ORIGINAL PAPER

GAP FUNCTION WITH ERROR BOUND FOR RANDOM GENERALIZED VARIATIONAL-LIKE INEQUALITY PROBLEMS

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Abstract. *In this paper, we used random generalized variational-like inequality problem and establish a new gap function for this problem in a fuzzy environment. In addition, we used this gap function, to evaluate an error bound for random generalized variational-like inequality problem in a fuzzy environment.*
Keywords: *Gap function; variational-like inequality; fuzzy mapping and error bound.*

1. INTRODUCTION

Zadeh [1] brought the concept of fuzzy set theory in 1965. In pure and applied science, the fuzzy set theory came to be very interesting branch of mathematics and very fascinating among authors. There are so many applications of fuzzy set theory such as artificial intelligence, optimization problems in mathematics, control engineering, decision theory and others.

The problem of variational inequality is a very important field of mathematics, specifically very useful in the field of mathematical programming. One of the main attraction of this theory is its so many applications to physical interest. Various class of variational inequalities have been studied by so many authors in past few decades, random generalized variational-like inequality problem is one of the generalized varitaional ineqialty problem, which we used in this paper.

In past years, problems of variational inequalities studied in fuzzy mapping environment and so many reformulation of variational inequalities have been done. So many authors are working on different types of variational inquality problems in fuzzy enviornment.

Varitional-like inequaltiy problem was introduced by Parida et al. [2] in 1989. Varitional-like inequality problem have been extended and generalized in so many directions by using techniques of fuzzy theory.

The theory of variational inequality with respect to fuzzy mapping was introduced by Chang and Zhu [3] in 1989. For fuzzy mapping, so many authors studied different classes of variational inequality problems and thier existence of solutions, iterative algorithms; see examples [4-15].

Gap function is very important tool in convex optimization. Gap function is traditional approach, one of the main tool is to convert a variational inequalty problem into identical optimization problem. In past few years, so many efforts have been done to construct gap functions for different class of variational inequality problems; see examples [16-32].

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Error bound can be easily derived using gap functions, so gap function is very useful to study error bounds. Let $f(x)=0$ be any arbitrary equation defined on a set X and $y \in X$ be a solution of $f(x)=0$ then for all $z \in X$, the distance between a solution y and z is known as error bound. Error bound are very useful in the analysis of local or global convergence analysis schemes for solving variational inequality problems. Gap function and error bound in a fuzzy environment was first studied by Khan et al [33].

In this paper, our focus to study gap function for random generalized variational-like inequality problem in the fuzzy mapping evironment and error bound for this problem in term of gap function.

2. PRELIMINARIES

In this paper throughout, let us suppose (Δ, D) be a measurable space, where Δ be a set and D denotes a σ-algebra of subsets of Δ. Suppose H be real Hilbert space such that $\| . \|$ denotes the norm on H and $\langle .,. \rangle$ denotes the inner product on H. Let $B(H)$ be the family of Borel σ-fields in H, 2^H be family of subsets of H, $CB(H)$ be the family of closed and bounded subsets of H and $H(.,.)$ be a Hausdorff metric on $CB(H)$.

Throughout in this paper, we let a map $S: H \to \mathfrak{A}(H)$ as a fuzzy mapping defined on H so that $S(p)$ (or S_p in the sequel) as a fuzzy set defined on H and a membership function denoted by, $S_p(q)$, q in S_p, where $\mathfrak{A}(H)$ be the family of fuzzy sets over H. For the set $J \in \mathfrak{A}(H)$ and $\alpha \in [0,1]$, $(J)_\alpha = \{ p \in H : J(p) \geq \alpha \}$ is called α-cut set of J.

2.1. DEFINITION

A map $p: \Delta \to H$ is known as measurable if for every $A \in B(H)$,

$$\{ s \in \Delta | \, p(s) \in A \} \in D.$$

2.2. DEFINITION

A map $h: \Delta \times H \to H$ is known as random operator if for every $p \in H$,

$$h(s, p(s)) = p(s) \text{ is measurable.}$$

A random operator $h(s,.): H \to H$ is continuous then h is continuous.

2.3. DEFINITION

A multivalued or set-valued mapping $S: \Delta \to 2^H$ is known as measurable if for every $A \in B(H)$, $S^{-1}(B) = \{ s \in \Delta : S(s) \bigcup A \neq \phi \} \in D$.

