ORIGINAL PAPER

SOLVABILITY AND ULAM-STABILITY FOR NONLINEAR DIFFERENTIAL PROBLEM WITH Phi-HILFER APPROACH

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Abstract. *In this article, we study the existenceof solutions and the stability in the sense of Ulam for new nonlinear differential problem via* $\varphi-$*Hilfer fractional derivative. Our approach is based on Krasnoselskii's fixed point theorem. An example is given to illustrate our results.*

Keywords: $\varphi-$*Hilfer fractional derivative; existence of solutions; fixed point; Ulam stability.*

1. INTRODUCTION

Recently, fractional differential equations with boundary conditions are being studied by many interested researchers. Indeed, the fractional differential equations describe many more phenomena than ordinary differential equations. Therefore, partial differential equations appear in many engineering and technological disciplines that include several sciences, see for example [1-11].

Currently, there are several different definitions of fractional integrals and derivatives, the most famous of which are the Riemann-Liouville derivative and Caputo derivative. A generalization of the derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [12], known as the fractional Hilfer derivative. There are many interesting results for fractional differential and partial differential equations involving Hilfer derivative are given in [13-17]. For some recent work on boundary value problems with fractional Hilfer derivatives, see [18-21]. In [22], C. Nuchpong et al. has been studied the following problem:

$$\begin{cases} {}^H\mathbf{D}^{\alpha,\beta}u(t) = h(t,u(t),\mathbf{I}^{\mu}u(t)), \ \ t\in J=(a,b] \\ u(a)=0, \int_a^b u(s)ds = \sum\limits_{i=1}^{n-2}\lambda_i u(\zeta_i), \end{cases}$$

where ${}^H\mathbf{D}^{\alpha,\beta;\varphi}$ is the Hilfer fractional derivative of order $\alpha, 1<\alpha<2$ and the parameter β, $0\le\beta\le1$, $\mathbf{I}^{\mu}$ is the Riemann-Liouville fractional integral of order $\mu>0$, $\lambda_i,\zeta_i\in(a,b], a>0$, are given constants, and $f: J\times\mathrm{R}\times\mathrm{R}\to\mathrm{R}^+$, is a continuous function. In view of the above considerations, we study the following fractional differential problem:

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$$\begin{cases} {}^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\phi}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u\right)(t)=h(t,u(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}u(t)),\quad t\in J=(a,b] \\ u(a)=0,u(b)=\sum_{i=1}^{n}\lambda_i u\left(\zeta_i\right), \\ {}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u(a)=0, \\ \text{and}\quad {}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u(b)=\mathbf{I}_{a^+}^{\rho;\phi}u\left(\zeta\right),\ a<\zeta<b, \end{cases} \tag{1}$$

where ${}^{H}\mathbf{D}_{0^+}^{\alpha_1,\beta;\varphi},{}^{H}\mathbf{D}_{0^+}^{\alpha_2,\beta;\varphi}$, are the $\varphi-$Hilfer fractional derivative of orders $\alpha_1,\alpha_2,1<\alpha_1,\alpha_2<2$ and the parameters β_1,β_2, $0\leq\beta_1,\beta_2\leq 1$, ${}^{RL}\mathbf{D}_{a^+}^{\kappa;\varphi}$ the φ -Riemann-Liouville fractional derivative of order μ where $\mu<\alpha_2$, $\mathbf{I}_{0^+}^{\rho;\varphi}$ the left-sided $\varphi-$Riemann Liouville fractional integral of order $\rho,$ where $\rho>0$, $\varphi : J\to \mathrm{R}$ be an increasing function such that $\varphi'(t)\neq 0$, for all $t\in J$, and $f : J\times\mathrm{R}\times\mathrm{R}\to\mathrm{R}^+$, is given function will be well defined later.

2. PRELIMINARY RESULT

In this section, we need to recall some definitions and lemmas which are very needed for our results. Let $\varphi : [a,b]\to \mathrm{R}$ be an increasing function with $\varphi'(t)\neq 0$, for all $t\in J$ and let $C([a,b],R)$ be the Banach space.

For all $\upsilon>-1$ and $s,t\in[0,\infty),(t\geq s)$, we pose $\varphi_\upsilon(t,s)=(\varphi(t)-\varphi(s))^\upsilon$.

Definition 2.1. *([6]) Let (a,b), ($-\infty\leq a<b\leq\infty$) be a finite or infinite interval of the half-axis $(0,\infty)$ and $\alpha>0$. In addition, let $\varphi(t)$ be a positive increasing function on $[a,b]$, which has a continuous derivative $\varphi'(t)$ on (a,b). The $\varphi-$Riemann-Liouville fractional integral of a function u with respect to another function φ on $[a,b]$ is defined by*

$$\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\varphi'(s)\varphi_{\alpha-1}(t,s)u(s)ds,$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *([6]) Let $n\in\mathrm{N}$ and let $\varphi,u\in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t)\neq 0$, for all $t\in(a,b]$. The left-sided $\varphi-$Riemann Liouville fractional derivative of a function u of order α is defined by*

$$\mathbf{D}_{a^+}^{\alpha;\varphi}u(t)=\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n\mathbf{I}_{a^+}^{n-\alpha;\varphi}u(t)$$
$$=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n\int_a^t\varphi'(s)\varphi_{n-\alpha-1}(t,s)u(s)ds,$$

where $n=[\alpha]+1$, $[\alpha]$ represents the integer part of the real number α.

