ORIGINAL PAPER

CONDITIONS IMPLYING NORMALITY AND HYPO-NORMALITY OF OPERATORS IN HILBERT SPACE

MOHAMMED MEZIANE[1], ABDELKADER BENALI[2*]

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Abstract. *The aim of this paper is to give sufficient conditions on two normal and hyponormal operators (bounded or not), defined on a Hilbert space, which make their algebraic sum hyponormal (only in bounded case). The results are accompanied by some interesting examples and counter examples.*

Keywords: *normal; hyponormal operators; Kaplansky theorem; bounded operators; unbounded operators.*

1. INTRODUCTION

Normal operators are a major class of bounded and unbounded operators. Among their virtues, they are the largest class of single operators for which the spectral theorem is proved [1, 2]. There are other classes of interesting non-normal operators, such as hyponormal and subnormal operators. They have been of interest to many mathematicians and have been extensively investigated, so that even monographs have been devoted to them [3, 4]. In this paper we are mainly interested in generalizing the following result to unbounded normal and bounded hyponormal operators: In this paper we are mainly interested in generalizing the following result to unbounded normal and bounded hyponormal operators:

Theorem 1.1. [5] *Let A and B be two bounded operators on a Hilbert space such that AB and A are normal. Then B commutes with AA^* iff BA is normal.*

Before recalling some essential background, we make the following convention: All operators are linear and are defined on a separable complex Hilbert space, which we will denote henceforth by H. A bounded operator A on H is said to be normal if $AA^* = A^*A$, and hyponormal if $AA^* \leq A^*A$, that is, $\| A^*x \| \leq \| Ax \|$ for all $x \in H$. Hence a normal operator is always hyponormal. Obviously, a hyponormal operator need not be normal. However, in a finite-dimensional setting, a hyponormal operator is normal. This is proved via a nice and simple trace argument [5]. An operator is said to be subnormal if it has a normal extension, and co-subnormal if its adjoint is subnormal. Another important class of non-normal operators is that of paranormal ones. We recall that an operator A is said to be paranormal if $\| A^2x \| \geq \| Ax \|^2$ for any unit vector x in H (it can easily be shown that hyponormality implies paranormality). Similarly, an operator A is called co-paranormal if its adjoint is paranormal. Since the paper is also concerned with unbounded operators, and for the reader's convenience, we recall some known notions and results about unbounded operators.

[1] The Higher School of Economics, 031000 Oran, Algeria. E-mail: m.meziane13@yahoo.fr.
[2] University of Hassiba Benbouali, Faculty of the Exact Sciences and Computer, Department of Mathematics, 02000 Chlef, Algeria. E-mail: Benali4848@gmail.com.

If A and B are two unbounded operators with domains $D(A)$ and $D(B)$ respectively, then B is said to be an extension of A (or A is a suboperator of B), and we write $A \subset B$, if $D(A) \subset D(B)$ and A and B coincide on each element of $D(A)$. An operator A is said to be densely defined if $D(A)$ is dense in H. The (Hilbert) adjoint of A is denoted by A^* and it is known to be unique if A is densely defined. An operator A is said to be closed if its graph is closed in $H \oplus H$. We say that an unbounded operator A is self-adjoint if $A = A^*$, and normal if A is closed and $AA^* = A^*A$.

An operator A is said to be formally normal if $D(A) \subset D(A^*)$ and $\| Ax \| = \| A^*x \|$ for all $x \in D(A)$. It is easy to see that a densely defined suboperator of a normal operator is formally normal. Recall also that the product BA is closed if for instance B is closed and A is bounded, and that if A, B and AB are densely defined, then only $B^*A^* \subset (AB)^*$ holds; and if further A is assumed to be bounded, then $B^*A^* = (AB)^*$. The notion of hyponormality extends naturally to unbounded operators. An unbounded A is called hyponormal if:

1. $D(A) \subset D(A^*)$
2. $\| A^*x \| \leq \| Ax \|$ for all $x \in D(A)$.

An operator A is said to be positive if $< Ax, x > \geq 0$ for every $x \in D(A)$ and write $A \geq 0$.

Proposition 1.2 [2] *If A is a positive self-adjoint operator on $D(A)$, then there is unique positive self-adjoint operator B such that $B^2 = A$. The operator B is called the positive square root of A and denoted by $A^{1/2}$.*

If A *is* densely defined, we denote by $|A| = (A^*A)^{1/2}$. Any other result or notion (such as the classical Fuglede–Putnam theorem, polar decomposition, subnormality etc.) will be assumed to be known by readers. For more details, the interested reader is referred to [2, 6-9]. For other works related to products of normal (bounded and unbounded) operators, the reader may consult [10-15] and the references therein.

