ON GAUSSIAN MERSENNE NUMBERS
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Abstract. In this paper, we define Gaussian Mersenne numbers and we give some properties of them. Moreover we present some relations among Gaussian Mersenne numbers, Gaussian Jacobsthal numbers and Gaussian Jacobsthal-Lucas numbers. We also present some results with matrices involving Gaussian Mersenne numbers.

Keywords: Gaussian Mersenne number; Gaussian Jacobsthal number; Gaussian Jacobsthal-Lucas number.

1. INTRODUCTION

In this study we consider Mersenne sequences, some studies about this sequence have been worked Koshy and Gaz in [1, 2] and Gay T. studied in [3]. In number theory, recall that a Mersenne number of order n, denoted by $M_n$ is a number of form $2^n - 1$, where $n$ is a non negative number. This identity is called the Binet formula for the Mersenne sequence. Mersenne numbers can also be defined recursively by

$$M_{n+1} = 2M_n + 1,$$  \hfill (1.1)

with initial conditions $M_0 = 0$ and $M_1 = 1$. Since this recurrence is inhomogeneous, substituting $n$ by $n + 1$, we obtain the new form

$$M_{n+2} = 2M_{n+1} + 1.$$ \hfill (1.2)

Subtracting (1.1) and (1.2), we find

$$M_{n+2} = 3M_{n+1} - 2M_n$$ \hfill (1.3)

the equality (1.3) states another form of the recurrence relation for the Mersenne sequence, with initial values $M_0 = 0$ and $M_1 = 1$. The roots of the respective characteristic equation $r^2 - 3r + 2 = 0$ are $r_1 = 1$ and $r_2 = 2$ and we easily get the Binet formula

$$M_n = 2^n - 1$$ \hfill (1.4)

We note that some Mersenne numbers are prime and the search for Mersenne Primes is an active field in number theory, computer science and coding theory. We remark that if $M_n$ is prime then $n$ is prime and the converse is false.

Horadam in [4] defined the Jacobsthal and Jacobsthal- Lucas sequences $J_n$ and $j_n$ by the following recurrence relations.

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The Gaussian Mersenne numbers are defined by the recurrence relations

\[ J_{n+2} = J_{n+1} + 2 J_n, \quad J_0 = 0, \quad J_1 = 1 \]  \hspace{1cm} (1.5)

and

\[ J_{n+2} = J_{n+1} + 2 J_n, \quad J_0 = 2, \quad J_1 = 1 \]  \hspace{1cm} (1.6)

respectively, where -1 and 2 are the roots of the characteristic equation associated with the above recurrence relations (1.5) and (1.6).

There are large number of sequences indexed in the Online Encyclopedia of Integer Sequences, being in this case \( \{J_n\} = \{0,1,1,3,5,11,21,43,85,171,\ldots\} : \text{A001045} \), \( \{j_n\} = \{2,1,5,7,17,31,65,127,257,\ldots\} : \text{A014551} \), \( \{M_n\} = \{0,1,3,7,15,31,63,127,255,\ldots\} : \text{A000225} \).


2. GAUSSIAN MERSENNE SEQUENCES

Firstly we give the definition of the Gaussian Mersenne sequence.

**Definition 1.** The Mersenne sequence is the sequence of complex numbers, \( GM_n \), defined by the initial values \( GM_0 = -i/2, GM_1 = 1 \) and the recurrence relation

\[ GM_n = 3GM_{n-1} - 2GM_{n-2}, \quad \text{for } n \geq 2. \]

The first few values of \( GM_n \) are \(-\frac{i}{2}, 1, 3+i, 7+3i, 15+7i, 31+15i, \ldots\).

We remark that the equality \( GM_n = M_n + iM_{n-1} \) is valid.

**Theorem 2.** The generating function of the Gaussian Mersenne sequence is

\[ f(x) = -\frac{i}{2} + \frac{(1 + \frac{3}{2} i)x}{1 - 3x + 2x^2}. \]

**Proof:** Let

\[ f(x) = \sum_{n=0}^{\infty} GM_n x^n = GM_0 + GM_1 x + GM_2 x^2 + \cdots + GM_n x^n + \cdots \]
be the generating function of the Gaussian Mersenne sequence. On the other hand, since

$$-3xf(x) = -3 \sum_{n=0}^{\infty} GM_n x^{n+1} = -3GM_0x - 3GM_1x^2 - 3GM_2x^3 - \cdots - 3GM_{n-1}x^n - \cdots$$

and

$$2x^2f(x) = 2 \sum_{n=0}^{\infty} GM_n x^{n+2} = 2GM_0x^2 + 2GM_1x^3 + 2GM_2x^4 + \cdots + 2GM_{n-2}x^n + \cdots$$

we have

$$(1 - 3x + 2x^2)f(x) = GM_0 + (GM_1 - 3GM_0)x + (GM_2 - 3GM_1 + 2GM_0)x^2 + \cdots + (GM_n - 3GM_{n-1} + 2GM_{n-2})x^n + \cdots$$

Now we consider $GM_0 = -\frac{i}{2}, GM_1 = 1, GM_n = 3GM_{n-1} - 2GM_{n-2}$. Thus we obtain

$$f(x) = \frac{-\frac{i}{2} + \left(1 + \frac{3}{2}i\right)x}{1 - 3x + 2x^2}$$

Thus the proof is complete.