2.4. DEFINITION

A map $x:\Delta\to H$ is known as measurable selection of a multivalued or set-valued map $S:\Delta\to 2^H$ if x is measurable and $x(s)\in S(s)$ for every $s\in\Delta$.

2.5. DEFINITION

A map $S:\Delta\times H\to 2^H$ is known as random multivalued or set-valued map if $S(.,p)$ is measurable, for every $p\in H$.

A random multivalued or set-valued map $S:\Delta\times H\to CB(H)$ is known as H-continuous if $S(s,.)$ is continuous in Hausdorff metric, for every $s\in\Delta$.

2.6. DEFINITION

A map $S:\Delta\to\mathfrak{A}(H)$ is known as measurable if for every $\alpha\in[0,1]$, $(S(.))_\alpha:\Delta\to 2^H$ is a measurable multivalued or set-valued map.

2.7. DEFINITION

A fuzzy map $S:\Delta\times H\to\mathfrak{A}(H)$ is known as random fuzzy mapping if $S(.,p):\Delta\to\mathfrak{A}(H)$ is measurable fuzzy map, for every $p\in H$.

From above definitions, we can clearly say that fuzzy maps, random multivalued or set-valued maps and multivalued or set-valued maps are some special cases of random fuzzy map.

Consider $\hat{S}:\Delta\times H\to\mathfrak{A}(H)$ be a random fuzzy map such that fulfil the following condition:

(P1): there exist a map $c:H\to[0,1]$ such that $(\hat{S}_{s,p})_{c(p)}\in CB(H)$, $\forall s\in\Delta,\forall p\in H$.

We can define S, a random multivalued or set-valued map by using $\hat{S}$, a random fuzzy map as follows:

$$S:\Delta\times H\to\mathfrak{A}(H),\ (s,p)\to(\hat{S}_{s,p})_{c(p)},\ \forall s\in\Delta,\forall p\in H.$$

S is known as random multivalued or set-valued map induced by the random fuzzy map $\hat{S}$.

Given map $c:H\to[0,1]$, the random fuzzy map $\hat{S}:\Delta\times H\to\mathfrak{A}(H)$ fulfils the condition (P1) and random operator $f:\Delta\times H\to H$ with $\mathrm{Img}(f)\bigcap\mathrm{dom}(\psi)\neq\phi$, we assume the following random generalized variational-like inequality problem, (in short we say RGVLIP):

To find the measurable maps $p,x:\Delta\to H$ such that for every $s\in\Delta$, $z(s)\in H$,

$$\hat{S}_{s,p}\left(x(s)\right)\geq c\left(x(s)\right),\ \left\langle x(s),\eta\left(z(s),f\left(s,p(s)\right)\right)\right\rangle+\psi\left(z(s)\right)-\psi\left(f\left(s,p(s)\right)\right)\geq 0, \qquad (1)$$

where $\eta : H \times H \to H$ is a map and $\psi : H \to \mathbb{R} \bigcup \{+\infty\}$ with closed effective domain, a proper, convex and lower semicontinuous function and the sub-differential of ψ is denoted by $\partial\psi$.

Also (p, x) is said to be random solution of RGVLIP (1).

SPECIAL CASES:

A. If we take $\eta\left(z(s), f\left(s, p(s)\right)\right) = z(s) - f\left(s, p(s)\right)$ in (1) then RGVLIP (1) reduces to random generalized variational inequality problem (RGVIP), in which to find $p(s) \in H$,

$$\left\langle x(s), z(s) - f\left(s, p(s)\right)\right\rangle + \psi\left(z(s)\right) - \psi\left(f\left(s, p(s)\right)\right) \geq 0, \ \forall z \in H. \tag{2}$$

B. If we consider c be a zero operator and $S : H \to H$ is a single-valued map then RGVLIP (1) reduces to generalized mixed variational inequality problem (GMVLIP), in which to find $p \in H$,

$$\left\langle S(p), \eta\left(z, f(p)\right)\right\rangle + \psi(z) - \psi\left(f(p)\right) \geq 0, \ \forall z \in H. \tag{3}$$

C. If $f(p) = p, \ \forall p$ then GMVLIP (3) reduces to mixed variational-like inequality problem (MVLIP), in which to find $p \in H$,

$$\left\langle S(p), \eta(z, p)\right\rangle + \psi(z) - \psi(p) \geq 0, \ \forall z \in H. \tag{4}$$

