*ORIGINAL PAPER*

# THE CONVERGENCE ON ALGEBRAIC LATTICE NORMED SPACES

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***Abstract.*** *The multiplicative convergence on Riesz algebras introduced and investigated with respect to norm and order convergences. If $X$ is a Riesz space and $E$ is a Riesz algebra then the vector norm $\mu: X \to E_+$ can be considered. Then $(X, \mu, E)$ is called algebraic lattice normed spaces. A net $(x_\alpha)_{\alpha \in A}$ in an $(X, \mu, E)$ is said to be multiplicative $\mu$-convergent to $x \in X$ if $\mu(x_\alpha - x) \cdot u \xrightarrow{o} 0$ holds for all $u \in E_+$. In this paper, the general properties of this convergence are studied.*

***Keywords:*** *Lattice normed space; Riesz space; Riesz Algebra.*

## 1. INTRODUCTION AND PRELIMINARIES

Riesz algebras and lattice normed spaces provide natural and efficient tools in the theory of functional analysis. However, as far as we know, the concept of Riesz algebras and lattice normed spaces have not been combined before. This paper aim to use the mo- and $mn$-convergences that were introduced by Aydın [1] for combining the concepts of the Riesz algebras and lattice normed spaces, and also, introduce a new convergence.

Let recal, first of all, some basic terminologies and notations which are used in the current paper. Let $E$ be a real-valued vector space. Thus, if there is an order relation "$\leq$" on $E$, i.e., it is antisymmetric, reflexive and transitive, then $E$ is called *ordered vector space* whenever, for every $x, y \in E$ such that $x \leq y$, the inequalities $x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold for all $z \in E$ and $\alpha \in \mathbb{R}$. Consider an ordered vector space $E$. Then it is called *Riesz space* or *vector lattice* if, for any two vectors $x, y \in E$, the infimum $x \wedge y$ and the supremum $x \vee y$ exist in $E$.

Let $E$ be a Riesz space. Then, for any $x \in E$, *the positive part* of $x$ is $x^+ := x \vee 0$, *the negative part* of $x$ is $x^- := (-x) \vee 0$ and the *absolute value* of $x$ is $|x| := x \vee (-x)$. Moreover, for any two elements $x,\ y$ in a Riesz space is called *disjoint* whenever $|x| \wedge |y| = 0$. If every nonempty bounded below subset has an infimum (or, every nonempty bounded above subset has a supremum) in a Riesz space $E$ then it is called *Dedekind complete* Riesz space.

A given partially ordered set $I$ is called directed if, for each $a_1, a_2 \in I$, there is another $a \in I$ such that $a \geq a_1$ and $a \geq a_2$. A function from a directed set $I$ into a set $E$ is called a $net$ in $E$. Thus, a Riesz space $E$ is Dedekind complete if and only if every $0 \leq x_\alpha \uparrow \leq x$ implies the existence of supremum of the net $(x_\alpha)_{\alpha \in A}$. A subset $A$ of a Riesz space $E$ is called *solid* if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A solid vector subspace of a Riesz space is referred to as an *ideal*. An order closed ideal is called a *band* [2-4].

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A net $(x_\alpha)_{\alpha\in A}$ in a Riesz space $E$ is said to *order convergent* to a vector $x \in E$ if there exists another net $(y_\beta)_{\beta\in B} \downarrow 0$ such that for every $\beta$, there is an index $\alpha_\beta$ such that $|x_\alpha - x| \leq y_\beta$ for all indexes $\alpha \geq \alpha_\beta$. In this case, it is abbreviated as $x_\alpha \xrightarrow{o} x$. A subset $A$ of a Riesz space is said to be *order closed* whenever $(x_\alpha)_{\alpha\in A}$ in $A$ and $x_\alpha \xrightarrow{o} x$ implies $x \in A$ [5, 6]. Recall that, in a Riesz space $E$, a net $(x_\alpha)_{\alpha\in A}$ is called *unbounded order convergent* (or, $uo$-convergent, for short) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in E_+$. In this case, write $x_\alpha \xrightarrow{uo} x$ [7-9].

