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SOME NOTES ON THE MODIFIED RIEMANNIAN EXTENSION $\tilde{g}_{\nabla,c}$ ON COTANGENT BUNDLE

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Abstract. *In this paper, we define the modified Riemannian extension $\tilde{g}_{\nabla,c}$ in the cotangent bundle T^*M, which is completely determined by its action on complete lifts of vector fields. Later, we obtain the covarient and Lie derivatives applied to the modified Riemannian extension with respect to the complete and vertical lifts of vector and kovector fields, respectively.*
Keywords: *covarient derivative; Lie derivative; modified Riemannian extension; complete lift; vertical lift.*

1. INTRODUCTION

1.1. THE COTANGENT BUNDLE

Let M be an $n-$dimensional smooth manifold and denote by $\pi: T^*M \to M$ its cotangent bundle whose fibres are cotangent spaces to M. Then T^*M is a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M can be used. Namely, a system of local coordinates $(U, x^i), i = 1, \ldots, n$ in M induces on T^*M a system of local coordinate $(\pi^{-1}(U), x^i, x^{\bar{\imath}} = p_i), \bar{\imath} = n + i = n + 1, \ldots, 2n$, where $x^{\bar{\imath}} = p_i$ are the components of covectors p in each cotangent space $T^*_x M, x \in U$ with respect to the natural coframe $\{dx^i\}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in U of a vector field X and a covector (1-form) field ω on M, respectively. Then the vertical lift ${}^V\omega$ of ω, the horizontal lift ${}^H X$ and the complete lift ${}^C X$ of X are given, with respect to the induced coordinates, by

$$ {}^V\omega = \omega_i \partial_{\bar{\imath}}, \tag{1.1}$$

$$ {}^H X = X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_{\bar{\imath}} \tag{1.2}$$

and

$$ {}^C X = X^i \partial_i - p_h \partial_i X^h \partial_{\bar{\imath}}, \tag{1.3}$$

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where $\partial_i = \frac{\partial}{\partial x^i}, \partial_{\bar{\imath}} = \frac{\partial}{\partial x^{\bar{\imath}}}$ and Γ_{ij}^h are the coefficients of a symmetric (torsion-free) affine connection ∇ in M.

Definition 1: The Lie bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas

$$\left[{}^H X, {}^H Y\right] = {}^H [X,Y] + {}^V\big(p \circ R(X,Y)\big) \tag{1.4}$$

$$\left[{}^H X, {}^V \omega\right] = {}^V (\nabla_X \omega)$$

$$\left[{}^V \theta, {}^V \omega\right] = 0$$

for any $X, Y \in \Im_0^1(M)$ and $\theta, \omega \in \Im_1^0(M)$, where R is the curvature tensor of the symmetric connection ∇ defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{X,Y}$ [1, 2].

1.2. MODIFIED RIEMANNIAN EXTENSION

Let M be an $n-$dimensional differentiable manifold and T^*M be its cotangent bundle. There is a well-known natural construction which yields, for any affine connection ∇ on M, a pseudo-Riemannian metric $\tilde{g}\nabla$ on T^*M. The metric $\tilde{g}\nabla$ is called the Riemannian extension of ∇. Riemannian extensions were originally defined by Patterson and Walker [3] and further studied by Afifi [4], thus relating pseudo-Riemannian properties of T^*M with the affine structure of the base manifold (M, ∇). Moreover, Riemannian extensions were also considered by Garcia-Rio et al. in [5] in relation to Osserman manifolds (see also Derdzinski [6]). Since Riemannian extensions provide a link between affine and pseudo-Riemannian geometries, some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemannian extension $\tilde{g}\nabla$. For instance, ∇ is projectively flat if and only if $\tilde{g}\nabla$ is locally conformally flat [4]. For Riemannian extensions, also see [7-15]. In [16, 17], the authors introduced a modification of the usual Riemannian extensions which is called the modified Riemannian extension.