D. If we take $\psi(.)$ as a characteristic function of closed set K in H and $\eta : K \times K \to H$ be a map then MVLIP (4) reduces to variational-like inequality problem (VLIP), in which to find $p \in K$,

$$\left\langle S(p), \eta(z, p)\right\rangle \geq 0, \ \forall z \in K. \tag{5}$$

E. If we take $\eta(z, p) = z - p$ in (5) then VLIP (5) reduces into classical variational inequality problem(VIP), in which to find $p \in K$,

$$\left\langle S(p), z - p\right\rangle \geq 0, \ \forall z \in K. \tag{6}$$

The sub-differential at $p \in H$ is denoted $\partial\psi(p)$ and defined by:

$$\partial\psi(p) = \left\{x \in H : \psi(z) \geq \psi(p) + \left\langle x, \eta(z, p)\right\rangle\right\}, \forall z \in H,$$

and the point $x \in \partial\psi(p)$ is called sub-gradient of ψ at p.

For measurable map $\gamma:\Delta \to (0,+\infty)$, define the proximal map $Q^{\psi}_{\gamma(s)}:\Delta \times H \to \mathrm{dom}(\psi)$, as

$$Q^{\psi}_{\gamma(s)}(s,w(s)) = \arg\min_{z(s)\in H}\left\{\psi\left(z(s)\right)+\frac{1}{2\gamma(s)}\left\|z(s)-w(s)\right\|^{2}\right\},\ w(s)\in H,\ s\in\Delta,$$

where $\arg\min_{T\in Y} T(v)$ denotes the set of minimizers of the map $T:H \to \mathbb{R}\cup\{+\infty\}$ over the set Y of H.

Motivated from Khan et al [33], we construct a function for the problem RGVLIP (1),

$$V^{\psi}_{\gamma(s)}\left(s,p(s)\right)=\eta\left(f\left(s,p(s)\right),Q^{\psi,p}_{\gamma(s)}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right]\right),\forall\, p(s)\in H. \tag{7}$$

We will show that function $V^{\psi}_{\gamma(s)}\left(s,p(s)\right)$ plays an important role in fuzzy environment for RGVLIP (1).

2.8. DEFINITION

A random multivalued or set-valued map $S:\Delta\times H \to CB(H)$ is called strongly f-monotone if $\exists$ a measurable map $a:\Delta \to (0,+\infty)$ such that,

$$\left\langle x_1(s)-x_2(s),\eta\left(f\left(s,p_1(s)\right),f\left(s,p_2(s)\right)\right)\right\rangle \ge a(s)\left\|p_1(s)-p_2(s)\right\|^2,$$
$$\forall\, p_1(s),p_2(s)\in H,\forall\, x_1(s),x_2(s)\in H, s\in\Delta.$$

2.9. DEFINITION

A random map $f:\Delta\times H \to H$ is called Lipschitz continuous if $\exists$ a measurable map $l:\Delta \to (0,+\infty)$ such that,

$$\left\|f\left(s,p_1(s)\right)-f\left(s,p_2(s)\right)\right\|\le l(s)\left\|p_1(s)-p_2(s)\right\|,\forall\, p_1(s),p_2(s)\in H, s\in\Delta.$$

2.10. DEFINITION

A random multivalued or set-valued map $S:\Delta\times H \to CB(H)$ is called $\hat{H}$-Lipschitz continuous if $\exists$ a measurable map $\theta:\Delta \to (0,+\infty)$ such that,

$$\hat{H}\left(S\left(s,p_1(s)\right),S\left(s,p_2(s)\right)\right)\le\theta(s)\left\|p_1(s)-p_2(s)\right\|,\forall\, p_1(s),p_2(s)\in H, s\in\Delta.$$

2.11. DEFINITION

A map $\eta : H \times H \to H$ is called skew in H if

$$\eta\left(p(s), q(s)\right) + \eta\left(q(s), p(s)\right) = 0, \forall\, p(s), q(s) \in H, s \in \Delta.$$

2.12. DEFINITION

A map $\eta : H \times H \to H$ is called strongly monotone if $\exists$ a measurable map $\zeta : \Delta \to (0, +\infty)$ such that,

$$\left\langle p_1(s) - p_2(s), \eta\left(p_1(s), p_2(s)\right)\right\rangle \geq \zeta(s)\left\|p_1(s) - p_2(s)\right\|^2, \forall\, p_1(s), p_2(s) \in H, s \in \Delta.$$

