ORIGINAL PAPER

A NEW FAMILIES OF GAUSS *k*-JACOBSTHAL NUMBERS AND GAUSS *k*-JACOBSTHAL-LUCAS NUMBERS AND THEIR POLYNOMIALS

ENGİN ÖZKAN[1], MERVE TAŞTAN[2]

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Abstract. *In this paper, we define the new families of Gauss k-Jacobsthal numbers and Gauss k-Jacobsthal-Lucas numbers. We obtain some exciting properties of the families. We give the relationships between the family of the Gauss k-Jacobsthal numbers and the known Gauss Jacobsthal numbers, the family of the Gauss k-Jacobsthal-Lucas numbers and the known Gauss Jacobsthal-Lucas numbers. We also define the generalized polynomials for these numbers. Further, we obtain some interesting properties of the polynomials. In addition, we give the relationships between the generalized Gauss k-Jacobsthal polynomials and the known Gauss Jacobsthal polynomials, the generalized Gauss k-Jacobsthal-Lucas polynomials and the known Gauss Jacobsthal-Lucas polynomials. Furthermore, we find the new generalizations of these families and the polynomials in matrix representation. Then we prove Cassini's Identities for the families and their polynomials.*

Keywords: *Gauss Jacobsthal polynomials; Gauss Jacobsthal-Lucas polynomials; Cassini's identity; generating function.*

1. INTRODUCTION

Fibonacci numbers have exciting properties and many applications in many branches of mathematics [1-4]. In [5], Mikkawy and Sogabe gave a new family of *k*-Fibonacci numbers. In [6], Özkan et al. defined a new family of *k*-Lucas numbers and gave some properties about the family of the numbers. There were some works on polynomials of the families of *k*-Fibonacci numbers and *k*-Lucas numbers [7,8]. In [9], Falcon and Plaza gave general *k*-Fibonacci numbers and showed properties of these numbers were related with elementary matrix algebra. In [10], Bolat and Köse found some important properties about *k*-Fibonacci number. The Jacobsthal numbers also have many applications to some field of science and many generalizations [11-16]. In [17], Koken and Bozkurt gave some identities and Binet formula for the Jacobsthal number by using matrix method.

We organize the paper into three parts. In Section 2, we give some known definitions and properties. We organize Section 3 into two parts. In Part 1, we define the new families of Gauss *k*-Jacobsthal numbers and Gauss *k*-Jacobsthal-Lucas numbers. Then we find generating functions of the families for *k=2* and *k=3*. We obtain some exciting properties of the families. We give the relationships between the family of the Gauss *k*-Jacobsthal numbers and the known Gauss Jacobsthal numbers, the family of the Gauss *k*-Jacobsthal-Lucas numbers and

[1] Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Department of Mathematics, Erzincan, Turkey. E-mail: eozkanmath@gmail.com; eozkan@erzincan.edu.tr.
[2] Erzincan Binali Yıldırım University, Graduate School of Natural and Applied Sciences, Erzincan, Turkey. E-mail: mervetastan24@gmail.com.

the known Gauss Jacobsthal-Lucas numbers. Further, we find the new generalizations of these families in matrix representation. Then we prove Cassini's Identities for the families.

In part 2, we define the generalized polynomials for these numbers. We obtain some interesting properties of the polynomials. We give the relationships between the generalized Gauss k-Jacobsthal polynomials and the known Gauss Jacobsthal polynomials, the generalized Gauss k-Jacobsthal-Lucas polynomials and the known Gauss Jacobsthal-Lucas polynomials. In addition, we find the new generalizations of these polynomials in matrix representation. Then we prove Cassini's Identities for the polynomials.

2. MATERIALS AND METHODS

2.1. DEFINITION

The Gauss Jacobsthal numbers $\{GJ_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$GJ_{n+1} = GJ_n + 2GJ_{n-1}, n \geq 1 \tag{1}$$

with initial conditions $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$.[18]

The Gauss Jacobsthal-Lucas numbers $\{Gj_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$Gj_{n+1} = Gj_n + 2Gj_{n-1} \tag{2}$$

with initial conditions $Gj_0 = 2 - \frac{i}{2}$ and $GJ_1 = 1 + 2i$.[18]

Binet formulas for GJ_n and Gj_n are given by as follows

$$GJ_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\,\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \tag{3}$$

and

$$Gj_n = \alpha^n + \beta^n + i(\alpha^{n-1} + \beta^{n-1}) \tag{4}$$

where $\alpha = 2$ and $\beta = -1$.[18]

Now we introduce the matrix Q and the matrix P. Let Q and P denote the 2×2 matrices defined as

$$Q = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1+i & 1 \\ 1 & \frac{i}{2} \end{bmatrix}.$$

For $n \geq 1$, we have

$$Q^n P = \begin{bmatrix} GJ_{n+2} & GJ_{n+1} \\ GJ_{n+1} & GJ_n \end{bmatrix}$$

where GJ_n is the nth Gauss Jacobsthal number. [17]

Now we introduce the matrix R. Let R denote the 2×2 matrix defined as

$$R = \begin{bmatrix} 5+i & 1+2i \\ 1+2i & 2-\frac{i}{2} \end{bmatrix}.$$

For $n \geq 1$, we have

$$Q^n R = \begin{bmatrix} Gj_{n+2} & Gj_{n+1} \\ Gj_{n+1} & Gj_n \end{bmatrix}$$

where GJ_n is the Gauss Jacobsthal-Lucas numbers.[17]