Let M_{2k} be a $2k-$dimensional differentiable manifold endowed with an almost complex structure J and a pseudo-Riemannian metric g of signature (k,k) such that $g(JX,Y) = g(X,JY)$ for arbitrary vector fields X and Y on M_{2k}. Then the metric g is called the Norden metric. Norden metrics are referred to as anti-Hermitian metrics or B-metrics. The study of such manifolds is interesting because there exists a difference between the geometry of a $2k-$dimensional almost complex manifold with Hermitian metric and the geometry of a $2k-$dimensional almost complex manifold with Norden metric. A notable difference between Norden metrics and Hermitian metrics is that $G(X,Y) = g(X,JY)$ is another Norden metric, rather than a differential 2-form. Some authors considered almost complex Norden structures on the cotangent bundle [18-20].

In this paper, we will use a deformation of the Riemannian extension on the cotangent bundle T^*M over (M,∇) by means of a symmetric tensor field c on M, where ∇ is a symmetric affine connection on M. The metric is the socalled modified Riemannian extenson. The article is constructed as follows, firstly, we define the modified Riemannian extension $\tilde{g}_{\nabla,c}$ in the cotangent bundle T^*M, which is completely determined by its action on complete lifts of vector fields. Later, we obtain the covariant and Lie derivatives applied to modified Riemannian extension with respect to the complete and vertical lifts of vector and kovector fields, respectively.

For a given symmetric connection ∇ on an $n-$dimensional manifold M, the cotangent bundle T^*M can be equipped with a pseudo-Riemannian metric $\tilde{g}\nabla$ of signature (n,n): the Riemannian extension of ∇ [3], given by

$$\tilde{g}_\nabla\left({}^C X, {}^C Y\right) = -\gamma(\nabla_X Y + \nabla_Y X),$$

where ${}^C X, {}^C Y$ denote the complete lifts to T^*M of vector fields X, Y on M. Moreover, for any $Z \in \Im_0^1(M)$, $Z = Z^i \partial_i, \partial Z$ is the function on T^*M defined by $\gamma Z = p_i Z^i$ [1]. The Riemannian extension is expressed by

$$\tilde{g}_\nabla = \begin{pmatrix} -2p_h \Gamma_{ij}^h & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}$$

with respect to the natural frame.

Now we give a deformation of the Riemannian extension above by means of a symmetric (0,2)-tensor field c on M whose metric is called the modified Riemannian extension. The modified Riemannian extension is expressed by

$$\tilde{g}_{\nabla,c} = g_\nabla + \pi^* c = \begin{pmatrix} -2p_h \Gamma_{ij}^h + c_{ij} & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix} \tag{1.5}$$

with respect to the natural frame. It follows that the signature of $\tilde{g}_{\nabla,c}$ is (n,n) [1].

Denote by ∇ the Levi-Civita connection of a semi-Riemannian metric g. In this section, we will consider T^*M equipped with the modified Riemannian extension $\tilde{g}_{\nabla,c}$ over a pseudo-Riemannian manifold (M,g). Since the vector fields ${}^H X$ and ${}^V \omega$ span the module of vector fields on T^*M, any tensor field is determined on T^*M by their actions on ${}^H X$ and ${}^V \omega$. The modified Riemannian extension $\tilde{g}_{\nabla,c}$ has the following properties [1]:

$$\tilde{g}_{\nabla,c}\left({}^H X, {}^H Y\right) = c(X,Y) \tag{1.6}$$

$$\tilde{g}_{\nabla,c}\left({}^H X X, {}^V \omega\right) = g_{\nabla,c}\left({}^V \omega, {}^H X\right) = \omega(X)$$

$$\tilde{g}_{\nabla,c}\left({}^V \omega, {}^V \theta\right) = 0$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, which characterize $\tilde{g}_{\nabla,c}$.

We know see, from (1.2) and (1.3), that the complete lift X^C of $X \in \Im_0^1(M)$ is expressed by

$$X^C = X^H - (p(\nabla X))^V, \tag{1.7}$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$. Using (1.6) and (1.7), we have

$$\tilde{g}_{\nabla,c}(X^C, Y^C) = c(X,Y) - p(\nabla Y)(X) - p(\nabla X)(Y). \tag{1.8}$$