Let $E$ be a Riesz space under an associative multiplication. If the multiplication with the usual properties makes $E$ an algebra, and also, the multiplication of two positive vectors in $E$ is positive, i.e., $x, y \in E_+$ implies $x \cdot y \in E_+$. Then $E$ is called a Riesz algebra (or, shortly, $l$-algebra). In addition, if $x \cdot y = y \cdot x$ holds for all $x, y \in E$ then $E$ is called commutative. For any Riesz algebra $E$, it can be seen from that if $x \wedge y = 0$ implies $(z \cdot x) \wedge y = (x \cdot z) \wedge y = 0$ for all $z \in E_+$ then $E$ is called $f$-algebra [10].

Recall that, for each real number $x > 0$, the Archimedean property; the sequence $(x_n)$ is unbounded above in $\mathbb{R}$. That means $\frac{1}{n}x \downarrow 0$ holds in $\mathbb{R}$ for each $x > 0$. Motivated by this property, a Riesz space $E$ is said to be Archimedean whenever $\frac{1}{n}x \downarrow 0$ holds in $E$ for each $x \in E_+$. Every Riesz space does not need to be Archimedean. To see this, give the following example.

**Example 1.1.** Consider the vector space $\mathbb{R}^2$ with the order, for any $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ or $x_1 = x_2$ and $y_1 \leq y_2$. Thus, $(\mathbb{R}^2, \leq)$ is a Riesz space, but it is not Archimedean. Indeed, $\frac{1}{n}(1,1) \downarrow$ in $\mathbb{R}^2$, but $\frac{1}{n}(1,1)$ is not decreasing to zero.

Next, let $E$ be arbitrary an Archimedean $f$-algebra which has a multiplicative unit vector $e$. Hence, from the equality $e = e \cdot e = e^2 \geq 0$, one can see that e is a positive element. Moreover, it can be seen that $e$ is a weak order unit because $x \wedge e = 0$ implies $x = x \wedge x = (x \cdot e) \wedge x = 0$. It is known that Archimedean implies the commutative. Thus, in the current paper, unless otherwise, assume that every Riesz spaces are real and Archimedean, and also, all $l$-algebras are assumed to be commutative. Recall that a net $(x_\alpha)_{\alpha\in A}$ in an $f$-algebra $E$ *multiplicative order converges* to $x \in E$ if $|x_\alpha - x| \cdot u \xrightarrow{o} 0$ for all $u \in E_+$. It is abbreviated as $(x_\alpha)_{\alpha\in A}$ $mo$-converges to $x$, or shortly $x_\alpha \xrightarrow{mo} x$. Also, the *$mo$-Cauchy*, *$mo$-complete* and *$mo$-continuous* are defined [11, 12]. On the other hand, a Riesz algebra $E$ is said to be *normed Riesz algebra* if it is Banach lattice and $\| x \cdot y \| \leq \| x \| \cdot \| y \|$ holds for all $x, y \in E$. Recall that the classical normed spaces are a function from a vector space to real numbers. However, when a *vector-valued norm* is taken, it means that the norm is a function from a vector space to Riesz space. Consider the structural properties of a vector space with some norm taking values in a Riesz space. So, it is called a *lattice normed space*; an *LNS* for short. Now, give some basic properties of lattice normed spaces [13, 14]. Let $X$ be a vector space and $E$ be a Riesz space. Then the map $\mu: X \to E_+$ is called a *vector norm* whenever it satisfies the following axioms:

1. $\mu(x) = 0$ if and only if $x = 0$ for all $x \in X$;

2. $\mu(\lambda x) = |\lambda|\mu(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$;

3. $\mu(x + y) \leq \mu(x) + \mu(y)$.

**Definition 1.2.** Let $(X, \mu, E)$ be a lattice normed space over $E$. Then it is called *algebraic lattice normed spaces* if $X$ is a Riesz space, $E$ is a Riesz algebra and the vector norm $\mu$ is monotone, i.e., $|x| \leq |y|$ implies $\mu(x) \leq \mu(y)$. Then, it is abbreviated the $(X, \mu, E)$ as $ALNS$.

Dealing with $ALNS$s, shall keep in mind also the following examples.