2.2. DEFINITION

The Gaussian Jacobsthal polynomials $\{GJ_n(x)\}_{n=0}^{\infty}$ are defined following recurrence relation

$$GJ_{n+1}(x) = GJ_n(x) + 2xGJ_{n-1}(x), n \geq 1 \tag{5}$$

with initial conditions $GJ_0(x) = \frac{i}{2}$ and $GJ_1(x) = 1$.[19]

The Gaussian Jacobsthal-Lucas polynomials $\{Gj_n(x)\}_{n=0}^{\infty}$ are defined following recurrence relation

$$Gj_{n+1}(x) = Gj_n(x) + 2xGj_{n-1}(x), n \geq 1 \tag{6}$$

with initial conditions $GJ_0(x) = 2-\frac{i}{2}$ and $GJ_1(x) = 1+2ix$.[19]

Now we introduce the matrix S. Let S denote the 2×2 matrix defined as

$$S = \begin{bmatrix} 1+ix & 1 \\ 1 & \frac{i}{2} \end{bmatrix}.$$

For $n \geq 1$, we get

$$Q^n S = \begin{bmatrix} GJ_{n+2}(x) & GJ_{n+1}(x) \\ GJ_{n+1}(x) & GJ_n(x) \end{bmatrix}$$

where $GJ_n(x)$ is the *n*th Gaussian Jacobsthal polynomials [16].

Now we introduce the matrix . Let T denote the 2×2 matrix defined as

$$T = \begin{bmatrix} 1+4x+ix & 1+2xi \\ 1+2xi & 2-\frac{i}{2} \end{bmatrix}.$$

For $n \geq 1$, we have

$$Q^n T = \begin{bmatrix} Gj_{n+2}(x) & Gj_{n+1}(x) \\ Gj_{n+1}(x) & Gj_n(x) \end{bmatrix}$$

where GJ_n is the Gauss Jacobsthal-Lucas polynomials.[16]

3. RESULTS AND DISCUSSION

3.1. THE GENERALIZED GAUSS K-JACOBSTHAL NUMBERS AND THE GENERALIZED GAUSS K-JACOBSTHAL-LUCAS NUMBERS

3.1.1. Definition

There are unique numbers *m* and *r* such that *n*=*mk*+*r* where m, k $(\neq 0)$ natural numbers and $0 \leq r < k$. The generalized Gauss *k*-Jacobsthal numbers $GJ_n^{(k)}$ are defined by

$$\mathrm{GJ}_n^{(k)} \coloneqq \left[\left(\frac{\alpha^m-\beta^m}{\alpha-\beta} + \mathrm{i}\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\right]^{k-r} \left[\left(\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} + \mathrm{i}\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)\right]^r, n = mk + r \tag{7}$$

where $\alpha = 2$ and $\beta = -1$.

Also, we can find the generalized Gauss *k*-Jacobsthal numbers by matrix methods. Indeed, it is clear that

$$GJ_n^{k-1} Q^n P = \begin{bmatrix} GJ_{kn+k+1}^{(k)} & GJ_{kn+k}^{(k)} \\ GJ_{kn+k}^{(k)} & GJ_{kn+k-1}^{(k)} \end{bmatrix}.$$

Let's give some values for the Gauss *k*-Jacobsthal numbers in Table 1.

Table 1. The generalized Gauss k-Jacobsthal numbers $GJ_n^{(k)}$ for some k and n

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$GJ_0^{(k)}$	$\frac{i}{2}$	$-\frac{1}{4}$	$-\frac{i}{8}$	$\frac{1}{16}$	$\frac{i}{32}$	$-\frac{1}{64}$
$GJ_1^{(k)}$	1	$\frac{i}{2}$	$-\frac{1}{4}$	$-\frac{i}{8}$	$\frac{1}{16}$	$\frac{i}{32}$
$GJ_2^{(k)}$	$1+i$	1	$\frac{i}{2}$	$-\frac{1}{4}$	$-\frac{i}{8}$	$\frac{1}{16}$
$GJ_3^{(k)}$	$3+i$	$1+i$	1	$\frac{i}{2}$	$-\frac{1}{4}$	$-\frac{i}{8}$
$GJ_4^{(k)}$	$5+3i$	$2i$	$1+i$	1	$\frac{i}{2}$	$-\frac{1}{4}$
$GJ_5^{(k)}$	$11+5i$	$2+4i$	$2i$	$1+i$	1	$\frac{i}{2}$

From (3) and (7), the following relation between the generalized Gauss *k*-Jacobsthal and the Gauss Jacobsthal numbers was obtained

$$GJ_n^{(k)} \coloneqq (GJ_m)^{k-r}(GJ_{m+1})^r, n = mk + r \tag{8}$$

Considering the case $k=1$ in (7), we see that $m=n$ and $r=0$. Therefore, $GJ_n^{(1)}$ is the ordinary Gauss Jacobsthal numbers GJ_n.

Throughout this paper, let $k, m \in \{1,2,3,\dots\}$.