Since the tensor field $\tilde{g}_{\nabla,c} \in \Im_2^0(T^*M)$ is completely determined also by its action on vector fields type X^C and Y^C, we have an alternative characterization of $\tilde{g}_{\nabla,c}$ on T^*M: $\tilde{g}_{\nabla,c}$ is completely determined by the condition (1.8). Similarly, we get the following results

$$\begin{aligned} \text{i) } \tilde{g}_{\nabla,c}(X^C, Y^C) &= (X^H - (p(\nabla X))^V, Y^H - (p(\nabla Y))^V) \\ &= \tilde{g}_{\nabla,c}(X^H, Y^H) - \tilde{g}_{\nabla,c}(X^H, (p(\nabla Y))^V) \\ &\quad - \tilde{g}_{\nabla,c}((p(\nabla X))^V, (p(\nabla Y))^V) - \tilde{g}_{\nabla,c}((p(\nabla X))^V, Y^H) \\ &= c(X,Y) - p(\nabla Y)(X) - p(\nabla X)(Y) \end{aligned} \tag{1.9}$$

$$\begin{aligned} \text{ii) } \tilde{g}_{\nabla,c}(X^C, \omega^V) &= \tilde{g}_{\nabla,c}(X^H - (p(\nabla X))^V, \omega^V) \\ &= \tilde{g}_{\nabla,c}(X^H, \omega^V) - \tilde{g}((p(\nabla X))^V, \omega^V) \\ &= \omega(X) \end{aligned}$$

$$\begin{aligned} \text{iii) } \tilde{g}_{\nabla,c}(\omega^V, Y^C) &= \tilde{g}_{\nabla,c}(\omega^V, Y^H - (p(\nabla Y))^V) \\ &= \tilde{g}_{\nabla,c}(\omega^V, Y^H) - \tilde{g}_{\nabla,c}(\omega^V, (p(\nabla Y))^V) \\ &= \omega(Y) \end{aligned}$$

$$\text{iv) } \tilde{g}_{\nabla,c}(\omega^V, \theta^V) = 0$$

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_q^p(T^*M)$ the corresponding set on the cotangent bundle T^*M.

2. RESULTS

Definition 2: Let M^n be an $\mathrm{n}-$dimensional diferentiable manifold. Differantial transformation of algebra $\mathrm{T}(\mathrm{M}^\mathrm{n})$, defined by

$$D = \nabla_X : T(M^n) \to T(M^n), X \in \Im_0^1(M^n)$$

is called as covariant derivation with respect to vector field X if

$$\nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t, \tag{2.1}$$

$$\nabla_X f = Xf,$$

where $\forall f, g \in \Im_0^0(M^n), \forall X, Y \in \Im_0^1(M^n), \forall t \in \Im(M^n)$ (see [21], p.123).

On the other hand, a transformation defined by

$$\nabla : \Im_0^1(M^n) \times \Im_0^1(M^n) \to \Im_0^1(M^n)$$

is called as an affine connection (see for details [21-25]).

Proposition 1: Covarient differentiation with respect to the complete lift ∇^C of a symetric affine connection ∇ in M^n to $T^*(M^n)$ has the following properties:

$$\nabla^C_{\omega^V}\theta^V = 0, \nabla^C_{\omega^V}Y^C = -\gamma(\omega o(\nabla Y)) = -(p(\omega o(\nabla Y)))^V, \nabla^C_{X^C}\theta^V = (\nabla_X\theta)^V,$$

$$\begin{aligned}\nabla^C_{X^C}Y^C &= (\nabla_X Y)^C + \gamma(\nabla(\nabla_X Y + \nabla_Y X)) - \gamma(\nabla_X\nabla Y + \nabla_Y\nabla X)\\ &= (\nabla_X Y)^C + (p((\nabla(\nabla_X Y + \nabla_Y X)) - (\nabla_X\nabla Y + \nabla_Y\nabla X)))^V\end{aligned}$$

for $X, Y \in \Im^1_0(M^n), \theta, \omega \in \Im^0_1(M^n)$[2].

Proposition 2 Covarient differentiation with respect to the horizontal lift ∇^H of a symetric affine connection ∇ in M^n to $T^*(M^n)$ satisfies

$$\nabla^H_{X^H}Y^H = (\nabla_X Y)^H, \nabla^H_{\theta^V}\omega^V = 0, \tag{2.2}$$

$$\nabla^H_{X^H}\omega^V = (\nabla_X\omega)^V, \nabla^H_{\theta^V}Y^H = 0,$$

for any $X, Y \in \Im^1_0(M^n), \theta, \omega \in \Im^0_1(M^n)$ [25].

