Now, we obtain the derivative formulas

$$
\begin{aligned}
\phi_s^h(s,t) &= (1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)\mathbf{t}_{\mathbf{q}} \\
&\quad -(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)\mathbf{n}_{\mathbf{q}} \\
&\quad +(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\mathbf{b}_{\mathbf{q}}, \\
\phi_t^h(s,t) &= \frac{\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}\mathbf{t}_{\mathbf{q}}+\frac{\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}\mathbf{n}_{\mathbf{q}}+\mathbf{b}_{\mathbf{q}}.
\end{aligned}
$$

Then, it is easy to see that

$$
\begin{aligned}
E^{\phi^h} &= (1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)^2 \\
&\quad +(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)^2 \\
&\quad +(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})^2, \\
F^{\phi^h} &= (1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
&\quad -(\frac{t\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)(\frac{\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
&\quad -(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}} \\
&\quad +(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)(\frac{\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
&\quad +(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}), \\
G^{\phi^h} &= 1+(\frac{\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})^2+(\frac{\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})^2.
\end{aligned}
$$

We instantly calculate

$$\begin{aligned}
\phi_{ss}^{h}(s,t) \;=\; & ((1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)_s \\
& +x_1(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s) \\
& -x_2(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))\mathbf{t_q} \\
& +(x_1(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s) \\
& -(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)_s \\
& -x_3(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))\mathbf{n_q} \\
& +(x_2(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s) \\
& +x_3(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s) \\
& (\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)\mathbf{b_q},
\end{aligned}$$

$$\begin{aligned}
\phi_{tt}^{h}(s,t) \;=\; & (\frac{\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_t\mathbf{t_q}+(\frac{\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_t\mathbf{n_q}, \\
\phi_{ts}^{h}(s,t) \;=\; & -(\grave{u}_2+(\frac{-\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s+(\frac{\grave{u}_1^2}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))\mathbf{t_q} \\
& -(x_3+\frac{-x_1\grave{u}_3}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{-\grave{u}_1}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)\mathbf{n_q} \\
& +(\frac{\grave{u}_3(x_1+x_2)}{H^{\mathbf{b_q}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\mathbf{b_q}.
\end{aligned}$$

With the help of the obtained equations, we express

$$
\begin{aligned}
n_h^{\mathbf{b}_{\mathbf{q}}} \quad = \quad & \frac{1}{\pi}[-(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s \\
& -(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
& (\frac{\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))\mathbf{t}_{\mathbf{q}}+((\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
& -\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\frac{\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}} \\
& -(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)\mathbf{n}_{\mathbf{q}} \\
& -(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s \\
& (\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})+(\frac{\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
& ((t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)))\mathbf{b}_{\mathbf{q}}].
\end{aligned}
$$

where $\pi=\left|\phi_s^h\wedge\phi_t^h\right|$.

Then

$$
\begin{aligned}
e^{\phi^h} \quad = \quad & \frac{1}{\pi}[((1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)_s \\
& +x_1(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s) \\
& -x_2(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})) \\
& (-(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s \\
& -(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}) \\
& (\frac{\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))+((\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{b}_{\mathbf{q}}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})
\end{aligned}
$$

$$-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\frac{\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}$$

$$-(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)$$

$$((x_1(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)$$

$$-(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)_s$$

$$-x_3(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})))$$

$$+((x_2(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)$$

$$+x_3(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)$$

$$(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)$$

$$(-(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s$$

$$(\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})+(\frac{\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})$$

$$((t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)))$$

$$
\begin{aligned}
f^{\varphi^h} = \; & \frac{1}{\pi}[(\grave{u}_2+(\frac{-\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s+(\frac{\grave{u}_1^2}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))\\
& (t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s\\
& -(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\\
& (\frac{\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))+((\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\\
& -\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\frac{\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}\\
& -(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)\\
& (-(x_3+\frac{-x_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{-\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s))\\
& +(\frac{\grave{u}_3(x_1+x_2)}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})(-(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\\
& -(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)(\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\\
& +(\frac{\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})((t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}\\
& +(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s)))
\end{aligned}
$$

$$
\begin{aligned}
g^{\phi^h} = \; & \frac{1}{\pi}[(-(t\grave{u}_3+\frac{t\grave{u}_1\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}+(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s\\
& -(\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\\
& (\frac{\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}))(\frac{\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_t)+((\frac{\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_t\\
& (\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\\
& -\frac{t\grave{u}_2\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})\frac{\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}}\\
& -(1-tx_2-\grave{u}_1(\frac{1-t\grave{u}_1}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})-(\frac{t\grave{u}_3}{H^{\mathbf{bq}}\sqrt{t^2\grave{u}_3^2+(1-t\grave{u}_2)^2}})_s).
\end{aligned}
$$

Consequently, using the definition of a Bonnet surface, we have proved the theorem.

Application to Helix

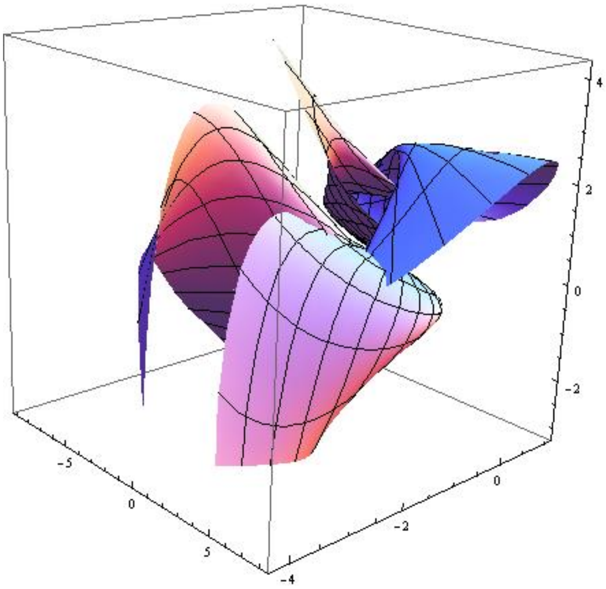

Figure1. The Quasi Binormal Surface of a Helix.

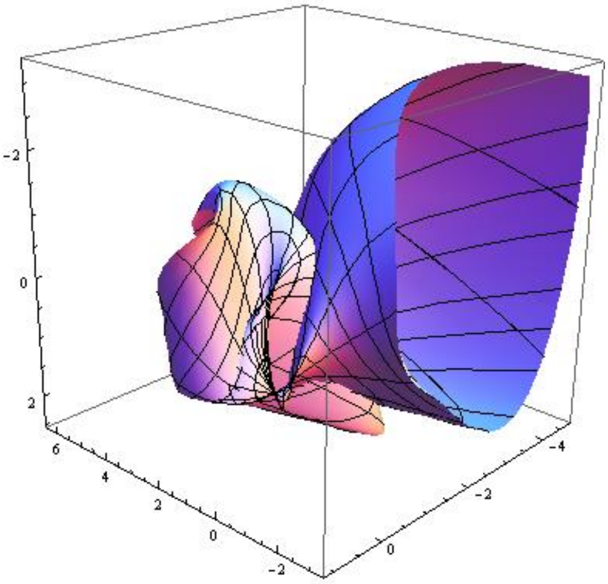

Figure2. A Harmonic Evolute Surface of ϕ^{b_q} .

4. CONCLUSION

In conclusion, a harmonic evolute surface of quasi binormal surface associated with quasi frame was studied throughout this paper. After constructing quasi binormal surface, we established harmonic evolute surface of quasi binormal surface by using quasi frame and mean curvature. Finally, our studies have enabled us to gain new results about these surfaces.

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