2.13. DEFINITION

A map $\eta : H \times H \to H$ is called Lipschitz continuous if $\exists$ a measurable map $\mu : \Delta \to (0, +\infty)$ such that,

$$\left\|\eta\left(p_1(s), p_2(s)\right)\right\| \leq \mu(s)\left\|p_1(s) - p_2(s)\right\|, \forall\, p_1(s), p_2(s) \in H, s \in \Delta.$$

2.14. DEFINITION

A funtion $g : H \to \mathbb{R}$ is known as gap function for RGVLIP (1), if it fulfils the following two conditions:

A. $g(p) \geq 0, \forall\, p \in H$,

B. $g(q) = 0$ if and only if $q \in H$ is a solution of RGVLIP (1).

Now we will show that the function $V_{\gamma(s)}^{\psi}\left(s, p(s)\right)$ is a gap function for RGVLIP (1).

3. MAIN RESULT

3.1. LEMMA

Let $\eta : H \times H \to H$ be a mapping such that $\eta\left(p(s), p(s)\right) = 0, \forall\, p(s) \in H, s \in \Delta.$ For every measurable map $\gamma : \Delta \to (0, +\infty)$ and for all $s \in \Delta, V_{\gamma(s)}^{\psi}\left(s, p(s)\right) = 0$ if and only if, the measurable map $p : \Delta \to H$ is a solution of RGVLIP (1).

Proof: Let $V^{\psi}_{\gamma(s)}\left(s, p(s)\right)=0$, this can be written as,

$$\eta\left(f\left(s, p(s)\right), Q^{\psi,p}_{\gamma(s)}\left[f\left(s, p(s)\right)-\gamma(s)x(s)\right]\right)=0,$$

if and only if,

$$f\left(s, p(s)\right)=Q^{\psi,p}_{\gamma(s)}\left[f\left(s, p(s)\right)-\gamma(s)x(s)\right],$$

which is clearly equivalent to,

$$f\left(s, p(s)\right)=\text{ arg min}_{z(s)\in H}\left\{\psi\left(z(s)\right)+\frac{1}{2\gamma(s)}z(s)-\left(f\left(s, p(s)\right)-\gamma(s)x(s)\right)^2\right\},$$

So by optimality conditions, which are necessary an sufficient (by convexity), this is equivalent to,

$$0\in\partial\psi\left(f\left(s, p(s)\right)\right)+\frac{1}{\gamma(s)}\left\{f\left(s, p(s)\right)-\left(f\left(s, p(s)\right)-\gamma(s)x(s)\right)\right\}=\partial\psi(f\left(s, p(s)\right)+x(s),$$

this implies, $-x(s)\in\partial\psi(f\left(s, p(s)\right)$, then using the definition of sub-gradient of ψ, the above is equivalent to,

$$\psi\left(z(s)\right)\geq\psi\left(f\left(s, p(s)\right)\right)+\left\langle -x(s), \eta\left(z(s), f\left(s, p(s)\right)\right)\right\rangle, \forall z(s)\in H, s\in\Delta,$$

that is,

$$\left\langle x(s), \eta\left(z(s), f\left(s, p(s)\right)\right)\right\rangle+\psi\left(z(s)\right)-\psi\left(f\left(s, p(s)\right)\right)\geq 0,$$

thus $p(s)$ solves RGVLIP (1).

3.2. LEMMA [32]

Let a random multivariable or set-valued map $S:\Delta\times H\to CB(H)$ is called $\hat{H}$-Lipschitz continuous then for every measurable map $p:\Delta\to H$, the multivalued or set-valued map $S(., p(.)):\Delta\to CB(H)$ is measurable.

3.3. LEMMA [32]

Let $S_1, S_2:\Delta\to CB(H)$ be two measurable multivalued or set-valued maps and $\delta>0$. Let $x_1:\Delta\to H$ be a measurable selection of S_1, then $\exists$ a measurable selection $x_2:\Delta\to H$ such that,

$$\left\|x_1(s)-x_2(s)\right\| \le (1+\delta)\hat{H}\left(S_1(s),S_2(s)\right), \forall s \in \Delta.$$

3.1. THEOREM

Let $\eta : H \times H \to H$ be a mapping such that $\eta\left(p(s),p(s)\right)=0, \forall p(s)\in H,\ s\in\Delta$. Let $p:\Delta \to H$ and $\gamma:\Delta\to(0,+\infty)$ be two measurable maps then for every $p(s)\in H$ and $s\in\Delta$, $V^{\psi}_{\gamma(s)}\left(s,p(s)\right)$ is a gap function for RGVLIP (1).