We do recall the celebrated Fuglede-Putnam Theorem though:

Theorem 1.3 [7] *Let $A \in B(H)$ and M, N are two normal non-necessarily bounded operators. Then*

$$AN \subset MA \Longrightarrow AN^* \subset M^*A.$$

Now, let us recall results obtained by Devinatz-Nussbaum (and von Neumann)

Theorem 1.4. [15] *Let A, B and C be self-adjoint operators. Then*

$$A \subset BC \Longrightarrow A = BC.$$

2. MAIN RESULTS

We note that the sum of two normal operators is not always normal as shown in

Example 2.1 *Consider for instance the matrices A and B defined as*

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Obviously A and B are both normal. However,

$$A + B = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \text{is not normal.}$$

The first result is quite elementary.

Proposition 2.2 *Let A and B be two normal operators. If A commutes with B^*, then $A + B$ is normal.*

Proof. We have

$$(A + B)^*(A + B) = (A^* + B^*)(A + B) = A^*A + A^*B + B^*A + B^*B \tag{1}$$

and

$$(A + B)(A + B)^* = AA^* + AB^* + BA^* + BB^*. \tag{2}$$

Since $AB^* = B^*A$, $BA^* = A^*B$. Combining these two equations with the normality of A and B yield the equality of Equations (1) and (2) and hence establishing the normality of $A + B$.

Example 2.3 *Let A and B be acting on the standard basis (e_n) of $\ell^2(\mathbb{N})$ by:*

$$Ae_n = \alpha_n e_n \text{ and } Be_n = e_{n+1}, \forall n \geq 1$$

respectively. Assume further that α_n is bounded, *real-valued* and *positive*, for all n. Hence A is self-adjoint (hence normal!) and positive. Then

$$ABe_n = \alpha_n e_{n+1}, \forall n \geq 1.$$

For convenience, let us carry out the calculations as infinite matrices. Then

$$AB = \begin{bmatrix} 0 & 0 & & & & 0 \\ \alpha_1 & 0 & 0 & & & \\ 0 & \alpha_2 & 0 & 0 & & \\ & 0 & \alpha_3 & 0 & \ddots & \\ & & 0 & \ddots & 0 & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \text{ so that } (AB)^* = \begin{bmatrix} 0 & \alpha_1 & & & & 0 \\ 0 & 0 & \alpha_2 & & & \\ 0 & 0 & 0 & \alpha_3 & & \\ & 0 & 0 & 0 & \ddots & \\ & & 0 & \ddots & 0 & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hence

$$AB(AB)^* = \begin{bmatrix} 0 & 0 & & & & 0 \\ 0 & \alpha_1^2 & 0 & & & \\ 0 & 0 & \alpha_2^2 & 0 & & \\ & 0 & 0 & \alpha_3^2 & \ddots & \\ & & 0 & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$(AB)^*AB = \begin{bmatrix} \alpha_1^2 & 0 & & & & 0 \\ 0 & \alpha_2^2 & 0 & & & \\ 0 & 0 & \alpha_3^2 & 0 & & \\ & 0 & 0 & \ddots & \ddots & \\ & & 0 & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It thus becomes clear that AB is hyponormal iff $\alpha_n \leq \alpha_{n+1}$.

Similarly,

$$BAe_n = \alpha_{n+1}e_{n+1}, \forall n \geq 1.$$

Whence the matrix representing BA is given by:

$$BA = \begin{bmatrix} 0 & 0 & & & & 0 \\ \alpha_2 & 0 & 0 & & & \\ 0 & \alpha_3 & 0 & 0 & & \\ & 0 & \alpha_4 & 0 & \ddots & \\ & & 0 & \ddots & 0 & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \text{ so that } (BA)^* = \begin{bmatrix} 0 & \alpha_2 & & & & 0 \\ 0 & 0 & \alpha_3 & & & \\ 0 & 0 & 0 & \alpha_4 & & \\ & 0 & 0 & 0 & \ddots & \\ & & 0 & \ddots & 0 & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Therefore,

$$BA(BA)^* = \begin{bmatrix} 0 & 0 & & & & 0 \\ 0 & \alpha_2^2 & 0 & & & \\ 0 & 0 & \alpha_3^2 & 0 & & \\ & 0 & 0 & \alpha_4^2 & \ddots & \\ & & 0 & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$(BA)^*BA = \begin{bmatrix} \alpha_2^2 & 0 & & & & 0 \\ 0 & \alpha_3^2 & 0 & & & \\ 0 & 0 & \alpha_4^2 & 0 & & \\ & 0 & 0 & \ddots & \ddots & \\ & & 0 & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Accordingly, BA is hyponormal iff $\alpha_n \leq \alpha_{n+1}$ (thankfully, this is the same condition for the hyponormality of AB).