3.1.2. Theorem

For k and m, we have

$$[(GJ_{m+1})^k-(GJ_m)^k]=[GJ_{(m+1)k}^{(k)}-GJ_{mk}^{(k)}].$$

Proof: By using (8), we get

$$\begin{aligned} GJ_{(m+1)k}^{(k)}-GJ_{mk}^{(k)} &= [(GJ_m)^{k-k}(GJ_{m+1})^k]-[(GJ_m)^{k-0}(GJ_{m+1})^0] \\ &= (GJ_{m+1})^k-(GJ_m)^k. \end{aligned}$$ □

For $k=2,3,4$ and n, we have the interesting following properties between these numbers.

i. $GJ_{2n}^{(2)}=GJ_n^2$,

ii. $GJ_{2n+1}^{(2)}=GJ_{2n}^{(2)}+2GJ_{2n-1}^{(2)}$,

iii. $GJ_{3n}^{(3)}=GJ_n^3$,

iv. $GJ_{3n+1}^{(3)}=GJ_n^2GJ_{n+1}$,

v. $GJ_{3n+2}^{(3)}=GJ_nGJ_{n+1}^2$,

vi. $GJ_{3n+1}^{(3)}=GJ_{3n}^{(3)}+2GJ_{3n-1}^{(3)}$,

vii. $GJ_{4n}^{(4)}=GJ_n^4$,

viii. $GJ_{4n+1}^{(4)}=GJ_n^3GJ_{n+1}$,

ix. $GJ_{4n+2}^{(4)}=GJ_n^2GJ_{n+1}^2$,

x. $GJ_{4n+3}^{(4)}=GJ_nGJ_{n+1}^3$,

xi. $GJ_{4n+1}^{(4)}=GJ_{4n}^{(4)}+2GJ_{4n-1}^{(4)}$.

Let's suppose that $GJ_{-n}^{(k)}(x)=0$ for $k=1,2,\dots$.

3.1.3. Theorem

For n, we have the relation

$$GJ_{kn+1}^{(k)}=GJ_{kn}^{(k)}+2GJ_{kn-1}^{(k)}.$$

Proof:

$$\begin{aligned} GJ_{kn}^{(k)}+2GJ_{kn-1}^{(k)} &= GJ_n^k+2(GJ_{n-1}GJ_n^{k-1}) \\ &= GJ_n^{k-1}(GJ_n+2GJ_{n-1}) \end{aligned}$$

$$= GJ_n^{k-1} GJ_{n+1}$$
$$= GJ_{kn+1}^{(k)}. \qquad \square$$

3.1.4. Theorem

(Cassini's Identity) Let $GJ_n^{(k)}$ be the generalized Gauss k-Jacobsthal numbers. For $n, k \geq 2$, Cassini's Identity for $GJ_n^{(k)}$ is as follows:

$$GJ_{kn+t}^{(k)} GJ_{kn+t-2}^{(k)} - \left(GJ_{kn+t-1}^{(k)}\right)^2 = \begin{cases} GJ_n^{2k-2}(-1)^n(3-i)2^{n-2}, & t = 1 \\ 0, & t \neq 1 \end{cases}.$$

Proof: By using (8), we get

$$\begin{aligned} GJ_{kn+t}^{(k)} GJ_{kn+t-2}^{(k)} - \left(GJ_{kn+t-1}^{(k)}\right)^2 &= (GJ_n^{k-1} GJ_{n+t})(GJ_n^{k-1} GJ_{n+t-2}) - (GJ_n^{k-1} GJ_{n+t-1})^2 \\ &= (GJ_n^{k-1})^2 (GJ_{n+t} GJ_{n+t-2} - (GJ_{n+t-1})^2) \\ &= GJ_n^{2k-2}((GJ_{n+t} GJ_{n+t-2} - (GJ_{n+t-1})^2). \end{aligned}$$

For $t = 1$:

$$\begin{aligned} &= GJ_n^{2k-2}((GJ_{n+1} GJ_{n-1} - (GJ_n)^2) \\ &= GJ_n^{2k-2}(-1)^n(3-i)2^{n-2}. \end{aligned}$$

For $t \neq 1$, $t = m$, ($m \in \mathbb{N}$):

$$\begin{aligned} &= GJ_n^{2k-2}((GJ_{n+m} GJ_{n+m-2} - (GJ_{n+m-1})^2) \\ &= GJ_n^{2k-2}(GJ_{2n+2m-2}^{(2)} - GJ_{2n+2m-2}^{(2)}) \\ &= 0. \end{aligned} \qquad \square$$

3.1.5. Definition

There are unique numbers m and r such that $n = mk + r$ where m, k ($\neq 0$)natural numbers and $0 \leq r < k$. The generalized Gauss k −Jacobsthal-Lucas numbers $Gj_n^{(k)}$ are defined by

$$Gj_n^{(k)} \coloneqq [(\alpha^m + \beta^m) + i(\alpha^{m-1} + \beta^{m-1})]^{k-r}[(\alpha^{m+1} + \beta^{m+1}) + i(\alpha^m + \beta^m)]^r,$$
$$n = mk + r \text{ where } \alpha = 2 \text{ and } \beta = -1.$$

Also, one can obtain the generalized Gauss k −Jacobsthal-Lucas numbers by matrix methods. Indeed, it is clear that

$$Gj_n^{k-1} Q^n R = \begin{bmatrix} j_{kn+k+1}^{(k)} & j_{kn+k}^{(k)} \\ j_{kn+k}^{(k)} & j_{kn+k-1}^{(k)} \end{bmatrix}.$$

Let's give some values for the Gauss k −Jacobsthal-Lucas numbers in Table 2.