**Theorem 3.** The Binet formula for the Gaussian Mersenne sequence is

$$GM_n = 2^{n-1}(2 + i) - (1 + i) \quad (2.1)$$

*Proof:* Considering the Binet formula for Mersenne sequences $M_n = 2^n - 1$ and $GM_n = M_n + iM_{n-1}$, it is easily seen

$$GM_n = 2^n - 1 + i(2^{n-1} - 1)$$

or

$$GM_n = 2^{n-1}(2 + i) - (1 + i)$$

**Theorem 4.** Catalan’s identity for Gaussian Mersenne sequence is

$$GM_{n+r}GM_{n-r} - GM_n^2 = -(1 + 3i)2^{n-r-1}(2^r - 1)^2 \quad (2.2)$$

where $n \geq r$ and $n, r \in \mathbb{Z}^+$.  

*Proof:* Using Binet formula for Gaussian Mersenne sequence, we have

$$GM_{n+r}GM_{n-r} - GM_n^2 = [2^{n+r-1}(2 + i) - (1 + i)][2^{n-r-1}(2 + i) - (1 + i)]$$

$$-\left[2^{n-1}(2 + i) - (1 + i)\right]^2$$

$$= -2^{n-1}(2 + i)(1 + i)(2^r + 2^{-r} - 2)$$

$$= -2^{n-1}(1 + 3i)(1 + i)\left(\frac{2^{2r+1} - 2^{2r}}{2^r}\right)$$
So the theorem is proved.

**Corollary 5.** (Cassini’s Identity for the Gaussian Mersenne Sequence) For $n \geq 1$, we have

$$GM_{n+1}GM_{n-1} - GM_n^2 = -(1 + 3i)(2^{n-2})$$

(2.3)

**Proof:** In fact, the equation (2.2), for $r = 1$ yields the equation (2.3).

**Theorem 6.** (d’Ocagne’s Identity) For the sequence $\{GM\}_{n=0}^{\infty}$, if $m > n$, we have,

$$GM_m GM_{n+1} - GM_{m+1} GM_n = (1 + 3i)(2^{m-1} - 2^{n-1}).$$

**Proof:** Considering the Binet formula for the Gaussian Mersenne sequence, the proof is immediately seen.

**Theorem 7.** The explicit formula for the Gaussian Mersenne sequence is

$$GM_n = \left( \sum_{k=0}^{n} \binom{n}{k} - 1 \right) + i \left( \sum_{k=0}^{n-1} \binom{n-1}{k} - 1 \right).$$

**Proof:** Using the sum of binomial coefficients of Mersenne numbers

$$M_n = \sum_{k=0}^{n} \binom{n}{k} - 1$$

and considering the equality

$$GM_n = M_n + iM_{n-1}$$

the proof is easily seen.

**Theorem 8.** (Summation formula of Gaussian Mersenne numbers)

$$\sum_{k=0}^{n} GM_k = GM_{n+1} - (n + 1)(1 + i) + \frac{i}{2}.$$  

**Proof:** Using the equality

$$GM_{n+1} = 2GM_n + 1 + i; \quad n \geq 0.$$  

We write

$$GM_1 = 2GM_0 + 1 + i$$

$$GM_2 = 2GM_1 + 1 + i$$

$$GM_3 = 2GM_2 + 1 + i$$

$$\vdots$$

$$GM_{n+1} = 2GM_n + 1 + i.$$
Thus from the above equations, we obtain
\[ \sum_{k=1}^{n+1} GM_k = 2 \sum_{k=0}^{n} GM_k + (n + 1)(1 + i) \]
which implies (using the fact that \( GM_0 = -\frac{i}{2} \)) that
\[ 2 \sum_{k=0}^{n} GM_k = \sum_{k=1}^{n+1} GM_k - (n + 1)(1 + i) \]
\[ = \sum_{k=0}^{n} GM_k - (n + 1)(1 + i) - GM_0 + GM_{n+1} \]
\[ = \sum_{k=0}^{n} GM_k - (n + 1)(1 + i) + \frac{i}{2} + GM_{n+1} \]
Consequently we obtain
\[ \sum_{k=0}^{n} GM_k = GM_{n+1} - (n + 1)(1 + i) + \frac{i}{2} \]

3. SOME RELATIONS AMONG GAUSSIAN MERSENNE NUMBERS, GAUSSIAN JACOBSTHAL NUMBERS AND GAUSSIAN JACOBSTHAL-LUCAS NUMBERS

**Definition 9.** Gaussian Jacobsthal numbers are defined by
\[ G_{J_n} = G_{J_{n-1}} + 2G_{J_{n-2}}, \quad n \geq 2. \]
with initial values \( G_{J_0} = \frac{i}{2}, G_{J_1} = 1. \)