The normality of unbounded products of normal operators has been studied recently. See e.g. [19] and the references therein. We recall

Theorem 2.4 *([17]) Let A, B be normal operators with $B \in B(H)$. If $BA \subset AB$, then AB and $\overline{BA}$ are both normal (and so $AB = \overline{BA}$).*

We start by this result on the normality.

Theorem 2.5 *Let A, B be normal operators with $B \in B(H)$. Assume that AB and $A^*B^*$are densely defined. If $B^*BA \subset AB^*B$ and $BA^*A \subset A^*AB$, then AB, $\overline{BA}$, AB^*B, $\overline{B^*BA}$, A^*AB, $\overline{BA^*A}$, A^*B^*B, $\overline{B^*BA^*}$, A^*AB^* and $\overline{B^*A^*A}$ are normal (and so $AB^*B = \overline{B^*BA}$, $A^*AB = \overline{BA^*A}$, $A^*B^*B = \overline{B^*BA^*}$ and $A^*AB^* = \overline{B^*A^*A}$).*

Proof. We have

$$B^*BA \subset AB^*B \Longrightarrow B^*BA^* \subset A^*B^*B$$

and

$$BA^*A \subset A^*AB \Longrightarrow B^*A^*A \subset A^*AB^*.$$

AB is closed and

$$(AB)(AB)^* \supset ABB^*A^* = AB^*B \supset B^*BAA^*,$$

since $(AB)(AB)^*$ is self-adjoint and by the boundedness of B^*B, we get

$$(AB)(AB)^* \subset AA^*B^*B$$

and Theorem 1.4 yields

$$(AB)(AB)^* = AA^*B^*B.$$

Similarly, we have

$$(\mathrm{AB})^*(\mathrm{AB}) = \mathrm{AA^*B^*B},$$

hence AB is normal.

Next we estblish the normality of $\overline{BA}$. First of all, $\overline{BA}$ exists because A^*B^* is densely defined. Indeed

$$\overline{BA}(\overline{BA})^* \supset BAA^*B^* = BA^*AB^* \supset B^*BAA^*,$$

since $\overline{BA}(\overline{BA})^*$ is self-adjoint and by the boundedness of B^*B , we get

$$\overline{BA}(\overline{BA})^* \subset AA^*B^*B,$$

by Theorem 1.4, we obtain

$$\overline{BA}(\overline{BA})^* = AA^*B^*B,$$

Similarly, we obtain

$$\overline{BA}(\overline{BA})^* = (\overline{BA})^*\overline{BA} = AA^*B^*B,$$

and this marks the end of the proof of the normality of $\overline{BA}$.

Since AA^* and B^*B are self-adjoint (hence normal!) and A^*, B^* are also normal, Theorem Fuglede-Putnam and Theorem 2.4 gives the rest of the desired result.

Corollary 2.6 *Let A, B be normal operators with $B \in B(H)$. Assume that AB and $A^*B^*$are densely defined. If $|B|A \subset A|B|$ and $B|A| \subset |A|B$, then AB, $\overline{BA}$, $A|B|$, $\overline{|B|A}$, $|A|B$, $\overline{B|A|}$, $A^*|B|$, $\overline{|B|A^*}$, $|A|B^*$, $\overline{B^*|A|}$, AB^*B, $\overline{B^*BA}$, A^*AB, $\overline{BA^*A}$, A^*B^*B, $\overline{B^*BA^*}$, A^*AB^* and $\overline{B^*A^*A}$ are normal (and so $A|B| = \overline{|B|A}$, $|A|B = \overline{B|A|}$, $A^*|B| = \overline{|B|A^*}$, $|A|B^* = \overline{B^*|A|}$, $AB^*B = \overline{B^*BA}$, $A^*AB = \overline{BA^*A}$, $A^*B^*B = \overline{B^*BA^*}$ and $A^*AB^* = \overline{B^*A^*A}$).*

Proof. By hypothesis, we have

$$|B|A \subset A|B| \Longrightarrow |B|^2A \subset A|B|^2 \Longrightarrow B^*BA \subset AB^*B$$

and

$$B|A| \subset |A|B \Longrightarrow BA^*A \subset A^*AB,$$

hence by the Theorem 2.5, we obtain the normality of AB and $\overline{BA}$. Since $|A|$ and $|B|$ are sel-adjoint (hence noraml!) and A^*, B^* are also normal, Theorem Fuglede-Putnam and Theorem 2.4 gives the rest of the desired result.