**Example 1.3.** Let $X$ be a Riesz algebra. Then $(X, |.|, X)$ is an $ALNS$.

**Example 1.4.** Let $X$ be a normed Riesz space with a norm $\|\cdot\|$. Then $(X, \|\cdot\|, \mathbb{R})$ is an $ALNS$.

Let $E$ be a Riesz space. Then consider the set $Orth(E) = \{T \in L_b(E): x \perp y \text{ implies } Tx \perp y\}$, where $L_b(E)$ is the set of all the order bounded operators on $E$. This set is not only a Riesz algebra but also an $f$-algebra.

**Example 1.5.** Let $X$ be a Riesz space and $Orth(X)$ be the set of orthomorphisms on $X$. Then define the map $\mu: X \to Orth(E)$ by $\mu(x)(f) = |f|(|x|)$. Thus, one can get that $(X, \mu, Orth(E))$ is an $ALNS$.

## 2. BASIC RESULTS

In this section, the concept of convergence is introduced on algebraic lattice normed spaces.

**Definition 2.1.** Let $(X, \mu, E)$ be an $ALNS$. Then a net $(x_\alpha)_{\alpha \in A}$ in $X$ is said to be multiplicative $\mu$-convergent to $x \in X$ if

$$\mu(x_\alpha - x) \cdot u \xrightarrow{o} 0$$

holds for all $u \in E_+$. Then it is abbreviated as $x_\alpha \xrightarrow{m\mu} x$.

It is clear that $x_\alpha \xrightarrow{m\mu} x$ is the same as saying $\mu(x_\alpha - x) \xrightarrow{mo} 0$. Also, for a Riesz algebra $E$, it follows that the $bo$-convergence coincides with the $m\mu$-convergence on the $ALNS$ $(X, |\cdot|, X)$ [14]. Also, for a lattice normed space $X$, the norm convergence is the same with the $m\mu$-convergence on the $ALNS$ $(X, \|\cdot\|, \mathbb{R})$. The following useful lemma is frequently used, so it is important to keep in mind it, which can be obtained by using the properties of Riesz algebras.

**Lemma 2.2.** If $y \leq x$ for $x$ and $y$ in a Riesz algebra $E$ then $u \cdot y \leq u \cdot x$ for all $u \in E_+$.

Recall that an element $x$ in an $f$-algebra $E$ is called *nilpotent* whenever $x^n = 0$ for some natural number $n$. The algebra $E$ is called *semiprime* if the only nilpotent element in E is the null element [10]. Now, let's begin with the next basic properties of the $m\mu$-convergence which directly can be gotten from Lemma 2.1. and the inequality $||x| - |y|| \leq |x - y| \leq |x| + |y|$ in Riesz spaces, so the proof of the following results are omitted.

**Proposition 2.3.** Let $x_\alpha \xrightarrow{m\mu} x$ and $y_\beta \xrightarrow{m\mu} y$ be in an $ALNS\ (X,\mu,E)$. Then

1. $x_\alpha \xrightarrow{m\mu} x$ if and only if $(x_\alpha - x) \xrightarrow{m\mu} 0$ ,
2. $x_{\alpha_\theta} \xrightarrow{m\mu} x$ for each subnet $(x_{\alpha_\theta})$ of $(x_\alpha)$,
3. $\lambda x_\alpha + \sigma y_\beta \xrightarrow{\mu_f} \lambda x + \sigma y$ for each $\lambda, \sigma \in \mathbb{R}$,
4. if $x_\alpha \xrightarrow{m\mu} x$ and $x_\alpha \xrightarrow{m\mu} y$ then $x = y$ whenever $E$ is semiprime $f$-algebra,
5. $|x_\alpha| \xrightarrow{m\mu} |x|$.

The $m\mu$-continuity of lattice operations in $ALNS$s are obtained by the following sense.

**Proposition 2.4.** Consider two nets $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ in an $ALNS\ (X,\mu,E)$. Then $x_\alpha \xrightarrow{m\mu} x$ and $y_\beta \xrightarrow{m\mu} y$ implies $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{m\mu} x \vee y$.