Theorem 1 Let $\tilde{g}_{\nabla,c}$ be the modified Riemannian extension, is defined by (1.9) and the complete lift ∇^C of symetric affine connection ∇ in M^n to $T^*(M^n)$. From Proposition (1) and Definition 2, we get the following results

i) $(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V, \xi^V) = 0,$

ii) $(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(\theta^V, \xi^V) = 0,$

iii) $(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V, Z^C) = 0,$

iv) $(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(\theta^V, Z^C) = 0,$

v) $(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(Y^C, \xi^V) = 0,$

vi) $(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(Y^C, \xi^V) = 0,$

vii) $(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(Y^C, Z^C) = p(\omega(\nabla Y))(Z) + p(\omega(\nabla Z))(Y),$

viii) $$\begin{aligned}(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(Y^C, Z^C) = {}& (\nabla_X c)(Y, Z) - (\nabla_X p(\nabla Z)Y) - (\nabla_X p(\nabla Y))Z\\ &+(p(\nabla(\nabla_X Y)))(Z) + (p(\nabla(\nabla_X Z)))(Y)\\ &-(p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))))(Z)\\ &-(p(\nabla(\nabla_X Z + \nabla_Z X) - (\nabla_X(\nabla Z) + \nabla_Z(\nabla X))))(Y),\end{aligned}$$

where the vertical lift $\omega^V \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ and complete lifts $X^C, \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$, defined by (1.1) and (1.3), respectively.

Proof:

i) $$(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) = \nabla^C_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^C_{\omega^V}\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^C_{\omega^V}\xi^V)$$
$$= 0$$

ii) $$(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) = \nabla^C_{X^C}\tilde{g}_{\nabla,c}(\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^C_{X^C}\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^C_{X^C}\xi^V)$$
$$= -\tilde{g}_{\nabla,c}((\nabla_X\theta)^V,\xi^V) - g_{\nabla,c}(\theta^V,(\nabla_X\xi)^V)$$
$$= 0$$

iii) $$(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^C) = \nabla^C_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,Z^C) - \tilde{g}_{\nabla,c}(\nabla^C_{\omega^V}\theta^V,Z^C) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^C_{\omega^V}Z^C)$$
$$= \nabla^C_{\omega^V}\theta(Z) + \tilde{g}_{\nabla,c}(\theta^V,(p(\omega(\nabla Z)))^V)$$
$$= \omega^V(\theta(Z))$$
$$= 0$$

iv) $$(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(\theta^V,Z^C) = \nabla^C_{X^C}\tilde{g}_{\nabla,c}(\theta^V,Z^C) - \tilde{g}_{\nabla,c}(\nabla^C_{X^C}\theta^V,Z^C) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^C_{X^C}Z^C)$$
$$= \nabla^C_{X^C}\theta(Z) - \tilde{g}_{\nabla,c}((\nabla_X\theta)^V,Z^C) - \tilde{g}_{\nabla,c}(\theta^V,(\nabla_X Z)^C$$
$$+(p(\nabla(\nabla_X Z + \nabla_Z X) - (\nabla_X(\nabla Z) + \nabla_Z(\nabla X))))^V)$$
$$= (\nabla_X\theta(Z)) - (\nabla_X\theta)(Z) - \tilde{g}_{\nabla,c}(\theta^V,(\nabla_X Z)^C)$$
$$-\tilde{g}_{\nabla,c}(\theta^V,(p(\nabla(\nabla_X Z + \nabla_Z X) - (\nabla_X(\nabla Z) + \nabla_Z(\nabla X))))^V)$$
$$= \nabla_X\theta(Z) - (\nabla_X\theta)(Z) - \theta(\nabla_X Z)$$
$$= 0$$

v) $$(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(Y^C,\xi^V) = \nabla^C_{\omega^V}\tilde{g}_{\nabla,c}(Y^C,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^C_{\omega^V}Y^C,\xi^V) - \tilde{g}(Y^C,\nabla^C_{\omega^V}\xi^V)$$
$$= \nabla^C_{\omega^V}\xi(Y) + \tilde{g}_{\nabla,c}((p(\omega(\nabla Y)))^V,\xi^V)$$
$$= \omega^V\xi(Y)$$
$$= 0$$