**Definition 10.** Gaussian Jacobsthal-Lucas numbers are defined by
\[ G_{J_n} = G_{J_{n-1}} + 2G_{J_{n-2}}, \quad n \geq 2. \]
with initial values \( G_{J_0} = 2 \left( 1 - \frac{1}{i} \right), G_{J_1} = 1 + 2i. \)

Binet formulas for these sequences are given by
\[ G_{J_n} = \frac{1}{3} \left[ 2^n - (-1)^n + i(2^{n-1} - (-1)^{n-1}) \right] \quad (3.1) \]
and
\[ G_{J_n} = 2^n + (-1)^n + i[2^{n-1} + (-1)^{n-1}] \quad (3.2) \]

**Theorem 11.** If \( GM_n, G_{J_n} \text{ and } G_{J_{n}} \) are the \( n \) th term of the Gaussian Mersenne sequence, Gaussian Jacobsthal sequence and Gaussian Jacobsthal- Lucas sequence, respectively, then we have
1. For an even subscript $n$

$$GM_n = 3GJ_n - 2i$$ \hfill (3.3)

$$GM_n = GJ_n - 2$$ \hfill (3.4)

2. For an odd subscript $n$

$$GM_n = 3GJ_n - 2$$ \hfill (3.5)

$$GM_n = GJ_n - 2i$$ \hfill (3.6)

**Proof:** Using Binet formulas for Gaussian Mersenne numbers, Gaussian Jacobsthal numbers and Gaussian Jacobsthal- Lucas numbers, the proof is immediately seen.

### 4. MATRICES WITH GAUSSIAN MERSENNE NUMBERS

**Theorem 12.** Let $Q, N$ and $K_n$ denote the $2 \times 2$ matrices defined as

$$Q = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 3 + i & 1 \\ 1 & -i/2 \end{bmatrix}$$

and $K_n = \begin{bmatrix} GM_{n+2} & GM_{n+1} \\ GM_{n+1} & GM_n \end{bmatrix}$.

Then for all $n \in \mathbb{Z}^+$,

$$Q^nN = K_n.$$ 

**Proof:** We can prove the theorem by induction on $n$. For $n = 1$, we have

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 + i & 1 \\ 1 & -i/2 \end{bmatrix} = \begin{bmatrix} 7 + 3i & 3 + i \\ 3 + i & 1 \end{bmatrix} = \begin{bmatrix} GM_3 & GM_2 \\ GM_2 & GM_1 \end{bmatrix}.$$ 

Assume that the theorem holds for $n = k$, that is

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 3 + i & 1 \\ 1 & -i/2 \end{bmatrix} = \begin{bmatrix} GM_{k+2} & GM_{k+1} \\ GM_{k+1} & GM_k \end{bmatrix}.$$ 

Then for $n = k + 1$, we have

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 3 + i & 1 \\ 1 & -i/2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 3 + i & 1 \\ 1 & -i/2 \end{bmatrix} = \begin{bmatrix} GM_{k+3} & GM_{k+2} \\ GM_{k+2} & GM_{k+1} \end{bmatrix}.$$ 

Thus the proof is complete.
Theorem 13. For all $n \in \mathbb{Z}^+$, we have

$$
\begin{bmatrix}
3 & -2 \\
1 & 0 \\
\end{bmatrix}^n 
\begin{bmatrix}
1 \\
\frac{i}{2} \\
\end{bmatrix} = 
\begin{bmatrix}
GM_{n+1} \\
GM_n \\
\end{bmatrix}
$$

Proof: Using the induction method on $n$, the proof is easily seen.

Theorem 14. Let $A_n$ denote the $n \times n$ tridiagonal matrix defined as

$$A_n = 
\begin{bmatrix}
1 & i & 0 & 0 & \cdots & 0 & 0 \\
-1 & 3 & -2 & 0 & \cdots & 0 & 0 \\
0 & -1 & 3 & -2 & \cdots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 3 & -2 \\
0 & 0 & 0 & 0 & \cdots & -1 & 3 \\
\end{bmatrix}
$$

and let $A_0 = -\frac{i}{2}$. Then

$$
\det A_n = GM_n, \ n \geq 1
$$

Proof: By the induction on $n$, we can prove the theorem. Since

$$
\det A_1 = 1 = GM_1
$$
$$
\det A_2 = 3 + i = GM_2
$$

It is true for $n = 1$ and $n = 2$. Assume that the theorem is true for $n - 1$ and $n - 2$, that is

$$
\det A_{n-1} = GM_{n-1}
$$

and

$$
\det A_{n-2} = GM_{n-2}
$$

Then, we have

$$
\det A_n = 3 \det A_{n-1} - 2 \det A_{n-2}
= 3GM_{n-1} - 2GM_{n-2}
= GM_n.
$$

So the theorem is proved.

4. CONCLUSIONS

In this study we defined a new sequence named the Gaussian Mersenne sequence. Further we obtained some properties of this sequence. Moreover we found some relations among Gaussian Mersenne Numbers, Gaussian Jacobsthal Numbers and Gaussian Jacobsthal-Lucas numbers.
REFERENCES