Table 2. The generalized Gauss k-Jacobsthal-Lucas numbers $Gj_n^{(k)}$ for some k and n

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$Gj_0^{(k)}$	$2-\frac{i}{2}$	$\frac{15}{4}-2\mathrm{i}$	$\frac{13}{2}-\frac{47}{8}\mathrm{i}$	$\frac{161}{16}-15\mathrm{i}$	$\frac{101}{8}-\frac{1121}{32}\mathrm{i}$	$\frac{495}{64}-\frac{611}{8}\mathrm{i}$
$Gj_1^{(k)}$	$1+2i$	$3+\frac{7}{2}\mathrm{I}$	$\frac{31}{4}+\frac{11}{2}\mathrm{i}$	$\frac{73}{4}+\frac{57}{8}\mathrm{i}$	$\frac{641}{16}+\frac{41}{8}\mathrm{i}$	$\frac{1323}{16}-\frac{313}{32}\mathrm{i}$
$Gj_2^{(k)}$	$5+i$	$-3+4i$	$-4+\frac{19}{2}\mathrm{i}$	$-\frac{13}{4}+21\mathrm{i}$	$4+\frac{349}{8}\mathrm{i}$	$\frac{477}{16}-\frac{341}{4}\mathrm{i}$
$Gj_3^{(k)}$	$7+5i$	$3+11i$	$-11-2i$	$-23+\frac{3}{2}\mathrm{i}$	$-\frac{181}{4}+\frac{29}{2}\mathrm{i}$	$\frac{333}{4}-\frac{413}{8}\mathrm{i}$
$Gj_4^{(k)}$	$17+7i$	$24+10i$	$-19+17i$	$-7-24i$	$-26-\frac{89}{2}\mathrm{i}$	$-\frac{297}{4}-76\mathrm{i}$
$Gj_5^{(k)}$	$31+17i$	$30+32i$	$4+58i$	$-53i-21i$	$41-38i$	$63-\frac{193}{2}\mathrm{i}$

From (4) and Definition 3.5, we have the relation

$$Gj_n^{(k)} \coloneqq (Gj_m)^{k-r}(Gj_{m+1})^r, n = mk + r \tag{9}$$

When $k = 1$ in last equation, we get that $m = n$ and $r = 0$ so $Gj_n^{(1)} = Gj_n$.

3.1.6. Theorem

For k and m, we have the following relation between the generalized Gauss k-Jacobsthal-Lucas numbers and the known Gauss Jacobsthal-Lucas numbers

$$[(Gj_{m+1})^k - (Gj_m)^k] = [Gj_{(m+1)k}^{(k)} - Gj_{mk}^{(k)}].$$

Proof: By using (9), we get

$$\begin{aligned} Gj_{(m+1)k}^{(k)} - Gj_{mk}^{(k)} &= [(Gj_m)^{k-k}(Gj_{m+1})^k] - [(Gj_m)^{k-0}(Gj_{m+1})^0] \\ &= (Gj_{m+1})^k - (Gj_m)^k. \end{aligned}$$ □

For $k = 2,3,4$ and n, we have the interesting following properties between these numbers.

i. $Gj_{2n}^{(2)} = Gj_n^2$,

ii. $Gj_{2n+1}^{(2)} = Gj_n Gj_{n+1}$,

iii. $Gj_n^{(2)} + Gj_{n+1}^{(2)} + Gj_{n+2}^{(2)} = Gj_{n+3}$,

iv. $Gj_{2n}^{(2)} = 2^{2n} + 2(-2)^n + 1$,

v. $Gj_{2n+1}^{(2)} = Gj_{2n}^{(2)} + 2Gj_{2n-1}^{(2)}$

vi. $Gj_{3n}^{(3)} = Gj_n^3$,

vii. $Gj_{3n+1}^{(3)} = Gj_n^2 Gj_{n+1}$,

viii. $Gj_{3n+2}^{(3)} = Gj_n Gj_{n+1}^2$,

ix. $Gj_{3n+1}^{(3)} = Gj_{3n}^{(3)} + 2Gj_{3n-1}^{(3)}$,

x. $Gj_{4n}^{(4)} = Gj_n^4$,

xi. $Gj_{4n+1}^{(4)} = Gj_n^3 Gj_{n+1}$,

xii. $Gj_{4n+2}^{(4)} = Gj_n^2 Gj_{n+1}^2$,

xiii. $Gj_{4n+3}^{(4)} = Gj_n Gj_{n+1}^3$,

xiv. $Gj_{4n+1}^{(4)} = Gj_{4n}^{(4)} + 2Gj_{4n-1}^{(4)}$.

3.1.7. Theorem

For n, we have the relation

$$Gj_{kn+1}^{(k)} = Gj_{kn}^{(k)} + 2Gj_{kn-1}^{(k)}.$$

Proof: By using (9), we get

$$\begin{aligned} Gj_{kn}^{(k)} + 2Gj_{kn-1}^{(k)} &= Gj_n^k + 2(Gj_{n-1}Gj_n^{k-1}) \\ &= Gj_n^{k-1}(Gj_n + 2Gj_{n-1}), \\ &= Gj_n^{k-1}Gj_{n+1} \\ &= Gj_{kn+1}^{(k)}. \end{aligned}$$ □

3.1.8. Theorem

(Cassini's Identity) Let $Gj_n^{(k)}$ be the generalized Gauss k-Jacobsthal-Lucas numbers. For $n, k \geq 2$, Cassini's Identity for $Gj_n^{(k)}$ is as follows:

$$Gj_{kn+t}^{(k)}Gj_{kn+t-2}^{(k)} - \left(Gj_{kn+t-1}^{(k)}\right)^2 = \begin{cases} Gj_n^{2k-2}(-1)^n 9(3-i), & t=1 \\ 0, & t \neq 1 \end{cases}.$$