Definition 2.3. *([23]) Let $n-1<\alpha<n$ with $n\in\mathrm{N},[a,b]$ is the interval such that $-\infty\leq a<b\leq\infty$ and $\varphi,u\in C^n([a,b],\mathrm{R})$ two functions such that φ is increasing and $\varphi'(t)\neq 0$, for all $t\in[a,b]$. The φ-Hilfer fractional derivative of a function u of order a and type $0\leq\beta\leq 1$ is defined by*

$$^{H}\mathbf{D}_{a^+}^{\alpha,\beta;\varphi}u(t)=\mathbf{I}_{a^+}^{\beta(n-\alpha);\varphi}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n\mathbf{I}_{a^+}^{(1-\beta)(n-\alpha);\varphi}u(t)=\mathbf{I}_{a^+}^{\gamma-\alpha;\varphi}\mathbf{D}_{a^+}^{\gamma;\varphi}u(t),$$

where $n=[\alpha]+1,\gamma-\alpha=\beta(n-\alpha)$.

Lemma 2.1. *([6]) Let* $\alpha,\rho>0$ *. Then, we have the following semi-group property*

$$\mathbf{I}_{a^+}^{\alpha;\varphi}\mathbf{I}_{a^+}^{\rho;\varphi}u(t)=\mathbf{I}_{a^+}^{\alpha+\rho;\varphi}u(t),\ \ t>a.$$

Next, we present the φ-fractional integral and derivatives of a power function.

Proposition 2.1. *([6,23]) Let* $\alpha\geq 0,\sigma>0$ *and* $t>a$. *Then,* φ*-fractional integral and derivative of a power function are given by*

1) $\mathbf{I}_{a^+}^{\alpha,\varphi}\varphi_{\sigma-1}(t,a)(t)=\frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}\varphi_{\sigma+\alpha-1}(t,a).$

2) $^{H}\mathbf{D}_{a^+}^{\alpha,\beta;\varphi}\varphi_{\sigma-1}(t,a)(t)=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}\varphi_{\sigma-\alpha-1}(t,a),n-1<\alpha<n,\sigma>n.$

Lemma 2.2. *([23]) If* $u\in C^n([a,b],R),n-1<\alpha<n,0\leq\beta\leq 1$ *and* $\gamma=\alpha+\beta(n-\alpha)$ *. Then*

$$\mathbf{I}_{a^+}^{\alpha,\varphi}(^{H}\mathbf{D}_{a^+}^{\alpha,\beta;\varphi}u)(t)=u(t)-\sum_{k=1}^{k=n}\frac{\varphi_{\gamma-k}(t,s)}{\Gamma(\gamma-k+1)}\nabla_{\varphi}^{[n-k]}\mathbf{I}_{a^+}^{(1-\beta)(n-\alpha);\varphi}u(a),\ t\in[a,b],$$

where $\nabla_{\varphi}^{[n]}u(t):=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n u(t).$

Lemma 2.3. *([24]) Let* $u\in C^n[a,b]$ *and* $0<q<1$, *we have*

$$\left|\mathbf{I}_{a^+}^{q;\varphi}u(t_2)-\mathbf{I}_{a^+}^{q;\varphi}u(t_1)\right|\leq\frac{2\|u\|}{\Gamma(q+1)}\varphi_q(t_2,t_1).$$

Lemma 2.4. *Let* $a\geq 0,1<\alpha_1,\alpha_2<2\ ,0\leq\beta_1,\beta_2\leq 1,$ *and* $2-\gamma_1=(1-\beta_1)(2-\alpha_1),2-\gamma_2=(1-\beta_2)(2-\alpha_2)$. *For* $f\in C(J,,R,R)$ *, the unique solution ofthe sequential* φ*-Hilfer fractional boundary value problem*

$$^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\phi}\left(^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u(t)\right)=f(t),\ \ t\in J=[a,b] \tag{2}$$

$$\begin{cases} u(a)=0,u(b)=\sum\limits_{i=1}^{n}\lambda_i u(\zeta_i), \\ ^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u(a)=0, \\ \text{and}\ \ ^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u(b)=\mathbf{I}_{a^+}^{\rho;\phi}u(\zeta),\ a<\zeta<b, \end{cases} \tag{3}$$

is given by

$$\begin{aligned} u(t)&=\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\phi'(s)\phi_{\alpha_1+\alpha_2-1}(t,s)f(s)ds \\ &+\frac{\Gamma(\gamma_1)}{\Gamma(\rho)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)}-\frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)}\right]\int_a^\zeta\phi'(s)\phi_\rho(\zeta,s)u(s)ds \\ &+\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)}-\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)}\right]\int_a^b\phi'(s)\phi_{\alpha_1-1}(b,s)f(s)ds \end{aligned}$$

$$+\frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)-\frac{\phi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_1+\alpha_2)\phi_{\gamma_2-1}(b,a)}\int_a^b \phi'(s)\phi_{\alpha_1+\alpha_2-1}(b,s)f(s)ds.$$