Corollary 2.7 *Let A, B be normal operators with $A, B \in B(H)$. If $B^*BA = AB^*B$ and $BA^*A = A^*AB$, then AB, BA, AB^*B and A^*B^*B are normal (so BA^*A and B^*A^*A are normal).*

Proof. The proof of the Corollary 2.7 uses Theorem 2.5 (replacing extension by equality).

The following result generalizes the Theorem 3 ([16]). The proof requires the following Lemma:

Lemma 2.8 ([10]) *Let A, B be bounded and positive operators. If $AB = BA$, then AB is positive.*

Theorem 2.9 *Let A, B be bounded and hyponormal operators. Assume further that $AB^*B = B^*BA$ and $BAA^* = AA^*B$. Then:*

1) *AB is hyponormal.*
2) *BB^*AA^*, B^*BAA^*, AA^*B^*B and A^*AB^*B are self-adjoints.*
3) *$BB^*AA^* \le B^*BAA^*$ and $AA^*B^*B \le A^*AB^*B$.*
4) *BA is hyponormal.*

Proof.

1) By hypothesis, We have

$$AB^*B = B^*BA \Longrightarrow A^*B^*B = B^*BA^*, \tag{1}$$

and

$$BAA^* = AA^*B \Longrightarrow B^*AA^* = AA^*B^* . \tag{2}$$

Hence

$$\begin{aligned}(AB)(AB)^* &= ABB^*A^* \\ &\le AB^*BA^* \quad \text{(because } B \text{ is hyponormal)} \\ &= B^*BAA^* \\ &= B^*AA^*B \\ &\le B^*A^*AB \quad \text{(because } A \text{ is hyponormal)} \\ &= (AB)^*AB,\end{aligned}$$

i.e. AB is hyponormal.

2) Also by hypothesis (1) and (2), we obtain:

$$(BB^*AA^*)^* = AA^*BB^* = BAA^*B^* = BB^*AA^*,$$

i.e. BB^*AA^* is self-adjoint.

Similarly, we obtain the self-adjontness of B^*BAA^*, AA^*B^*B *and* A^*AB^*B.

3) We have

$$\begin{aligned}(BB^*AA^* - B^*BAA^*)^* &= BB^*AA^* - B^*BAA^* = (BB^* - B^*B)AA^* \\ &= AA^*(BB^* - B^*B).\end{aligned}$$

Since B is hyponormal and AA^* is positive, Lemma 2.8 yields

$$(BB^* - B^*B)AA^* \leq 0,$$

i.e.

$$BB^*AA^* \leq B^*BAA^*.$$

Similarly,

$$AA^*B^*B \leq A^*AB^*B.$$

4) As above, we get

$$\begin{aligned}(BA)(BA)^* &= BAA^*B^* \\ &= BB^*AA^* \\ &\leq B^*BAA^* \\ &= AB^*BA^* \\ &= AA^*B^*B \\ &\leq A^*AB^*B \\ &= A^*B^*BA \\ &= (BA)^*BA.\end{aligned}$$

I.e. BA is hyponormal, as needed.

Corollary 2.10 *Let A and B be bounded and hyponormal operators, suppose A commutes with $|B|$and B commutes with $|A^*|$, then AB and BA are hyponormal.*

Proof. Clearly

$$A|B| = |B|A \Longrightarrow AB^*B = B^*BA$$

and

$$B|A^*| = |A^*|B \Longrightarrow BAA^* = AA^*B ,$$

by Theorem 2.9, we obtain the hypernormality of AB and BA.

Remark 2.11 *If an operator T defined on the space produced by hyponormal operators does not imply that T is hyponormal. Here is an example:*

Example 2.12 *Let A, B be bounded opérators and hyponormal in Hilbert space H. Let S, T two opérators defined on $H \oplus H$ by*

$$S(x,y) = (Ax, By) \quad and \quad T(x,y) = (Ay, Bx).$$

We will see that S is hyponormal but T is not hyponrmal. Consider the following matrix blocks:

$$\tilde{S} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad and \quad \tilde{T} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

We can prove that $\tilde{S}^*$ and $\tilde{T}^*$ are defined by

$$\tilde{S}^* = \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix} \quad and \quad \tilde{T}^* = \begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix}$$

hence

$$\tilde{S}\tilde{S}^* - \tilde{S}^*\tilde{S} = \begin{pmatrix} AA^* - A^*A & 0 \\ 0 & BB^* - B^*B \end{pmatrix}$$

and

$$\tilde{T}\tilde{T}^* - \tilde{T}^*\tilde{T} = \begin{pmatrix} AA^* - B^*B & 0 \\ 0 & BB^* - A^*A \end{pmatrix}$$

It is obvious that $\tilde{S}$ is hyponormal but e $\tilde{T}$ becomes hyponormal if $AA^* - B^*B \leq 0$ and $BB^* - A^*A \leq 0$.