Proof: By using (9), we get

$$\begin{aligned} Gj_{kn+t}^{(k)}Gj_{kn+t-2}^{(k)} - \left(Gj_{kn+t-1}^{(k)}\right)^2 &= (Gj_n^{k-1})(Gj_n^{k-1}Gj_{n+t-2}) - (Gj_n^{k-1}Gj_{n+t-1})^2 \\ &= (Gj_n^{k-1})^2(Gj_{n+t}Gj_{n+t-2} - (Gj_{n+t-1})^2) \\ &= Gj_n^{2k-2}((Gj_{n+t}Gj_{n+t-2} - (Gj_{n+t-1})^2). \end{aligned}$$

For $t = 1$;

$$\begin{aligned} &= Gj_n^{2k-2}((Gj_{n+1}Gj_{n-1} - (Gj_n)^2) \\ &= Gj_n^{2k-2}(-1)^n 9(3-i). \end{aligned}$$

For $t \neq 1,\ t = m, (m \in \mathbb{N})$;

$$\begin{aligned} &= Gj_n^{2k-2}((Gj_{n+m}Gj_{n+m-2} - (Gj_{n+m-1})^2) \\ &= Gj_n^{2k-2}(Gj_{2n+2m-2}^{(2)} - Gj_{2n+2m-2}^{(2)}) \\ &= 0. \end{aligned} \qquad \square$$

For n, there are exciting relations between the generalized Gauss k-Jacobsthal numbers and the known Jacobsthal numbers, the generalized Gauss k-Jacobsthal-Lucas and the known Jacobsthal-Lucas numbers $(m \in \mathbb{N})$;

$$GJ_{nk+t}^{(k)} = GJ_n^{k-t} GJ_{n+1}^{t}$$

and

$$Gj_{nk+t}^{(k)} = Gj_n^{k-t} Gj_{n+1}^{t}.$$

3.2. THE GENERALIZED GAUSS k-JACOBSTHAL POLYNOMIALS AND THE GENERALIZED GAUSS k-JACOBSTHAL-LUCAS POLYNOMIALS

3.2.1. Definition

There are unique numbers m and r such that $n = mk + r$ where $m,\ k\ (\neq 0)$ natural numbers and $0 \leq r < k$. The generalized Gauss k-Jacobsthal polynomials $GJ_n^{(k)}(x)$ are defined by

$$GJ_n^{(k)} := \left[\frac{\alpha^m(x)-\beta^m(x)}{\alpha(x)-\beta(x)} + ix\frac{\alpha^{m-1}(x)-\beta^{m-1}(x)}{\alpha(x)-\beta(x)}\right]^{k-r}\left[\frac{\alpha^{m+1}(x)-\beta^{m+1}(x)}{\alpha(x)-\beta(x)} + ix\frac{\alpha^{m}(x)-\beta^{m}(x)}{\alpha(x)-\beta(x)})\right]^{r},$$

$$n = mk + r$$

where $\alpha(x) = \frac{1+\sqrt{1+8x}}{2}$ and $\beta(x) = \frac{1-\sqrt{1+8x}}{2}$.

Also, we can find the generalized Gauss k-Jacobsthal polynomials by matrix methods. Indeed, it is clear that

$$GJ_{n+1}^{k-1}(x)Q^nS = \begin{bmatrix} GJ_{kn+k+1}^{(k)}(x) & GJ_{kn+k}^{(k)}(x) \\ GJ_{kn+k}^{(k)}(x) & GJ_{kn+k-1}^{(k)}(x) \end{bmatrix}$$

where $Q^nS = \begin{bmatrix} GJ_{n+2}(x) & GJ_{n+1}(x) \\ GJ_{n+1}(x) & GJ_n(x) \end{bmatrix}$.

Let's give some values for the Gauss k-Jacobsthal polynomials in Table 3.

Table 3. The generalized Gauss k-Jacobsthal polynomials $GJ_n^{(k)}(x)$ for some k and n

	$k=1$	$k=2$	$k=3$	$k=4$
$GJ_0^{(k)}(x)$	$\frac{i}{2}$	$-\frac{1}{4}$	$-\frac{l}{8}$	$\frac{1}{16}$
$GJ_1^{(k)}(x)$	1	$\frac{i}{2}$	$-\frac{1}{4}$	$-\frac{l}{8}$
$GJ_2^{(k)}(x)$	$1+ix$	1	$\frac{i}{2}$	$-\frac{1}{4}$
$GJ_3^{(k)}(x)$	$1+2x+ix$	$1+ix$	1	$\frac{i}{2}$
$GJ_4^{(k)}(x)$	$1+4x+(1+2x)ix$	$1-x^2+2ix$	$1+ix$	1
$GJ_5^{(k)}(x)$	$1+6x+4x^2+(1+4x)ix$	$1+2x+x^2+(2x+2x^2)i$	$1-x^2+2ix$	$1+ix$

The generalized Gauss k-Jacobsthal polynomials and the known Gauss Jacobsthal polynomials are related by

$$GJ_n^{(k)}(x) := (GJ_m(x))^{k-r}(GJ_{m+1}(x))^r, n = mk + r \tag{10}$$

If $k = 1$ in last equation, we get that $m = n$ and $r = 0$ so $GJ_n^{(1)}(x) = GJ_n(x)$.