Proof: Assume that u is a solution of the sequential nonlocal boundary value Problems (2) and (3). Applying the two operators $\mathtt{I}_{a^+}^{\alpha_1;\varphi}, \mathtt{I}_{a^+}^{\alpha_2;\varphi}$ to both sides of Equation (2) and using Lemma 2.2 and Proposition 2.1, we obtain

$$^H\mathtt{D}_{a^+}^{\alpha_2,\beta_2;\phi}u(t)=\mathtt{I}_{a^+}^{\alpha_1;\phi}f(t)+\frac{m_0}{\Gamma(\gamma_1-1)}\phi_{\gamma_1-2}(t,a)+\frac{m_1}{\Gamma(\gamma_1)}\phi_{\gamma_1-1}(t,a), \tag{4}$$

where $m_0, m_1 \in R$, and $2-\gamma_1=(1-\beta_1)(2-\alpha_1)$.

From the condition $^H\mathtt{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(a)=0,$ we get $m_0=0.$

And by $^H\mathtt{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(b)=\mathtt{I}_{a^+}^{\rho;\varphi}u(\zeta),$ we obtain

$$m_1=\frac{\Gamma(\gamma_1)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathtt{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathtt{I}_{0^+}^{\alpha_1;\varphi}f(b)\right).$$

By (4), we have

$$u(t)=\mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}f(t)+\frac{m_1}{\Gamma(\gamma_1)}\mathtt{I}_{a^+}^{\alpha_2;\varphi}\varphi_{\gamma_1-1}(t,a)+\frac{m_2}{\Gamma(\gamma_2-1)}\varphi_{\gamma_2-2}(t,a)+\frac{m_3}{\Gamma(\gamma_2)}\varphi_{\gamma_2-1}(t,a),$$

where $m_2, m_3 \in R$, and $2-(1-\beta_2)(2-\alpha_2)=\gamma_2$, From the condition, we get

$$m_2=0.$$

So,

$$u(t)=\mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}f(t)+\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)}\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathtt{I}_{a^+}^{\rho;\varphi}u(\zeta)-\mathtt{I}_{a^+}^{\alpha_1;\varphi}f(b)\right)+m_3\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\gamma_2)}.$$

By conditions $u(b)=\sum_{i=1}^{n}\lambda_i u(\zeta_i),$ we get

$$m_3=\frac{\Gamma(\gamma_2)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)-\frac{\Gamma(\gamma_2)}{\varphi_{\gamma_2-1}(b,a)}\mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}f(b)$$
$$-\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)}{\varphi_{\gamma_2-1}(b,a)}\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1+\alpha_2)}\left(\mathtt{I}_{a^+}^{\rho;\varphi}u(\zeta)-\mathtt{I}_{a^+}^{\alpha_1;\varphi}f(b)\right).$$

Thus,

$$\begin{aligned}
u(t)=\ &\mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}f(t)\\
&+\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}-\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\mathtt{I}_{a^+}^{\rho;\varphi}u(\zeta)\\
&-\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}-\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\mathtt{I}_{a^+}^{\alpha_1;\varphi}f(b)\\
&+\left[\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)\right]-\left[\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}f(b)\right].
\end{aligned}$$

The reverse will be done by Proposition 2.1 and direct computation.This ends the proof.

By derivation we get:

$$
\begin{aligned}
&{}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u(t)\\
&= \mathbf{I}_{a^+}^{\alpha_1+\alpha_2-\mu;\varphi} f(t) - \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\varphi} f(b) + \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)\\
&+\left[\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2-\mu)}\frac{\varphi_{\gamma_1+\alpha_2-\mu-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)} - \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\mathbf{I}_{a^+}^{\rho;\varphi}u(\zeta)\\
&+\left[\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2-\mu)}\frac{\varphi_{\gamma_1+\alpha_2-\mu-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)} - \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\mathbf{I}_{a^+}^{\alpha_1;\varphi} f(b).
\end{aligned}
$$

3. MAIN RESULTS

In this section, we present our main results on the existence and the stability for the above problem. We begin by considering the space

$$C_\varphi^\mu = \left\{u \;:\; u, {}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u \in C([a,b],\mathrm{R})\right\},$$

with the norm

$$\|u\|_{C_\varphi^\mu} = \|u\|_C + \left\|{}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u\right\|_C,$$

such that

$$\|u\|_C = \sup_{t\in[a,b]}|u(t)|, \text{ and } \left\|{}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u\right\|_C = \sup_{t\in[a,b]}\left|{}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u(t)\right|.$$

3.1. EXISTENCE OF SOLUTIONS

Now, we need to make the following assumptions:

$\mathbf{H}_1$) h is continuous function.