*Proof:* Assume $x_\alpha \xrightarrow{m\mu} x$ and $y_\beta \xrightarrow{m\mu} y$. Then the assumption implies that there are two nets $(t_\gamma)_{\gamma\in\Gamma} \downarrow 0$ and $(z_\lambda)_{\lambda\in\Lambda} \downarrow 0$ in $E$. Also, for each $(\gamma,\lambda) \in \Gamma \times \Lambda$ there exist $\alpha_\gamma \in A$ and $\beta_\lambda \in B$ so that $\mu(x_\alpha - x)\cdot u \le z_\gamma$ and $\mu(y_\beta - y)\cdot u \le w_\lambda$ for all $\alpha \ge \alpha_\gamma$ and $\beta \ge \beta_\lambda$, and for each positive vector $u$. By applying the inequality $|x \vee y - x \vee z| \le |y - z|$ [4], one can see that

$$\begin{aligned}\mu(x_\alpha \vee y_\beta - x \vee y)\cdot u &= \mu(|x_\alpha \vee y_\beta - x_\alpha \vee y + x_\alpha \vee y - x \vee y|)\cdot u \\ &\le \mu(|x_\alpha \vee y_\beta - x_\alpha \vee y|)\cdot u + \mu(|x_\alpha \vee y - x \vee y|)\cdot u \\ &\le \mu(|y_\beta - y|)\cdot u + \mu(|x_\alpha - x|)\cdot u \le w_\lambda + z_\gamma\end{aligned}$$

for all $\alpha \ge \alpha_\gamma$ and $\beta \ge \beta_\lambda$ and for every $u \in E_+$. Hence, $\mu(x_\alpha \vee y_\beta - x \vee y)\cdot \mathrm{u} \xrightarrow{o} 0$ because of $(w_\lambda + z_\gamma) \downarrow 0$, that is, one can get $(x_\alpha \vee y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{m\mu} x \vee y$.

Let $E^\delta$ be a Dedekind complete Riesz space and $E$ be another Riesz space. Then $E^\delta$ is called order completion of $E$ if $E$ is isomorphic to a majorizing order dense Riesz space subspace of $E^\delta$. It is known that Riesz space has a unique order completion if it is Archimedean [2]. Thus, the following work can be given.

**Theorem 2.5.** Let $(x_\alpha)_{\alpha\in A}$ be a net in an $ALNS\ (X,\mu,E)$. Then $x_\alpha \xrightarrow{m\mu} 0$ in $E$ if and only if $x_\alpha \xrightarrow{m\mu} 0$ in the $ALNS\ (X,\mu^\delta,E^\delta)$, where $\mu^\delta(x) = \sup\{y \in E: y \le \mu(x)\}$ for all $x \in X$.

*Proof:* Suppose that $(x_\alpha)$ $m\mu$-converges to zero in $(X,\mu,E)$. That is, $\mu(x_\alpha)\cdot u \xrightarrow{o} 0$ in $E$ for all $u \in E_+$. Thus, $\mu(x_\alpha)\cdot u \xrightarrow{o} 0$ in $E^\delta$ for all $u \in E_+$ [9]. Consider an arbitrary positive vector $w \in E^\delta$. Since $E^\delta$ is order completion of $E$, $E$ majorizes $E^\delta$ i.e., there is a positive element $u_w \in E_+$ such that $w \le u_w$. Thus, one can get that $\mu^\delta(x_\alpha)\cdot w \le \mu(x_\alpha)\cdot u_w$ because of $\mu^\delta(x_\alpha) \le \mu(x_\alpha)$. Hence, it can be seen that $\mu^\delta(x_\alpha)\cdot w \xrightarrow{o} 0$ in $E^\delta$. It means that $x_\alpha \xrightarrow{m\mu} 0$ in the order completion $E^\delta$ because $w \in E^\delta_+$ is arbitrary.