3.2.2. Theorem

For k and m, the generalized Gauss k-Jacobsthal polynomials and the known Gauss Jacobsthal polynomials satisfy

$$[(GJ_{m+1}(x))^k - (GJ_m(x))^k] = [GJ_{(m+1)k}^{(k)}(x) - GJ_{mk}^{(k)}(x)].$$

Proof: By using (10), we have

$$\begin{aligned} GJ_{(m+1)k}^{(k)}(x) - GJ_{mk}^{(k)}(x) &= [(GJ_m(x))^{k-k}(GJ_{m+1}(x))^k] - [(GJ_m(x))^{k-0}(GJ_{m+1}(x))^0] \\ &= (GJ_{m+1}(x))^k - (GJ_m(x))^k. \end{aligned}$$ □

For $k = 2,3,4$ and n, we have the interesting following properties the generalized Gauss k-Jacobsthal polynomials and the known Gauss Jacobsthal polynomials.

i. $GJ_{2n}^{(2)}(x) = GJ_n^2(x)$,

ii. $GJ_{2n+1}^{(2)}(x) = GJ_n(x)GJ_{n+1}(x)$,

iii. $GJ_{2n+1}^{(2)}(x) = GJ_{2n}^{(2)}(x) + 2GxJ_{2n-1}^{(2)}(x)$,

iv. $GJ_{3n}^{(3)}(x) = GJ_n^3(x)$,

v. $GJ_{3n+1}^{(3)}(x) = GJ_n^2(x)GJ_{n+1}(x)$,

vi. $GJ_{3n+2}^{(3)}(x) = GJ_n(x)GJ_{n+1}^2(x)$,

vii. $GJ_{3n+1}^{(3)}(x) = GJ_{3n}^{(3)}(x) + 2xGJ_{3n-1}^{(3)}(x)$,

viii. $GJ_{4n}^{(4)}(x) = GJ_n^4(x)$,

ix. $GJ_{4n+1}^{(4)}(x) = GJ_n^3(x)GJ_{n+1}(x)$,

x. $GJ_{4n+2}^{(4)}(x) = GJ_n^2(x)GJ_{n+1}^2(x)$,

xi. $GJ_{4n+3}^{(4)}(x) = GJ_n(x)GJ_{n+1}^3(x)$,

xii. $GJ_{4n+1}^{(4)}(x) = GJ_{4n}^{(4)}(x) + 2xGJ_{4n-1}^{(4)}(x)$.

3.2.3. Theorem

For n, we have the following relation

$$GJ_{kn+1}^{(k)}(x) = GJ_{kn}^{(k)}(x) + 2xGJ_{kn-1}^{(k)}(x).$$

Proof: By using (10), we have

$$\begin{aligned} GJ_{kn}^{(k)}(x) + 2GJ_{kn-1}^{(k)}(x) &= GJ_n^k(x) + 2x(GJ_{n-1}(x)GJ_n^{k-1}(x)) \\ &= GJ_n^{k-1}(x)(GJ_n(x) + 2xGJ_{n-1}(x)) \\ &= GJ_n^{k-1}(x)GJ_{n+1}(x) \\ &= GJ_{kn+1}^{(k)}(x). \end{aligned}$$

□

(Cassini's Identity) Let $GJ_n^{(k)}(x)$ be the generalized Gauss *k*-Jacobsthal polynomials. For $n, k \geq 2$, Cassini's Identity for $GJ_n^{(k)}(x)$ is as follows:

$$GJ_{kn+t}^{(k)}(x)GJ_{kn+t-2}^{(k)}(x) - \left(GJ_{kn+t-1}^{(k)}(x)\right)^2 =$$

$$\begin{cases} GJ_n^{2k-2}(x)(-1)^n 2^{n-2} x^{n-1}(2 + x - i), & t = 1 \\ 0, & t \neq 1 \end{cases}.$$

Proof: By using (10), we get

$$\begin{aligned} &GJ_{kn+t}^{(k)}(x)GJ_{kn+t-2}^{(k)}(x) - \left(GJ_{kn+t-1}^{(k)}(x)\right)^2 \\ &= \left(GJ_n^{k-1}(x)GJ_{n+t}(x)\right)\left(GJ_n^{k-1}(x)GJ_{n+t-2}(x)\right) - (GJ_n^{k-1}(x)GJ_{n+t-1}(x))^2 \\ &= (GJ_n^{k-1}(x))^2(GJ_{n+t}(x)GJ_{n+t-2}(x) - \left(GJ_{n+t-1}(x)\right)^2) \\ &= GJ_n^{2k-2}(x)((GJ_{n+t}(x)GJ_{n+t-2}(x) - \left(GJ_{n+t-1}(x)\right)^2) \end{aligned}$$

For $t = 1$:

$$\begin{aligned} &= GJ_n^{2k-2}(x)((GJ_{n+1}(x)GJ_{n-1}(x) - \left(GJ_n(x)\right)^2) \\ &= GJ_n^{2k-2}(x)(-1)^n 2^{n-2} x^{n-1}(2 + x - i). \end{aligned}$$

For $t \neq 1,\ t = m, (m \in \mathbb{N})$:

$$\begin{aligned}&= GJ_n^{2k-2}(x)((GJ_{n+m}(x)GJ_{n+m-2}(x) - \left(GJ_{n+m-1}(x)\right)^2)\\ &= GJ_n^{2k-2}(x)(GJ_{2n+2m-2}^{(2)}(x) - GJ_{2n+2m-2}^{(2)}(x))\\ &= 0.\end{aligned}$$