$\mathbf{H}_2$) There exists a constant $\Upsilon > 0,$ such that

$$|h(t,u,v) - h(t,x,y)| \le \Upsilon(|u-x| + |v-y|),$$

for all $t\in[a,b], (u,v,x,y)\in \mathrm{R}^4.$

Now, we define the following quantities:

$$
\begin{aligned}
M &= \phi(b) - \phi(a)\\
\Lambda_1 &= \frac{4\Upsilon M^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{2\Upsilon\Gamma(\gamma_1)\left[M^{\gamma_1+\alpha_1+\alpha_2-1} + M^{\alpha_1+\alpha_2}\right]}{\Gamma(\alpha_1+1)\Gamma(\gamma_1+\alpha_2)}\\
\Lambda_2 &= \frac{2\Upsilon M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)} + \frac{2\Upsilon\Gamma(\gamma_2)M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\gamma_2-\mu)}\\
&\quad + \frac{2\Upsilon\Gamma(\gamma_1)}{\Gamma(\alpha_1+1)}\left[\frac{M^{\alpha_2+\alpha_1-\mu}}{\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2-\mu)} + \frac{M^{\gamma_1+\alpha_1+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\right]\\
\Lambda_3 &= \frac{\Gamma(\gamma_1)\left[M^{\alpha_2+\rho+1} + M^{\gamma_1+\alpha_2+\rho}\right]}{(\rho+1)\Gamma(\rho)\Gamma(\gamma_1+\alpha_2)} + \sum_{i=1}^{n}|\lambda_i|
\end{aligned} \tag{5}
$$

$$\Lambda_4 = \frac{M^{-\mu}\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\left(\sum_{i=1}^{n}|\lambda|_i\right)$$
$$+\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\rho+1)}\left[\frac{M^{\alpha_2+\rho-\mu}}{\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2-\mu)}+\frac{M^{\gamma_1+\rho+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\right],$$
$$\Upsilon_1 = \frac{\left[M^{\alpha_2}+M^{\gamma_1+\alpha_2-1}\right]M^{\rho+1}\Gamma(\gamma_1)}{(\rho+1)\Gamma(\rho)\Gamma(\gamma_1+\alpha_2)}+\left(\sum_{i=1}^{n}|\lambda_i|\right),$$
$$\Upsilon_2 = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\rho+1)}\left[\frac{M^{\alpha_2+\rho-\mu}}{\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2-\mu)}+\frac{M^{\gamma_1+\rho+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\right]$$
$$+\frac{M^{-\mu}\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\left(\sum_{i=1}^{n}|\lambda_i|\right).$$

Based on the above hypotheses, we present to the reader the following result.

Now, consider the following operator $G\ :\ C_{\varphi}^{\mu}\to C_{\varphi}^{\mu}$ by:

$$(Gu)(t) = \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_1+\alpha_2-1}(t,s)h_u(s)ds$$
$$+\frac{\Gamma(\gamma_1)}{\Gamma(\rho)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}-\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\int_a^{\zeta}\varphi'(s)\varphi_{\rho}(\zeta,s)u(s)ds$$
$$+\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}-\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\right]\int_a^b\varphi'(s)\varphi_{\alpha_1-1}(b,s)h_u(s)ds$$
$$+\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)-\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_1+\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b\varphi'(s)\varphi_{\alpha_1+\alpha_2-1}(b,s)h_u(s)ds.$$

where

$$h_u(t) = h(t,u(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u(t)).$$

Here, we divide the operator $(Gu)(t)$ as follows:

$$(\mathrm{G}u)(t) = (\mathrm{G}_1u)(t)+(\mathrm{G}_2u)(t) \tag{6}$$

where

$$(G_1u)(t) = \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_1+\alpha_2-1}(t,s)h_u(s)ds$$
$$+\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}-\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\right]\int_a^b\varphi'(s)\varphi_{\alpha_1-1}(b,s)h_u(s)ds$$
$$-\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_1+\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b\varphi'(s)\varphi_{\alpha_1+\alpha_2-1}(b,s)h_u(s)ds,$$

and

$$\left(G_2u\right)(t)=\frac{\Gamma\left(\gamma_1\right)}{\Gamma(\rho)\Gamma\left(\gamma_1+\alpha_2\right)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}\left(t,a\right)}{\varphi_{\gamma_1-1}\left(b,a\right)}-\frac{\varphi_{\gamma_1+\alpha_2-1}\left(b,a\right)\varphi_{\gamma_2-1}\left(t,a\right)}{\varphi_{\gamma_2-1}\left(b,a\right)}\right]\int_a^{\zeta}\varphi'(s)\varphi_{\rho}(\zeta,s)u\left(s\right)ds$$
$$+\frac{\varphi_{\gamma_2-1}\left(t,a\right)}{\varphi_{\gamma_2-1}\left(b,a\right)}\sum_{i=1}^{n}\lambda_i u\left(\zeta_i\right).$$

Our first result concerning the existence of solutions of the problem (1) for which we have used the fixed point theorem of Krasnoselskii's is as follows