Conversely, suppose $x_\alpha \xrightarrow{m\mu} 0$ in $E^\delta$. That is, $\mu^\delta(x_\alpha)\cdot u \xrightarrow{o} 0$ in $E^\delta$ for every $u \in E_+^\delta$. In particular, $\mu^\delta(x_\alpha)\cdot v \xrightarrow{o} 0$ in $E^\delta$ for each $v \in E_+$. As a result, $\mu^\delta(x_\alpha)\cdot v \xrightarrow{o} 0$ in $E$ holds for all $v \in E_+$ [9]. Hence, $\mu(x_\alpha)\cdot v \xrightarrow{o} 0$ in $E$ for all $v \in E_+$, and so, $x_\alpha \xrightarrow{m\mu} 0$ hold in $E$.

**Definition 2.6.** Let $(X,\mu,E)$ be an $ALNS$ and $Y$ be a subset of $X$. Then $Y$ is said to be $m\mu$-closed if, for any net $(x_\alpha)_{\alpha\in A}$ in $Y$ that is $m\mu$-convergent to $x \in X$, it satisfies that $x \in Y$.

**Remark 2.7.** In $LNFA$s, every band is $m\mu$-closed. Indeed, Let $B$ be given a band in an $LNFA$ $(X,\mu,E)$. Take a net $(x_\alpha)_{\alpha\in A}$ in $B$ such that $x_\alpha \xrightarrow{m\mu} x$. Thus, by using Proposition 2.4., $x_\alpha| \wedge |z| \xrightarrow{m\mu} |x| \wedge |z|$ for any $z \in B^\perp$. Since $(x_\alpha)$ in $B$ and $z \in B^\perp$, $|x_\alpha| \wedge |z| = 0$ for all $\alpha$. Thus, $|x| \wedge |y| = 0$, and so $x \in B^{\perp\perp} = B$.

The following useful property can be directly gotten as a result of Proposition 2.4.

**Proposition 2.8.** The positive cone $X_+$ in an $ALNS$ $(X,\mu,E)$ is $m\mu$-closed.

**Proposition 2.9.** Every monotone $m\mu$-convergent net in an $ALNS$ order converges to its $m\mu$-limit.

*Proof:* Assume $(x_\alpha)_{\alpha\in A}$ is an increasing net in an $ALNS$ $(X,\mu,E)$. Then it is enough to show that $x_\alpha \xrightarrow{m\mu} x$ implies $x_\alpha \uparrow x$. Let's fix an arbitrary index $\alpha$. Then $x_\beta - x_\alpha \in X_+$ for $\beta \geq \alpha$. From Proposition 2.8., $x_\beta - x_\alpha \xrightarrow{m\mu} x - x_\alpha \in X_+$. Then it can be seen that $x \geq x_\alpha$ for each $\alpha$. Thus $x$ is an upper bound of $(x_\alpha)$ because $\alpha$ is arbitrary. Take $y$ as another upper bound of $(x_\alpha)$, i.e., $y \geq x_\alpha$ for all $\alpha$. Then, again by Proposition 2.8., $y - x_\alpha \xrightarrow{m\mu} y - x \in X_+$, or $y \geq x$. Therefore, one can obtain $x_\alpha \uparrow x$.

Let $(X,\mu,E)$ be an $ALNS$ and $F$ be a Riesz subalgebra of $E$. For a net $(x_\alpha)_{\alpha\in A}$ in $X$, $y_\alpha \xrightarrow{m\mu} 0$ in $(X,\mu,E)$ implies $x_\alpha \xrightarrow{m\mu} 0$ in $(X,\mu,F)$.

**Theorem 2.10.** Let $(X,\mu,E)$ be an $ALNS$ and $F$ be a Riesz subalgebra of $E$. Assume $(x_\alpha)_{\alpha\in A}$ is a net in $X$ such that $x_\alpha \xrightarrow{m\mu} 0$ in $(X,\mu,F)$. Then $y_\alpha \xrightarrow{m\mu} 0$ in $(X,\mu,E)$ if each one of the following cases holds: $F$ is majorizing in $E$, $F$ is a projection band in $E$, or if, for each $u \in E$, there are element $f_1, f_2 \in F$ such that $|u - f_1| \leq |f_2|$.

*Proof:* Suppose $(x_\alpha)_{\alpha\in A}$ is an $m\mu$-convergent to zero net in $(X,\mu,F)$. Take a fixed $u \in E_+$.