Theorem 2.13 *Let A, B be two opérators such that $B \in B(H)$ is hyponormal and A is normal. If $|B|A \subset A|B|$ and $B|A^*| \subset |A^*|B$, then BA is hyponormal.*

Proof. Recall that for any bounded operator or normal opertator T defined on $\in D(T)$, we have:

$$\| Tx \| = \| (T^*T)^{1/2} x \| = \| |T|x \|, \; x \in D(T).$$

Since $D(B) = H$ and $D(A^*) = D(A)$, it folows that

$$\begin{aligned} D(BA)^* = D(A^*B^*) &= \{x \in D(B^*) : B^*x \in D(A^*)\} \\ &= \{x \in D(B^*) : B^*x \in D(A)\} \\ &= D(B^*A) \\ &= D(A) \\ &= D(BA). \end{aligned}$$

Let $x \in D(BA)$, therfore

$$\begin{aligned} \| (BA)^*x \| &= \| A^*B^*x \| \\ &= \| |A^*|B^*x \| \\ &= \| B^*|A^*|x \| \\ &\leq \| B \; |A^*|x \| \text{ (because } B \text{ is hyponormal)} \\ &= \| |B||A^*|x \| \\ &= \| |A^*||B|x \| \text{ (because } |A^*| \text{ and } |B| \text{ commutes)} \\ &= \| A^*|B|x \| \\ &= \| A \; |B|x \| \text{ (because } A \text{ is normal)} \\ &= \| |B| \; Ax \| \\ &= \| BAx \|. \end{aligned}$$

Hence BA is hyponormal.
Finishing the proof.

Proposition 2.14 *Let A, B be non necessarily bounded operators such that B is hyponormal and A is self-adjoint with $A \in B(H)$. Assume further that $AB \subset BA$. Then AB is hyponormal.*

Proof. Since $D(B^*) \supset D(B)$, we obtain

$$\begin{aligned} D(AB)^* = D(B^*A) &= \{x \in D(A): Ax \in D(B^*)\} \\ &\supset \{x \in D(A): Ax \in D(B)\} \\ &= D(BA) \\ &\supset D(AB). \end{aligned}$$

Let $x \in D(AB)$, we have

$$\begin{aligned} \| (AB)^*x \| &= \| B^*Ax \| \\ &\leq \| BAx \| \end{aligned}$$

Since $BA = AB$ on $D(AB)$, then

$$\| (AB)^*x \| \leq \| ABx \|,$$

i.e. AB is hyponormal.

Proposition 2.15 *Let A, B be non necessarily bounded operators such that B is hyponormal and $A \in B(H)$. If $AB \subset BA^*$, then AB is hyponormal.*

Proof. Since $D(B^*) \supset D(B)$, we get

$$\begin{aligned} (AB)^* &= D(B^*A^*) \\ &= \{x \in D(A^*): A^*x \in D(B^*)\} \\ &\supset \{x \in D(A^*): A^*x \in D(B)\} \\ &= D(BA^*) \\ &\supset D(AB). \end{aligned}$$

Now, let $x \in D(AB)$, we have

$$\begin{aligned} \| (AB)^*x \| &= \| B^*A^*x \| \\ &\leq \| BA^*x \|. \end{aligned}$$

Since $BA^* = AB$ on $D(AB)$, we obtain

$$\| (AB)^*x \| \leq \| ABx \|,$$

proving the hypernormality of AB.

3. CONCLUSION

With an eye toward questions about products of normal and hypo-normal operators, these results are naturally tied to questions about operator product adjoints of AB and the closure of AB in the case of normality.

Then for conditions for operator products AB and BA to normal or hypo-normal, commutativity of operators is unavoidable. We established both sufficient conditions. In the bounded case, we lightened the sufficient conditions of the commutativity of opérators with $|B|$ and $|A^*|$ to have the hypo-normality of $AB$and BA, i.e. we just used the commutativity of operators with $|B|^2$ and $|A^*|^2$.

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