Theorem 3.1. *We assume that the hypotheses* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *are satisfied. If*

$$\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4\leq 1,\quad \text{and}\quad \Upsilon_1+\Upsilon_2<1,$$

where $\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4,\Upsilon_1,$ *and* Υ_2 *are defined in (5). Then the equation (1) has a solution.*

Proof: The proof will be given in several steps. Let $\mathbf{U}_r=\left\{u\in C_{\varphi}^{\mu}\ :\ \|u\|_{C_{\varphi}^{\mu}}\leq r\right\}$

Step 1. We prove that $\left\|(Gu)\right\|_C\leq r$. From (6), we obtain

$$\left\|(Gu)\right\|_C\leq\left\|(G_1u)\right\|_C+\left\|(G_2u)\right\|_C \tag{7}$$

Let $u\in\mathbf{U}_r$, then

$$\sup_{t\in[a,b]}\left|\left(G_1u\right)(t)\right|$$
$$\leq\sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\phi'(s)\phi_{\alpha_1+\alpha_2-1}(t,s)h_u(s)ds\right.$$
$$+\frac{\Gamma\left(\gamma_1\right)}{\Gamma(\alpha_1)\Gamma\left(\gamma_1+\alpha_2\right)}\left[\frac{\phi_{\gamma_1+\alpha_2-1}\left(b,a\right)\phi_{\gamma_2-1}\left(t,a\right)}{\phi_{\gamma_2-1}\left(b,a\right)}-\frac{\phi_{\gamma_1+\alpha_2-1}\left(t,a\right)}{\phi_{\gamma_1-1}\left(b,a\right)}\right]\int_a^b\phi'(s)\phi_{\alpha_1-1}(b,s)h_u(s)ds$$
$$\left.-\frac{\phi_{\gamma_2-1}\left(t,a\right)}{\Gamma(\alpha_1+\alpha_2)\phi_{\gamma_2-1}\left(b,a\right)}\int_a^b\phi'(s)\phi_{\alpha_1+\alpha_2-1}(b,s)h_u(s)ds\right| \tag{8}$$
$$\leq\left[\frac{2\Upsilon M^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{\Gamma\left(\gamma_1\right)\Upsilon\left[M^{\gamma_1+\alpha_1+\alpha_2-1}+M^{\alpha_1+\alpha_2}\right]}{\Gamma(\alpha_1+1)\Gamma\left(\gamma_1+\alpha_2\right)}\right]\sup_{t\in[a,b]}\left|u(t)\right|$$
$$+\left[\frac{\Gamma\left(\gamma_1\right)\Upsilon\left[M^{\gamma_1+\alpha_1+\alpha_2-1}+M^{\alpha_1+\alpha_2}\right]}{\Gamma(\alpha_1+1)\Gamma\left(\gamma_1+\alpha_2\right)}+\frac{2\Upsilon M^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\right]\sup_{t\in[a,b]}\left|{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}u(t)\right|$$
$$\leq\Lambda_1 r.$$

Also, we have

$$\sup_{t\in[a,b]}\left|\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}\mathrm{G}_1u\right)(t)\right|$$
$$\leq\left[\frac{M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}+\frac{\Gamma\left(\gamma_2\right)M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma\left(\gamma_2-\mu\right)}\right]\sup_{t\in[a,b]}\left|h_u(t)\right| \tag{9}$$
$$+\frac{\Gamma\left(\gamma_1\right)}{\Gamma(\alpha_1+1)}\left[\frac{M^{\alpha_2+\alpha_1-\mu}}{\Gamma\left(\gamma_2\right)\Gamma\left(\gamma_1+\alpha_2-\mu\right)}+\frac{M^{\gamma_1+\alpha_1+\alpha_2-\mu-1}}{\Gamma\left(\gamma_1+\alpha_2\right)\Gamma\left(\gamma_2-\mu\right)}\right]\sup_{t\in[a,b]}\left|h_u(t)\right|$$

$$
\begin{aligned}
&\leq \left\{ \frac{\Upsilon M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)} + \frac{\Upsilon\Gamma(\gamma_2) M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\gamma_2-\mu)} \right.\\
&\quad \left. + \frac{\Upsilon\Gamma(\gamma_1)}{\Gamma(\alpha_1+1)} \left[\frac{M^{\alpha_2+\alpha_1-\mu}}{\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2-\mu)} + \frac{M^{\gamma_1+\alpha_1+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)} \right] \right\} \sup_{t\in[a,b]} |u(t)| \\
&\quad + \left\{ \frac{\Upsilon M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)} + \frac{\Upsilon\Gamma(\gamma_2) M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\gamma_2-\mu)} \right.\\
&\quad \left. + \frac{\Upsilon\Gamma(\gamma_1)}{\Gamma(\alpha_1+1)} \left[\frac{M^{\alpha_2+\alpha_1-\mu}}{\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2-\mu)} + \frac{M^{\gamma_1+\alpha_1+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)} \right] \right\} \sup_{t\in[a,b]} \left| {}^{RL}\mathbf{D}_{a^+}^{\mu;\phi} u(t) \right| \\
&\leq \Lambda_2 r.
\end{aligned}
$$