Proof: A. Cleary we can see that, $\left\|V^{\psi}_{\gamma(s)}\left(s,p(s)\right)\right\|\ge 0,\ \forall p(s)\in H,\ s\in\Delta$.

B. So now for $q(s)\in H$, we let $\left\|V^{\psi}_{\gamma(s)}\left(s,q(s)\right)\right\|=0$, which is equivalent, $V^{\psi}_{\gamma(s)}\left(s,q(s)\right)=0$, this can be written as,

$$\eta\left(f\left(s,q(s)\right),Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right)=0,$$

if and only if,

$$f\left(s,q(s)\right),=Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right],$$

by using optimality conditions which are necessary and sufficient (by convexity) with respect to $Q^{\psi,q}_{\gamma(s)}$, above is equivalent to,

$$0\in\partial\psi\left(Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right)+\frac{1}{\gamma(s)}\left\{Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]-\left(f\left(s,q(s)\right)-\gamma(s)x(s)\right)\right\},$$

this implies,

$$-x(s)+\frac{1}{\gamma(s)}\left\{f\left(s,q(s)\right)-Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right\}\in\partial\psi\left(Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right),$$

by definition of sub-gradient, equivalently, we can write,

$$\psi\left(z(s)\right)\ge\psi\left(Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right)$$
$$+\left\langle -x(s)+\frac{1}{\gamma(s)}\left\{f\left(s,q(s)\right)-Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right\},\eta\left(z(s),Q^{\psi,q}_{\gamma(s)}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right)\right\rangle,$$
$$\forall z(s)\in H, s\in\Delta,$$

This shows $q(s)$ is a solution of RGVLIP (1).

3.4. LEMMA (UNIQUENESS OF SOLUTION)

Let $\eta : H\times H \to H$ be a mapping such that $\eta\left(p(s), p(s)\right) = 0, \forall\, p(s) \in H,\ s\in\Delta$ and η is skew in H. Let $f : \Delta\times H \to H$ be a random map and $\psi : H \to \mathbb{R}\cup\{+\infty\}$ be real-valued map. Also let $\hat{S} : \Delta\times H \to \mathfrak{A}(H)$ be a random fuzzy map such that fulfil the condition (P1), and a random fuzzy map $\hat{S}$ induced, a random multi-valued or set-valued map $S:\Delta\times H \to CB(H)$, which is f-strongly monotone with a measurable map $a:\Delta\to(0,+\infty)$, then the RGVLIP (1) admits a unique solution.

Proof: Let two measurable maps $p_1, p_2 : \Delta \to H$ be solutions of RGVLIP (1) then, for $z(s)\in H,\ s\in\Delta$, we have,

$$\left\langle x_1(s), \eta\left(z(s), f\left(s, p_1(s)\right)\right)\right\rangle + \psi\left(z(s)\right) - \psi\left(f\left(s, p_1(s)\right)\right) \geq 0, \tag{8}$$

$$\left\langle x_2(s), \eta\left(z(s), f\left(s, p_2(s)\right)\right)\right\rangle + \psi\left(z(s)\right) - \psi\left(f\left(s, p_2(s)\right)\right) \geq 0, \tag{9}$$

taking $z(s) = f\left(s, p_1(s)\right)$ in (9) and $z(s) = f\left(s, p_2(s)\right)$ in (8), also using skew property of η then adding, we get,

$$\left\langle x_2(s) - x_1(s), \eta\left(f\left(s, p_1(s)\right), f\left(s, p_2(s)\right)\right)\right\rangle \geq 0,$$

using the definition of S, that is f-strongly monotone property of S for the measurable map $a:\Delta\to(0,+\infty)$, then we can write above as,

$$0 \leq -\left\langle x_1(s) - x_2(s), \eta\left(f\left(s, p_1(s)\right), f\left(s, p_2(s)\right)\right)\right\rangle \leq -a(s)\left\| p_1(s) - p_2(s)\right\|^2,$$

this implies, $\left\| p_1(s) - p_2(s)\right\|^2 = 0$, so that we can write, $p_1(s) = p_2(s)$, which is evidence that RGVLIP (1) has a unique solution.