Firstly, assume $F$ is majorizing in $E$. Thus, there is $f \in F_+$ such that $u \leq f$. So, one can see the following inequality

$$\mu(x_\alpha)\cdot u \leq \mu(x_\alpha)\cdot f.$$

Since $\mu(x_\alpha)\cdot f \xrightarrow{o} 0$, $\mu(x_\alpha)\cdot u \xrightarrow{o} 0$. That is, $x_\alpha \xrightarrow{m\mu} 0$ in $(X,\mu,E)$.

Secondly, let $F$ be a projection band in $E$. Then, by Theorem 1.41.(1) [4], $E = F \oplus F^\perp$ and $F = F^{\perp\perp}$. So, one can get positive vectors $u_1 \in F_+$ and $u_2 \in F_+^\perp$ such that $u = u_1 + u_2$. It is known that $(x_\alpha)$ in $F$ and $u_2 \in F^\perp$, and so $x_\alpha \wedge u_2 = 0$ for all indexes $\alpha$. Thus, $x_\alpha \cdot u = 0$ for each $\alpha$ [6]. As a result, by the following equality $\mu(x_\alpha)\cdot u = \mu(x_\alpha)\cdot(u_1 + u_2) = \mu(x_\alpha)\cdot u_1 \xrightarrow{o} 0$, $\mu(x_\alpha)\cdot u \xrightarrow{o} 0$ in $E$. Therefore, $x_\alpha \xrightarrow{m\mu} 0$ in $(X,\mu,E)$.

Lastly, assume there exist some elements $f_1, f_2 \in F$ such that $|u-y| \leq |x|$ for the given $u \in E_+$. Then

$$\mu(x_\alpha) \cdot u \leq \mu(x_\alpha) \cdot |u - f_1| + \mu(x_\alpha) \cdot |f_1| \leq \mu(x_\alpha) \cdot |f_2| + \mu(x_\alpha) \cdot |f_1|.$$

Thus, by using the $m\mu$-convergence of $(x_\alpha)_{\alpha\in A}$ in $(X, \mu, F)$, it is known $\mu(x_\alpha) \cdot |f_1| \xrightarrow{o} 0$ and $\mu(x_\alpha) \cdot |f_2| \xrightarrow{o} 0$, and so $\mu(x_\alpha) \cdot u \xrightarrow{o} 0$. It means that $x_\alpha \xrightarrow{m\mu} 0$ in $(X, \mu, E)$ because $u$ is arbitrary in $E_+$.

**Definition 2.11.** Let $(X, \mu, E)$ be an $ALNS$, $A$ be a subset of $X$ and $z$ be a vector in $X$. Then

1. $z$ is said to be *$m\mu$-unit* if, for any $x \in X_+$, $\mu(x - x \wedge nz) \cdot u$ order converges to zero for all $u \in E_+$;

2. $A$ is said to be *$m\mu$-dense* in $X$ if, for any $a \in A$ and for any $0 \neq w \in \mu(X)$, there is $y \in X$ such that $\mu(a - y) \cdot u \leq w$ for all $u \in E_+$.

**Theorem 2.12.** Let $(X, \mu, E)$ be an $ALNS$, $z \in X_+$, and $I_z$ be the order ideal generated by $z$ in $X$. If $I_z$ is $m\mu$-dense in $X$ then $z$ is a $m\mu$-unit.

*Proof:* Take a non zero element $w \in \mu(X)$. Let's fix a positive vector $x \in X_+$. So, there is $y \in I_z$ such that $p(x - y) \cdot u \leq w$ for all $u \in E_+$ because $I_z$ is $\mu_f$-dense in $X$. One can observe the following inequality [4];

$$|y^+ \wedge x - x \wedge x| \leq |y^+ - x| = |y^+ - x^+| \leq |y - x|.$$

Hence, by replacing $y$ by $y^+ \wedge x$. Thus, assume without loss of generality that there is $y \in I_z$ such that $0 \leq y \leq x$ and $\mu(x - y) \cdot u \leq w$ for all $u \in E_+$. Therefore, for any $k \in \mathbb{N}$, there is $y_k \in I_z$ such that $0 \leq y_k \leq x$ and $\mu(x - y_k) \cdot u \leq \frac{1}{k} w$ for every $u \in E_+$. Then there is $j = j(k) \in \mathbb{N}$ such that $0 \leq y_k \leq jz$ because of $y_k \in I_z$. So, $0 \leq y_k \leq jz \wedge x$ holds.