By (7) - (9), we get

$$
\|G_1 u\|_{C_\phi^\mu} \leq (\Lambda_1+\Lambda_2) r \tag{10}
$$

Similarly, if $v \in \mathbf{U}_r$. Then

$$
\begin{aligned}
&\sup_{t\in[a,b]} |(G_2 v)(t)| \\
&\leq \sup_{t\in[a,b]} \left| \frac{\Gamma(\gamma_1)}{\Gamma(\rho)\Gamma(\gamma_1+\alpha_2)} \left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)} - \frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \right] \int_a^\zeta \phi'(s)\phi_\rho(\zeta,s) v(s)\, ds \right.\\
&\quad \left. + \frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \sum_{i=1}^n \lambda_i v(\zeta_i) \right| \\
&\leq \left[\frac{\Gamma(\gamma_1)\left[M^{\alpha_2+\rho+1} + M^{\gamma_1+\alpha_2+\rho}\right]}{(\rho+1)\Gamma(\rho)\Gamma(\gamma_1+\alpha_2)} + \sum_{i=1}^n |\lambda_i| \right] \sup_{t\in[a,b]} |v(t)| \\
&\leq \Lambda_3 r.
\end{aligned} \tag{11}
$$

Also, we have

$$
\begin{aligned}
&\sup_{t\in[a,b]} \left| \left({}^{RL}\mathbf{D}_{a^+}^{\mu;\phi} \mathrm{G}_2 v \right)(t) \right| \\
&\leq \left\{ \frac{M^{-\mu}\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)} \left(\sum_{i=1}^n |\lambda|_i \right) \right.\\
&\quad \left. + \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\rho+1)} \left[\frac{M^{\alpha_2+\rho-\mu}}{\Gamma(\gamma_2)\Gamma(\gamma_1+\alpha_2-\mu)} + \frac{M^{\gamma_1+\rho+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)} \right] \right\} \sup_{t\in[a,b]} |u(t)| \\
&\leq \Lambda_4 r.
\end{aligned} \tag{12}
$$

By (11) and (12), we get

$$
\|G_2 v\|_{C_\phi^\mu} \leq (\Lambda_3+\Lambda_4) r. \tag{13}
$$

Utilizing (7), (10) and (13), we obtain

$$\left\|G_1u+G_2v\right\|_{C_\phi^\mu} \le \left(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4\right)r \le r \tag{14}$$

Step 2. G_2 is a contraction.

Let $u,v \in \mathbf{U}_r$, we have the following estimate

$$\begin{aligned}&\left|\left(G_2u\right)(t)-\left(G_2v\right)(t)\right|\\ &\le \frac{\Gamma\left(\gamma_1\right)}{\Gamma(\rho)\Gamma\left(\gamma_1+\alpha_2\right)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}\left(t,a\right)}{\varphi_{\gamma_1-1}\left(b,a\right)}+\frac{\varphi_{\gamma_1+\alpha_2-1}\left(b,a\right)\varphi_{\gamma_2-1}\left(t,a\right)}{\varphi_{\gamma_2-1}\left(b,a\right)}\right]\int_a^\zeta \varphi'(s)\varphi_\rho(t,s)\left|u\left(s\right)-v(s)\right|ds\\ &+\frac{\varphi_{\gamma_2-1}\left(t,a\right)}{\varphi_{\gamma_2-1}\left(b,a\right)}\sum_{i=1}^n\lambda_i\left|u\left(\zeta_i\right)-v\left(\zeta_i\right)\right|.\end{aligned}$$

So,

$$\begin{aligned}&\sup_{t\in[a,b]}\left|\left(G_2u\right)(t)-\left(G_2v\right)(t)\right|\\ &\le\left[\frac{\left[M^{\alpha_2}+M^{\gamma_1+\alpha_2-1}\right]M^{\rho+1}\Gamma\left(\gamma_1\right)}{(\rho+1)\Gamma(\rho)\Gamma\left(\gamma_1+\alpha_2\right)}+\left(\sum_{i=1}^n\left|\lambda_i\right|\right)\right]\sup_{t\in[a,b]}\left(\left|u\left(t\right)-v(t)\right|\right)\\ &\le \Upsilon_1\left\|u-v\right\|_{C_\phi^\mu}.\end{aligned} \tag{15}$$

Also,

$$\begin{aligned}&\sup_{t\in[a,b]}\left|\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_2u\right)(t)-\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_2v\right)(t)\right|_C\\ &\le\left[\frac{\Gamma\left(\gamma_1\right)\Gamma\left(\gamma_2\right)}{\Gamma\left(\rho+1\right)}\left[\frac{M^{\alpha_2+\rho-\mu}}{\Gamma\left(\gamma_2\right)\Gamma\left(\gamma_1+\alpha_2-\mu\right)}+\frac{M^{\gamma_1+\rho+\alpha_2-\mu-1}}{\Gamma\left(\gamma_1+\alpha_2\right)\Gamma\left(\gamma_2-\mu\right)}\right]\right.\\ &\left.+\frac{M^{-\mu}\Gamma\left(\gamma_2\right)}{\Gamma\left(\gamma_2-\mu\right)}\left(\sum_{i=1}^n\left|\lambda_i\right|\right)\right]\sup_{t\in[a,b]}\left|u(t)-v(t)\right|.\end{aligned}$$