vi) $$(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(Y^C,\xi^V) = \nabla^C_{X^C}\tilde{g}_{\nabla,c}(Y^C,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^C_{X^C}Y^C,\xi^V) - \tilde{g}_{\nabla,c}(Y^C,\nabla^C_{X^C}\xi^V)$$
$$= \nabla^C_{X^C}\xi(Y) - \tilde{g}_{\nabla,c}((\nabla_X Y)^C - \tilde{g}_{\nabla,c}(Y^C,(\nabla_X\xi)^V)$$
$$+(p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))))^V,\xi^V)$$
$$= X^C(\xi(Y)) - \tilde{g}_{\nabla,c}((\nabla_X Y)^C,\xi^V) - (\nabla_X\xi)(Y)$$
$$-\tilde{g}_{\nabla,c}((p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))))^V,\xi^V)$$
$$= (\nabla_X\xi(Y) - \xi(\nabla_X Y) - (\nabla_X\xi)(Y))^V$$
$$= 0$$

vii) $$(\nabla^C_{\omega^V}\tilde{g}_{\nabla,c})(Y^C,Z^C) = \nabla^C_{\omega^V}\tilde{g}_{\nabla,c}(Y^C,Z^C) - \tilde{g}_{\nabla,c}(\nabla^C_{\omega^V}Y^C,Z^C) - \tilde{g}_{\nabla,c}(Y^C,\nabla^C_{\omega^V}Z^C)$$
$$= \omega^V(c(X,Y) - p(\nabla Y)(X) - p(\nabla X)(Y))$$
$$+\tilde{g}_{\nabla,c}((p(\omega(\nabla Y)))^V,Z^C) + \tilde{g}_{\nabla,c}(Y^C,(p(\omega(\nabla Z)))^V)$$

$$= p(\omega(\nabla Y))(Z) + p(\omega(\nabla Z))(Y)$$

viii) $$(\nabla^C_{X^C}\tilde{g}_{\nabla,c})(Y^C, Z^C) = \nabla^C_{X^C}\tilde{g}_{\nabla,c}(Y^C, Z^C) - \tilde{g}_{\nabla,c}(\nabla^C_{X^C}Y^C, Z^C) - \tilde{g}_{\nabla,c}(Y^C, \nabla^C_{X^C}Z^C)$$
$$= X^C(c(Y,Z) - p(\nabla Z)(Y) - p(\nabla Y)(Z))$$
$$-\tilde{g}_{\nabla,c}((\nabla_X Y)^C + (p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))))^V, Z^C)$$
$$-\tilde{g}_{\nabla,c}(Y^C, (\nabla_X Z)^C + (p(\nabla(\nabla_X Z + \nabla_Z X) - (\nabla_X(\nabla Z) + \nabla_Z(\nabla X))))^V)$$
$$= (\nabla_X c(Y,Z)) - (\nabla_X p(\nabla Z)(Y) + p(\nabla Y)(Z))$$
$$-c((\nabla_X Y), Z) + p(\nabla Z)(\nabla_X Y) + p(\nabla(\nabla_X Y))(Z)$$
$$-(p(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))))(Z)$$
$$-c(Y, (\nabla_X Z)) + (p(\nabla(\nabla_X Z)))(Y) + (p(\nabla Y))(\nabla_X Z)$$
$$-(p(\nabla(\nabla_X Z + \nabla_Z X) - (\nabla_X(\nabla Z) + \nabla_Z(\nabla X))))(Y)$$
$$= (\nabla_X c)(Y,Z) - (\nabla_X p(\nabla Z))Y - (\nabla_X p(\nabla Y))Z$$
$$+(p(\nabla(\nabla_X Y)))(Z) + (p(\nabla(\nabla_X Z)))(Y)$$
$$-(P(\nabla(\nabla_X Y + \nabla_Y X) - (\nabla_X(\nabla Y) + \nabla_Y(\nabla X))))(Z)$$
$$-P(\nabla(\nabla_X Z + \nabla_Z X) - (\nabla_X(\nabla Z) + \nabla_Z(\nabla X))))(Y),$$

where $f^V(\tilde{P}) = f(P)$ ($P, \tilde{P}$ are points of M and $T^*(M)$ respectively), $c(Y,Z)$ and $\omega(Y)$ are function on $T^*(M), X, Y, Z \in \Im^1_0(M)$, $\omega, \theta, \xi \in \Im^0_1(M), f \in \Im^0_0(M)$ and $\nabla_X\theta(Z) = (\nabla_X\theta)Z + \theta(\nabla_X Z)$.