□

3.2.5. Definiton

There are unique numbers m and r such that $n = mk + r$ where m, k $(\neq 0)$ natural numbers and $0 \leq r < k$. The generalized Gauss k-Jacobsthal-Lucas polynomials $Gj_n^{(k)}(x)$ are defined by

$$Gj_n^{(k)} := [(\alpha^m(x) + \beta^m(x)) + ix(\alpha^{m-1}(x) + \beta^{m-1}(x))]^{k-r}\left[(\alpha^{m+1}(x) + \beta^{m+1}(x)) + ix\left(\alpha^m(x) + \beta^m(x)\right)\right]^r$$

$$n = mk + r \text{ where } \alpha(x) = \frac{1+\sqrt{1+8x}}{2} \text{ and } \beta(x) = \frac{1-\sqrt{1+8x}}{2}.$$

Also, one can obtain the generalized Gauss k-Jacobsthal-Lucas polynomials by matrix methods. Indeed, it is clear that

$$Gj_{n+1}^{k-1}(x)Q^nT = \begin{bmatrix} Gj_{kn+k+1}^{(k)}(x) & Gj_{kn+k}^{(k)}(x) \\ Gj_{kn+k}^{(k)}(x) & Gj_{kn+k-1}^{(k)}(x) \end{bmatrix}$$

where

$$Q^nT = \begin{bmatrix} Gj_{n+2}(x) & Gj_{n+1}(x) \\ Gj_{n+1}(x) & Gj_n(x) \end{bmatrix}.$$

Table 4. The generalized Gauss k-Jacobsthal Lucas polynomials $Gj_n^{(k)}(x)$ for some k and n

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$Gj_0^{(k)}$(x)	$2 - \frac{i}{2}$	$\frac{15}{4} - 2i$	$\frac{13}{2} - \frac{47}{8}i$	$\frac{161}{16} - 15\text{i}$
$Gj_1^{(k)}$(x)	$1 + 2ix$	$x + 2 + \left(4x - \frac{1}{2}\right)\text{i}$	$4x + \frac{15}{4} + (\frac{15x}{2} - 2)i$	$\frac{13}{2} + \frac{47x}{4} + (-\frac{47}{8} + 13x)i$
$Gj_2^{(k)}$(x)	$1 + 4x + ix$	$-4x^2 + 1 + 4ix$	$-8x^2 + 2x + 1 + (-\frac{1}{2} + 8\text{x} + 2x^2)i$	$\frac{15}{4} - 15x^2 + (-2 + 15x + 8x^2)i$
$Gj_3^{(k)}$(x)	$1 + 6x + (1 + 4x)ix$	$-2x^2 + 4x + 1 + (3x + 8x^2)i$	$1 - 12x^2 + (6x - 8x^3)i$	$2 + 3x - 24x^2 - 4x^3 + (-\frac{1}{2} + 12\text{x} + 6x^2 - 16x^3)i$

	$k=1$	$k=2$	$k=3$	$k=4$
$Gj_4^{(k)}$(x)	$1+8x+8x^2+(1+6x)ix$	$15x^2+8x+1+(2x+8x^2)i$	$1+4x-8x^2-16x^3+(5x+16x^2-4x^3)i$	$1-24x^2+16x^4+(8x-32x^3)i$
$Gj_5^{(k)}$(x)	$1+10x+20x^2+(1+8x+8x^2)ix$	$1+10x+23x^2-4x^3+(2x+14x^2+16x^3)i$	$1+8x+11x^2-16x^3+(4x+24x^2+30x^3)i$	$1+4x-18x^2-48x^3+8x^4+(7x+24x^2-20x^3-32x^4)i$

The generalized Gauss k-Jacobsthal-Lucas polynomials and the known Gauss Jacobsthal-Lucas polynomials are related by

$$Gj_n^{(k)}(x) \coloneqq (Gj_m(x))^{k-r}(Gj_{m+1}(x))^r,\ n=mk+r \tag{11}$$

If $k=1$ in last equation, we get that $m=n$ and $r=0$ so $Gj_n^{(1)}(x)=Gj_n(x)$.

3.2.6. Theorem

For fixed k and m, we have

$$[(Gj_{m+1}(x))^k-(Gj_m(x))^k]=[Gj_{(m+1)k}^{(k)}(x)-Gj_{mk}^{(k)}(x)].$$

Proof: By using (11), we get

$$\begin{aligned} Gj_{(m+1)k}^{(k)}(x)-Gj_{mk}^{(k)}(x) &= [(Gj_m(x))^{k-k}(Gj_{m+1}(x))^k]-[(Gj_m(x))^{k-0}(Gj_{m+1(x)})^0] \\ &= (Gj_{m+1}(x))^k-(Gj_m(x))^k. \end{aligned}$$ □