So,

$$\sup_{t\in[a,b]}\left|\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}Gu\right)(t)-\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}Gv\right)(t)\right|_C \le \Upsilon_2\left\|u-v\right\|_{C_\phi^\mu}. \tag{16}$$

By (15) and (16), yields the following inequality

$$\left\|G_2u-G_2v\right\|_{C_\varphi^\mu} \le \left(\Upsilon_1+\Upsilon_2\right)\left\|u-v\right\|_{C_\varphi^\mu}.$$

Step 3. G_1 is compact and continuous. Since h is a continuous function, this implies that the operator G_1 is continuous on $\mathbf{U}_r$. Moreover, G_1u is uniformly bounded by (14). Next, we show equicontinuity. Let $u \in \mathbf{U}_r$ and $t_1,t_2 \in [a;b]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
&\sup_{t\in[a,b]}\left|\left(G_1u\right)(t_2)-\left(G_1u\right)(t_1)\right| \\
&\le \sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_1+\alpha_2)}\left(\int_a^{t_2}\phi'(s)\phi_{\alpha_1+\alpha_2-1}(t,s)h_u(s)ds-\int_a^{t_1}\phi'(s)\phi_{\alpha_1+\alpha_2-1}(t,s)h_u(s)ds\right)\right. \\
&+\frac{\Gamma(\gamma_1)\phi_{\gamma_1+\alpha_2-1}(b,a)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\phi_{\gamma_2-1}(t_2,a)-\phi_{\gamma_2-1}(t_1,a)}{\phi_{\gamma_2-1}(b,a)}\right]\int_a^b\phi'(s)\phi_{\alpha_1-1}(b,s)h_u(s)ds \\
&+\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2)}\left[\frac{\phi_{\gamma_1+\alpha_2-1}(t_2,a)-\phi_{\gamma_1+\alpha_2-1}(t_1,a)}{\phi_{\gamma_1-1}(b,a)}\right]\int_a^b\phi'(s)\phi_{\alpha_1-1}(b,s)h_u(s)ds \\
&\left.+\frac{\phi_{\gamma_2-1}(t_2,a)-\phi_{\gamma_2-1}(t_1,a)}{\Gamma(\alpha_1+\alpha_2)\phi_{\gamma_2-1}(b,a)}\int_a^b\phi'(s)\phi_{\alpha_1+\alpha_2-1}(b,s)h_u(s)ds\right|.
\end{aligned}$$

By Lemma 2.3, we obtain

$$\begin{aligned}
&\sup_{t\in[a,b]}\left|\left(G_1u\right)(t_2)-\left(G_1u\right)(t_1)\right| \\
&\le \frac{\sup_{t\in[a,b]}\left|h_u(t)\right|}{\Gamma(\alpha_1+\alpha_2+1)}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_1+\alpha_2} \\
&+\left(+\frac{\varphi_{\alpha_1+\alpha_2}(b,a)\sup_{t\in[a,b]}\left|h_u(t)\right|}{\Gamma(\alpha_1+\alpha_2+1)\varphi_{\gamma_2-1}(b,a)}\right)\left(\varphi(t_2)-\varphi(t_1)\right)^{\gamma_2-1}.
\end{aligned}$$

Hence,

$$\sup_{t\in[a,b]}\left|\left(G_1u\right)(t_2)-\left(G_1u\right)(t_1)\right|\to 0,\ \text{ as }\ t_2\to t_1.$$

Also, we can say that

$$\begin{aligned}
&\sup_{t\in[a,b]}\left|\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_1u\right)(t_2)-\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_1u\right)(t_1)\right| \\
&\le \sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_1+\alpha_2-\mu)}\left(\int_a^{t_2}\varphi'(s)\varphi_{\alpha_1+\alpha_2-\mu-1}(t,s)h_u(s)ds-\int_a^{t_1}\varphi'(s)\varphi_{\alpha_1+\alpha_2-\mu-1}(t,s)h_u(s)ds\right)\right. \\
&+\frac{\Gamma(\gamma_2)}{\Gamma(\alpha_1+\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\left[\varphi_{\gamma_2-\mu-1}(t_2,a)-\varphi_{\gamma_2-\mu-1}(t_1,a)\right]}{\varphi_{\gamma_2-1}(b,a)}\int_a^b\varphi'(s)\varphi_{\alpha_1+\alpha_2-1}(b,s)h_u(s)ds \\
&+\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2-\mu)}\left[\frac{\varphi_{\gamma_1+\alpha_2-\mu-1}(t_2,a)-\varphi_{\gamma_1+\alpha_2-\mu-1}(t_1,a)}{\varphi_{\gamma_1-1}(b,a)}\right]\int_a^b\varphi'(s)\varphi_{\alpha_1-1}(b,s)h_u(s)ds \\
&\left.+\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\varphi_{\gamma_1+\alpha_2-1}(b,a)}{\Gamma(\alpha_1)\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\left[\frac{\varphi_{\gamma_2-\mu-1}(t_2,a)-\varphi_{\gamma_2-\mu-1}(t_1,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\int_a^b\varphi'(s)\varphi_{\alpha_1-1}(b,s)h_u(s)ds\right|.
\end{aligned}$$