As a result, for $n \geq j$, $x - x \wedge nz \leq x - x \wedge jz \leq x - y_k$, and so $\mu(x - x \wedge nz) \cdot u \leq \mu(x - y_k) \cdot u \leq \frac{1}{k} w$ for all $u \in E_+$. Therefore, $\mu(x - x \wedge nz) \cdot u \xrightarrow{o} 0$ for each $u \in E_+$, and so $z$ is a $m\mu$-unit.

**Remark 2.13.** Let $(X, \mu, E)$ be an $ALNS$. Then the followings hold:

1. Let $z$ be $m\mu$-unit in $X$. Take a positive reel number $\lambda$ and a positive vector $y \in X_+$. For any $x \in X_+$, it can be observed that

   $$\mu(x - n\alpha z \wedge x) = \alpha\mu(\frac{x}{\alpha} - nz \wedge \frac{x}{\alpha}) \text{ and } \mu(x - n(z + y) \wedge x) \leq \mu(x - x \wedge nz).$$

   Therefore, it can be easily seen that $\alpha e$ and $e + z$ are both $\mu$-units.

2. A positive vector $z$ in a Riesz space $X$ is called *strong order unit* if, for each $x \in X$, there exists an integer $m$ such that $|x| \leq mz$. If $z \in X$ is a strong unit, then $z$ is a $m\mu$-unit. Indeed, fix an element $x \in X_+$, then there is $j \in \mathbb{N}$ such that $x \leq jz$, so

$$\mu(x - x \wedge nz) \cdot u = 0$$

for any $n \geq j$ and for all $u \in E_+$.

3. A positive vector $z$ in a Riesz space $X$ is said to be a *weak order unit* whenever the band generated by $z$ satisfies $B_e = E$. Assume $z \in X$ is a $m\mu$-unit. Then $z$ is a weak unit. Suppose $x \wedge z = 0$. Thus, $x \wedge kz = 0$ for any $k \in \mathbb{N}$. So,

$$\mu(x - x \wedge kz) \cdot u = \mu(x) \cdot u = 0$$

for all $u \in E_+$ because $z$ is a $m\mu$-unit. Hence, $x = 0$. Therefore, by the fact that a vector $e > 0$ is a weak order unit if and only if $x \wedge z = 0$ implies $x = 0$ [4], $z$ is is a weak unit.

**Proposition 2.14.** Let $(X, \mu, E)$ be an $ALNS$. Then, for each $m\mu$-unit $z$ in $X \neq \{0\}$, $z > 0$ holds.

*Proof:* Suppose $z \neq 0$ is a $m\mu$-unit in $X \neq \{0\}$. It is enough to show that $z^- = 0$ because of $z = z^+ - z^-$. Assume $e^- > 0$. Then, for $x := z^-$, it can be obtained that

$$\mu(x - x \wedge nz) \cdot u = \mu\left(z^- - \left(z^- \wedge n(z^+ - z^-)\right)\right) \cdot u = \mu\left(z^- - (-nz^-)\right) \cdot$$

$$= \mu((n+1)z^-) \cdot u = (n+1)\mu(z^-) \cdot u$$

for all $u \in E_+$. Thus, $(n+1)\mu(z^-) \cdot u$ is not order convergent to zero. This is impossible because $z$ is a $m\mu$-unit. Therefore, $z^- = 0$, it means that $z > 0$.

## 3. CONCLUSION

The concept of $ALNS$ and the $m\mu$-convergence are introduced. The continuity of lattice operations in $ALNS$s with the $m\mu$-convergence are given. A relation of the $m\mu$-convergence between Riesz spaces and their Dedekind completions is shown. A relation between the $m\mu$- and the order convergences is seen. The notions of $m\mu$-unit and $m\mu$-density are defined. Some basic properties of $m\mu$-unit and $m\mu$-density are proved.

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