For $k=2,3,4$ and n, the generalized Gauss k-Jacobsthal-Lucas polynomials and the known Gauss Jacobsthal-Lucas polynomials satisfy

i. $Gj_{2n}^{(2)}(x)=Gj_n^2(x)$,

ii. $Gj_{2n+1}^{(2)}(x)=Gj_n(x)Gj_{n+1}(x)$,

iii. $Gj_{2n+1}^{(2)}(x)=Gj_{2n}^{(2)}(x)+2xGj_{2n-1}^{(2)}(x)$

iv. $Gj_{3n}^{(3)}(x)=Gj_n^3(x)$,

v. $Gj_{3n+1}^{(3)}(x)=Gj_n^2(x)Gj_{n+1}(x)$,

vi. $Gj_{3n+2}^{(3)}(x)=Gj_n(x)Gj_{n+1}^2(x)$,

vii. $Gj_{3n+1}^{(3)}(x)=Gj_{3n}^{(3)}(x)+2xGj_{3n-1}^{(3)}(x)$

viii. $Gj_{4n}^{(4)}(x)=Gj_n^4(x)$,

ix. $Gj_{4n+1}^{(4)}(x)=Gj_n^3(x)Gj_{n+1}(x)$,

x. $Gj_{4n+2}^{(4)}(x)=Gj_n^2(x)Gj_{n+1}^2(x)$,

xi. $Gj_{4n+3}^{(4)}(x) = Gj_n(x)Gj_{n+1}^3(x)$,

xii. $Gj_{4n+1}^{(4)}(x) = Gj_{4n}^{(4)}(x) + 2xGj_{4n-1}^{(4)}(x)$.

3.2.7. Theorem

For fixed natural numbers n, we have

$$Gj_{kn+1}^{(k)}(x) = Gj_{kn}^{(k)}(x) + 2xGj_{kn-1}^{(k)}(x).$$

Proof: By using (11), we get

$$\begin{aligned} Gj_{kn}^{(k)}(x) + 2xGj_{kn-1}^{(k)}(x) &= Gj_n^k(x) + 2x(Gj_{n-1}(x)Gj_n^{k-1}(x)) \\ &= Gj_n^{k-1}(x)(Gj_n(x) + 2xGj_{n-1}(x)) \\ &= Gj_n^{k-1}(x)Gj_{n+1}(x) \\ &= Gj_{kn+1}^{(k)}(x). \end{aligned}$$ □

3.2.8. Theorem

(Cassini's Identity) Let $Gj_n^{(k)}(x)$ be the generalized Gauss k-Jacobsthal-Lucas polynomials. For $n, k \geq 2$, Cassini's Identity for $Gj_n^{(k)}(x)$ is as follows:

$$Gj_{kn+t}^{(k)}(x)Gj_{kn+t-2}^{(k)}(x) - \left(Gj_{kn+t-1}^{(k)}(x)\right)^2 = \begin{cases} Gj_n^{2k-2}(x)(-1)^{n-1}2^{n-2}x^{n-1}(2+x-i)(1+8x), & t = 1 \\ 0, & t \neq 1 \end{cases}.$$

Proof: By using (11), we get

$$\begin{aligned} & Gj_{kn+t}^{(k)}(x)Gj_{kn+t-2}^{(k)}(x) - \left(Gj_{kn+t-1}^{(k)}(x)\right)^2 \\ &= (Gj_n^{k-1}(x)Gj_{n+t}(x))(Gj_n^{k-1}(x)Gj_{n+t-2}(x)) - (Gj_n^{k-1}(x)Gj_{n+t-1}(x))^2 \\ &= (Gj_n^{k-1}(x))^2(Gj_{n+t}(x)Gj_{n+t-2}(x) - \left(Gj_{n+t-1}(x)\right)^2 \\ &= Gj_n^{2k-2}(x)\left((Gj_{n+t}(x)Gj_{n+t-2}(x) - \left(Gj_{n+t-1}(x)\right)^2\right). \end{aligned}$$

For $t = 1$;

$$\begin{aligned} &= Gj_n^{2k-2}(x)((Gj_{n+1}(x)Gj_{n-1}(x) - \left(Gj_n(x)\right)^2) \\ &= Gj_n^{2k-2}(x)(-1)^{n-1}2^{n-2}x^{n-1}(x+2-i)(1+8x). \end{aligned}$$

For $t \neq 1$, $t = m$, $(m \in \mathbb{N})$;

$$\begin{aligned} &= Gj_n^{2k-2}(x)((Gj_{n+m}(x)Gj_{n+m-2}(x) - \left(Gj_{n+m-1}(x)\right)^2) \\ &= Gj_n^{2k-2}(x)(Gj_{2n+2m-2}^{(2)}(x) - Gj_{2n+2m-2}^{(2)}(x)) \\ &= 0. \end{aligned}$$ □

For fixed natural numbers n, there are the following interesting relations between the generalized Gauss k-Jacobsthal polynomials and the known Gauss Jacobsthal polynomials and the generalized Gauss k-Jacobsthal-Lucas polynomials and the known Gauss Jacobsthal-Lucas polynomials: $(t \in \mathbb{N})$

$$GJ_{nk+t}^{(k)}(x) = GJ_n^{k-t}(x)GJ_{n+1}^{t}(x)$$

and

$$Gj_{nk+t}^{(k)}(x) = Gj_n^{k-t}(x)Gj_{n+1}^{t}(x).$$

4. CONCLUSIONS

In the present paper, we defined the new families of Gauss k-Jacobsthal numbers, Gauss k-Jacobsthal-Lucas numbers and their polynomials. We gave some relations among these families and the known Gauss Jacobsthal numbers and the known Gauss Jacobsthal-Lucas numbers. In addition to, we obtained the relationships among their polynomials and the known Gauss Jacobsthal polynomials and the known Gauss Jacobsthal-Lucas polynomials. Further, we find the new generalizations of these families and the polynomials in matrix representation. Then we prove Cassini's Identities for the families and their polynomials. We would like to thank the editor and referees for their valuable comments and remarks which led to a great improvement of the article.

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