Definition 3: Let M^n be an $n-$dimensional diferentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im^1_0(M^n)$ if

$$L_X f = Xf, \forall f \in \Im^0_0(M^n), \tag{2.3}$$

$$L_X Y = [X, Y], \forall X, Y \in \Im^1_0(M^n).$$

$[X, Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [23, 24, 2]

$$(L_X F)Y = [X, FY] - F[X, Y]. \tag{2.4}$$

Proposition 3: If $X \in \Im^1_0(M), \omega \in \Im^0_1(M)$ and $F, G \in \Im^1_1(M)$, then [2]

$$F^C\omega^V = (\omega o F)^V, F^C\gamma G = \gamma(GF),$$

$$F^C X^C = (FX)^C + \gamma(L_X F).$$

Theorem 2: Let $\tilde{g}_{\nabla,c}$ be the modified Riemannian extension, is defined by (1.9) and L_X the operator Lie derivation with respect to X. From (1.9), Definition (1) and Definition (3), we get the following results

i) $(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V, \xi^V) = 0,$

ii) $(L_{X^C}\tilde{g}_{\nabla,c})(\theta^V,\xi^V)=0$,

iii) $(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^C)=0$,

iv) $(L_{X^C}\tilde{g}_{\nabla,c})(\theta^V,Z^C)=0$,

v) $(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^C,\xi^V)=0$,

vi) $(L_{X^C}\tilde{g}_{\nabla,c})(Y^C,\xi^V)=0$,

vii) $(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^C,Z^C)=(L_Y\omega)(Z)+(L_Z\omega)(Y)$,

viii) $$\begin{aligned}(L_{X^C}\tilde{g}_{\nabla,c})(Y^C,Z^C)=&(L_Xc)(Y,Z)-(L_Xp(\nabla Z))(Y)-(L_Xp(\nabla Y))(Z)\\&+p(\nabla(L_XY))(Z)+p(\nabla(L_XZ))(Y),\end{aligned}$$

where the vertical lift $\omega^V\in\Im_0^1(T^*M^n)$ of $\omega\in\Im_1^0(M^n)$ and complete lifts $X^C,\in\Im_0^1(T^*M^n)$ of $X\in\Im_0^1(M^n)$, defined by (1.1) and (1.3), respectively.

Proof:

i) $$\begin{aligned}(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,\xi^V)&=L_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,\xi^V)-\tilde{g}(L_{\omega^V}\theta^V,\xi^V)-\tilde{g}_{\nabla,c}(\theta^V,L_{\omega^V}\xi^V)\\&=0\end{aligned}$$

ii) $$\begin{aligned}(L_{X^C}\tilde{g}_{\nabla,c})(\theta^V,\xi^V)&=L_{X^C}\tilde{g}_{\nabla,c}(\theta^V,\xi^V)-\tilde{g}_{\nabla,c}(L_{X^C}\theta^V,\xi^V)-\tilde{g}_{\nabla,c}(\theta^V,L_{X^C}\xi^V)\\&=-\tilde{g}_{\nabla,c}((L_X\theta)^V,\xi^V)-\tilde{g}_{\nabla,c}(\theta^V,(L_X\xi)^V)\\&=0\end{aligned}$$

iii) $$\begin{aligned}(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^C)&=L_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,Z^C)-\tilde{g}_{\nabla,c}(L_{\omega^V}\theta^V,Z^C)-\tilde{g}_{\nabla,c}(\theta^V,L_{\omega^V}Z^C)\\&=L_{\omega^V}\theta(Z)+\tilde{g}_{\nabla,c}(\theta^V,(L_Z\omega)^V)\\&=\omega^V\theta(Z)\\&=0\end{aligned}$$