Now we will find the error bound for the solution of RGVLIP (1) with the help of $V^{\psi}_{\gamma(s)}\left(s, p(s)\right)$.

3.2. THEOREM

Let $\eta : H\times H \to H$ be a mapping such that,

A. η is skew in H,

B. $\eta\left(p(s), q(s)\right) + \eta\left(q(s), r(s)\right) = \eta\left(p(s), r(s)\right), \forall\, p(s), q(s), r(s) \in H, s\in\Delta$.

Let $p_0(s)$ solves the RGVLIP (1), for all $s\in\Delta$. Let $\hat{S}:\Delta\times H\to\mathfrak{A}(H)$ be a random fuzzy map such that fulfil the condition (P1), and a random fuzzy map $\hat{S}$ induced, a random multi-valued or set-valued map $S:\Delta\times H\to CB(H)$. Let $f:\Delta\times H\to H$ be a random map and $\psi:H\to\mathbb{R}\cup\{+\infty\}$ be real-valued map such that,

A. for every $s\in\Delta$, the measurable map $\hat{S}$, which is strongly f-monotone and $\hat{H}$-Lipschitz continuous with measurable maps $a,\theta:\Delta\to(0,+\infty)$, respectively;

B. for every $s\in\Delta$, the Lipschitz continuous map $f(s,.)$ with measurable map $l:\Delta\to(0,+\infty)$;

then for every $p(s)\in H, s\in\Delta$ we get,

$$\|p(s)-p_0(s)\|\le\frac{\theta(s)\gamma(s)(1+\delta)\left\|V_{\gamma(s)}^{\psi}(s,p(s))\right\|+\mu(s)l(s)\left\|f(s,p(s))-Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right]\right\|}{a(s)\gamma(s)}.$$

Poof: $\forall s\in\Delta$, since $p_0(s)$ is the solution of the RGVLIP (1) then,

$$\left\langle x_0(s),\eta\left(z(s),f(s,p_0(s))\right)\right\rangle+\psi(z(s))-\psi\left(f(s,p_0(s))\right)\ge 0,\forall z(s)\in H,s\in\Delta,$$

taking $z(s)=Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right]$ in above inequality, then,

$$\begin{aligned}&\left\langle x_0(s),\eta\left(Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right],f(s,p_0(s))\right)\right\rangle\\&+\psi\left(Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right]\right)-\psi\left(f(s,p_0(s))\right)\ge 0,\end{aligned}\tag{10}$$

for fixed $p(s)\in H, s\in\Delta$ and measurable map $\gamma:\Delta\to(0,+\infty)$, we can write,

$$\begin{aligned}\left[f(s,p(s))-\gamma(s)x(s)\right]&\in(I+\gamma(s)\partial\psi)(I+\gamma(s)\partial\psi)^{-1}\left[f(s,p(s))-\gamma(s)x(s)\right]\\&=(I+\gamma(s)\partial\psi)Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right],\end{aligned}$$

this implies,

$$-x(s)+\frac{1}{\gamma(s)}\left\{f(s,p(s))-Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right]\right\}\in\partial\psi\left(Q_{\gamma(s)}^{\psi,p_0}\left[f(s,q(s))-\gamma(s)x(s)\right]\right),$$

by definition of sub-gradient,

$$\begin{aligned}&\psi(z(s))\ge\psi\left(Q_{\gamma(s)}^{\psi,p_0}\left[f(s,q(s))-\gamma(s)x(s)\right]\right)\\&+\left\langle -x(s)+\frac{1}{\gamma(s)}\left\{f(s,p(s))-Q_{\gamma(s)}^{\psi,p_0}\left[f(s,p(s))-\gamma(s)x(s)\right]\right\},\eta\left(z(s),Q_{\gamma(s)}^{\psi,p_0}\left[f(s,q(s))-\gamma(s)x(s)\right]\right)\right\rangle,\end{aligned}$$

now putting $z(s) = f\left(s, p_0(s)\right)$, then we can write above inequality as,

$$\left\langle x(s) - \frac{1}{\gamma(s)}\left\{f\left(s, p(s)\right) - Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, p(s)\right) - \gamma(s)x(s)\right]\right\}, \eta\left(f\left(s, p_0(s)\right), Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right)\right\rangle$$

$$-\psi\left(f\left(s, p_0(s)\right)\right) + \psi\left(Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right) \geq 0,$$