iv) $$\begin{aligned}(L_{X^C}\tilde{g}_{\nabla,c})(\theta^V,Z^C)&=L_{X^C}\tilde{g}_{\nabla,c}(\theta^V,Z^C)-\tilde{g}_{\nabla,c}(L_{X^C}\theta^V,Z^C)-\tilde{g}_{\nabla,c}(\theta^V,L_{X^C}Z^C)\\&=L_{X^C}\theta(Z)-\tilde{g}_{\nabla,c}((L_X\theta)^V,Z^C)-\tilde{g}_{\nabla,c}(\theta^V,(L_XZ)^C)\\&=(L_X\theta(Z)-(L_X\theta)Z-\theta(L_XZ))^V\\&=0\end{aligned}$$

v) $$\begin{aligned}(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^C,\xi^V)&=L_{\omega^V}\tilde{g}_{\nabla,c}(Y^C,\xi^V)-\tilde{g}_{\nabla,c}(L_{\omega^V}Y^C,\xi^V)-\tilde{g}_{\nabla,c}(Y^C,L_{\omega^V}\xi^V)\\&=L_{\omega^V}\xi(Y)+\tilde{g}_{\nabla,c}((L_Y\omega)^V,\xi^V)\\&=0\end{aligned}$$

vi) $$\begin{aligned}(L_{X^C}\tilde{g}_{\nabla,c})(Y^C,\xi^V)&=L_{X^C}\tilde{g}_{\nabla,c}(Y^C,\xi^V)-\tilde{g}_{\nabla,c}(L_{X^C}Y^C,\xi^V)-\tilde{g}_{\nabla,c}(Y^C,L_{X^C}\xi^V)\\&=X^C(\xi(Y))-\tilde{g}_{\nabla,c}((L_XY)^C,\xi^V)-\tilde{g}_{\nabla,c}(Y^C,(L_X\xi)^V)\end{aligned}$$

$$= (L_X\xi(Y) - \xi(L_XY) - (L_X\xi)Y)^V$$
$$= 0$$

vii) $$(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^C,Z^C) = L_{\omega^V}\tilde{g}_{\nabla,c}(Y^C,Z^C) - \tilde{g}_{\nabla,c}(L_{\omega^V}Y^C,Z^C) - \tilde{g}_{\nabla,c}(Y^C,L_{\omega^V}Z^C)$$
$$= L_{\omega^V}(c(Y,Z) - p(\nabla Z)(Y) - p(\nabla Y)(Z))$$
$$+\tilde{g}_{\nabla,c}((L_Y\omega)^V,Z^C) + \tilde{g}_{\nabla,c}(Y^C,(L_Z\omega)^V)$$
$$= \omega^V(c(Y,Z) - p(\nabla Z)(Y) - p(\nabla Y)(Z)) + (L_Y\omega)(Z)$$
$$+(L_Z\omega)(Y)$$
$$= (L_Y\omega)(Z) + (L_Z\omega)(Y)$$

viii) $$(L_{X^C}\tilde{g}_{\nabla,c})(Y^C,Z^C) = L_{X^C}\tilde{g}_{\nabla,c}(Y^C,Z^C) - \tilde{g}_{\nabla,c}(L_{X^C}Y^C,Z^C) - \tilde{g}_{\nabla,c}(Y^C,L_{X^C}Z^C)$$
$$= X^C(c(Y,Z) - p(\nabla Z)(Y) - p(\nabla Y)(Z))$$
$$-\tilde{g}_{\nabla,c}((L_XY)^C,Z^C) - \tilde{g}_{\nabla,c}(Y^C,(L_XZ)^C)$$
$$= (L_Xc(Y,Z)) - c((L_XY),Z) - c(Y,(L_XZ))$$
$$-(L_Xp(\nabla Z)(Y) - p(\nabla Z)(L_XY))$$
$$-(L_Xp(\nabla Y)(Z) - p(\nabla Y)(L_XZ))$$
$$+p(\nabla(L_XY))(Z) + p(\nabla(L_XZ))(Y)$$
$$= (L_Xc)(Y,Z) - (L_Xp(\nabla Z))(Y) - (L_Xp(\nabla Y))(Z)$$
$$+p(\nabla(L_XY))(Z) + p(\nabla(L_XZ))(Y),$$

where $L_X\theta(Z) = (L_X\theta)Z + \theta(L_XZ), L_X\theta \in \Im_1^0(M), \theta(Z) \in \Im_0^0(M)$.

3. CONCLUSION

In this paper, obtain the covarient and Lie derivatives applied to the modified Riemannian extension with respect to the complete and vertical lifts of vector and kovector fields, respectively.

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