now adding (10) and above inequality, we get, (also using skew property of η in (10)),

$$\left\langle \begin{array}{l} x(s) - x_0(s) - \dfrac{1}{\gamma(s)}\left\{f\left(s, p(s)\right) - Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, p(s)\right) - \gamma(s)x(s)\right]\right\} \\ , \eta\left(f\left(s, p_0(s)\right), Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right) \end{array} \right\rangle \geq 0,$$

using property (B) of η,

$$\gamma(s)\left\langle x(s) - x_0(s), \eta\left(f\left(s, p_0(s)\right), f\left(s, p(s)\right)\right)\right\rangle$$

$$+\gamma(s)\left\langle x(s) - x_0(s), \eta\left(f\left(s, p(s)\right), Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right)\right\rangle$$

$$-\left\langle f\left(s, p(s)\right) - Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, p(s)\right) - \gamma(s)x(s)\right], \eta\left(f\left(s, p_0(s)\right), f\left(s, p(s)\right)\right)\right\rangle$$

$$-\left\langle f\left(s, p(s)\right) - Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, p(s)\right) - \gamma(s)x(s)\right], \eta\left(f\left(s, p(s)\right), Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right)\right\rangle \geq 0,$$

using skew property of η, the above inequality can be written as,

$$\gamma(s)\left\langle x(s) - x_0(s), \eta\left(f\left(s, p(s)\right), Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right)\right\rangle$$

$$+\left\langle f\left(s, p(s)\right) - Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, p(s)\right) - \gamma(s)x(s)\right], \eta\left(f\left(s, p(s)\right), f\left(s, p_0(s)\right)\right)\right\rangle$$

$$\geq \gamma(s)\left\langle x(s) - x_0(s), \eta\left(f\left(s, p(s)\right), f\left(s, p_0(s)\right)\right)\right\rangle$$

$$+\left\langle f\left(s, p(s)\right) - Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, p(s)\right) - \gamma(s)x(s)\right], \eta\left(f\left(s, p(s)\right), Q_{\gamma(s)}^{\psi, p_0}\left[f\left(s, q(s)\right) - \gamma(s)x(s)\right]\right)\right\rangle,$$

by Cauchy-Schwarz inequality, we have,

$$\gamma(s)\left\|x(s)-x_0(s)\right\|\left\|\eta\left(f\left(s,p(s)\right),Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right)\right\|$$

$$+\left\|f\left(s,p(s)\right)-Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right]\right\|\left\|\eta\left(f\left(s,p(s)\right),f\left(s,p_0(s)\right)\right)\right\|$$

$$\geq\gamma(s)\left\langle x(s)-x_0(s),\eta\left(f\left(s,p(s)\right),f\left(s,p_0(s)\right)\right)\right\rangle$$

$$+\left\langle f\left(s,p(s)\right)-Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right],\eta\left(f\left(s,p(s)\right),Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,q(s)\right)-\gamma(s)x(s)\right]\right)\right\rangle,$$

using the definition of $\hat{H}$-Lipschitz continuity and strong monotonicity of S and η, and Lipschitz continuity of f and η we have,

$$\gamma(s)\theta(s)(1+\delta)\left\|p(s)-p_0(s)\right\|\left\|V_{\gamma(s)}^{\psi}\left(s,p(s)\right)\right\|$$

$$+\mu(s)l(s)\left\|f\left(s,p(s)\right)-Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right]\right\|\left\|p(s)-p_0(s)\right\|$$

$$\geq\gamma(s)a(s)\left\|p(s)-p_0(s)\right\|^2+\zeta(s)\left\|f\left(s,p(s)\right)-Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right]\right\|^2$$

that is,

$$\gamma(s)\theta(s)(1+\delta)\left\|p(s)-p_0(s)\right\|\left\|V_{\gamma(s)}^{\psi}\left(s,p(s)\right)\right\|$$

$$+\mu(s)l(s)\left\|f\left(s,p(s)\right)-Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right]\right\|\left\|p(s)-p_0(s)\right\|$$

$$\geq\gamma(s)a(s)\left\|p(s)-p_0(s)\right\|^2,$$

this implies our result, that is,

$$\left\|p(s)-p_0(s)\right\|\leq\frac{\theta(s)\gamma(s)(1+\delta)\left\|V_{\gamma(s)}^{\psi}\left(s,p(s)\right)\right\|+\mu(s)l(s)\left\|f\left(s,p(s)\right)-Q_{\gamma(s)}^{\psi,p_0}\left[f\left(s,p(s)\right)-\gamma(s)x(s)\right]\right\|}{a(s)\gamma(s)}.$$

4. CONCLUSION AND REMARK

In this paper, our clear focus to study the theory of gap function for random generalized variational-like inequality problems and estimated error bound using definition of our gap function.

Our results are generalization of results given by Khan et al [33].

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REFERENCES

[1] Zadeh, L.A., *Information and Control*, **8**, 338, 1965.
[2] Parida, J., Sahoo, M., Kumar, A., *Bulletin of the Australian Mathematical Society*, **39**, 225, 1989.
[3] Chang, S.S., Zhu, Y.G., *Fuzzy Sets and Systems*, **32**, 359, 1989.
[4] Chang, S.S., *Fuzzy Sets and Systems*, **61**, 359, 1994.
[5] Chang, S.S., Huang, N.J., *Fuzzy Sets and Systems*, **55**, 227, 1993.
[6] Noor, M.A., *Fuzzy Sets and Systems*, **55**, 309, 1993.
[7] Noor, M.A., *Fuzzy Sets and Systems*, **97**, 101, 1998.
[8] Noor, M.A., *Fuzzy Sets and Systems*, **110**, 101, 2000.
[9] Huang, N.J., *Fuzzy Sets Syst.*, **105**, 437, 1999.
[10] Dai, H.X., *J. Comput. Appl. Math.*, **224**, 20, 2009.
[11] Huang, N.J., *Journal of Liaoning Normal University*, **18**, 89, 1995.
[12] Husain, T, Tarafdar, E, Yuan, X.Z., *Far East Journal of Mathematical Sciences*, **2**, 35, 1994.
[13] Yuan, X.Z., *Journal of Applied Mathematics and Stochastic Analysis*, **7**, 467, 1994.
[14] Rockafellar, R.T., *SIAM Journal on Control and Optimization*, **14**, 877, 1976.
[15] Chang, S.S., *Fixed Point Theory with Applications*, Chongqing Publishing House, Chongqing, 1984.
[16] Aussel, D., Dutta, J., *Journal of Optimization Theory and Applications*, **149**, 513, 2011.
[17] Aussel, D., Correa, R., Marechal, M., *Journal of Optimization Theory and Applications*, **151**, 474, 2011.
[18] Gupta, R., Mehra, A., *Journal of Global Optimization*, **53**, 737, 2012.
[19] Fukushima, M., *Mathematical Programming*, **53**, 99, 1992.
[20] Aussel, D., Gupta, R., Mehra, A., *Journal of Mathematical Analysis and Applications*, **407**, 270, 2013.
[21] Huang, L.R., Ng, K.F., *Journal of Optimization Theory and Applications*, **125**, 299, 2015.
[22] Khan, S.A., Chen, J.W., *Applied Mathematics and Computation,* **260**, 71, 2015.
[23] Khan, S.A., Chen, J.W., *Journal of Optimization Theory and Applications*, **166**, 767, 2015.
[24] Khan, S.A., Iqbal, J., Shehu, Y., *Journal of Inequalities and Applications*, **2015**, 417, 2015.
[25] Noor, M.A., *Journal of Mathematical Analysis and Applications*, **316**, 736, 2006.

[26] Qu, B., Wang, C.Y., Zhang, J.Z., *Journal of Optimization Theory and Applications*, **119**, 535, 2003.
[27] Solodov, M.V., Tseng, P., *Computational Optimization and Applications*, **17**, 255, 2000.
[28] Solodov, M.V., *Journal of Mathematical Analysis and Applications*, **287**, 405, 2003.
[29] Tang, G.J., Huang, N.J., *Taiwanese Journal of Mathematics*, **17**, 1267, 2013.
[30] Yamashita, N., Fukushima, M., *SIAM Journal on Control and Optimization*, **35**, 273, 1997.
[31] Yamashita, N., Taji, K., Fukushima, M., *Journal of Optimization Theory and Applications*, **92**, 439, 1997.
[32] Wu, J.H., Florian, M., Marcotte, P., *Mathematical Programming*, **61**, 281, 1993.
[33] Khan, S.A., Iqbal, J., Shehu, Y., *Journal of Inequalities and Applications*, **2016**, 47, 2016.