By Lemma 2.3, we obtain

$$\sup_{t\in[a,b]}\left|\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_1u\right)(t_2)-\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_1u\right)(t_1)\right|$$

$$\leq \frac{\sup_{t\in[a,b]}|h_u(t)|}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_1+\alpha_2-\mu}$$

$$+\left[\frac{\varphi_{\alpha_1+\alpha_2}(b,a)\Gamma(\gamma_2)\sup_{t\in[a,b]}|h_u(t)|}{\Gamma(\gamma_2-\mu)\varphi_{\gamma_2-1}(b,a)}\right]\left(\frac{1}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{\Gamma(\gamma_1)\varphi_{\gamma_1-1}(b,a)}{\Gamma(\alpha_1+1)\Gamma(\gamma_1+\alpha_2)}\right)\left(\varphi(t_2)-\varphi(t_1)\right)^{\gamma_2-\mu-1}$$

$$+\frac{\Gamma(\gamma_1)\varphi_{\alpha_1}(b,a)}{\Gamma(\alpha_1+1)\Gamma(\gamma_1+\alpha_2-\mu)}\left[\frac{\sup_{t\in[a,b]}|h_u(t)|}{\varphi_{\gamma_1-1}(b,a)}\right]\left(\varphi(t_2)-\varphi(t_1)\right)^{\gamma_1+\alpha_2-\mu-1}.$$

Consequently,

$$\sup_{t\in[a,b]}\left|\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_1u\right)(t_2)-\left({}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}G_1u\right)(t_1)\right|\to 0, \text{ as } t_2\to t_1.$$

This shows that $G_1\,\mathbf{U}_r$ is equicontinuous. Hence, by Arzelià-Ascoli theorem G_1 is completely continuous on $\mathbf{U}_r$. As a consequence of Krasnoselskii's fixed point theorem, we conclude that has a fixed point which is a solution of (1.1).

The proof of Theorem 3.1 is thus completely achieved.

3.2. ULAM TYPE STABILITY

We introduce the following two definitions

Definition 3.4. *The problem (1) is Ulam-Hyers stable if* $\exists\ \lambda\in\mathrm{R}_+^*$, *such that for each* $\varepsilon>0, t\in J$, *and for each* $u\in C_\varphi^\mu$ *solution of the following inequality*

$$\left\|{}^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\phi}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u\right)(t)-h(t,u(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}u(t))\right\|_{C_\phi^\mu}<\varepsilon, \tag{17}$$

$\exists v\in C_\varphi^\mu$ *solution of (1), i.e.*

$${}^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\phi}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}v\right)(t)=h(t,v(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}v(t)), \tag{18}$$

such that, the inequality

$$\|u(t)-v(t)\|_{C_\varphi^\mu}\leq\lambda\varepsilon.$$

holds.

Definition3.5. *The equation (1) is generalized stable in sense of Ulam-Hyers if*
$\exists\ \varphi\in C(J,\mathrm{R}_+)$, *such that for each* $\varepsilon>0, t\in J$, *and for each* $u\in C_\varphi^\mu$ *solution of:*

$$\left\| {}^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\phi}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u\right)(t) - h(t,u(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}u(t))\right\|_{C_\phi^\mu} < \varepsilon \tag{19}$$

$\exists v \in C_\varphi^\mu$ *solution of (1) that satisfies*

$$\|u(t)-v(t)\|_{C_\varphi^\mu} \leq \varepsilon\varphi(t).$$

In the light of the first definition and using the above existence and uniqueness theorem, we present to the reader the following result.

Theorem3.2.*If the assumptions ($\mathbf{H}_2$) are satisfied, then Eq (1) is Ulam-Hyers stable under the condition that* $N_1 + N_2 < 1$, *where*

$$N_1 = \Upsilon M^{\alpha_1+\alpha_2}\left(\frac{2}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(1+M)\Gamma(\gamma_1)}{M\Gamma(\alpha_1+1)\Gamma(\gamma_1+\alpha_2)}\right),$$

and

$$N_2 = \Upsilon M^{\alpha_1+\alpha_2-\mu}\left(\frac{1}{\Gamma(\alpha_1+\alpha_2-\mu+1)} + \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\gamma_2-\mu)}\right)$$
$$+\frac{\Upsilon\Gamma(\gamma_1)M^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\gamma_1+\alpha_2-\mu)} + \frac{\Gamma(\gamma_2)}{M\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\right).$$

Proof. Let $u \in C_\varphi^\mu$ be a solution of the inequality (17), i.e.

$$\left\| {}^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\phi}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\phi}u\right)(t) - h(t,u(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}u(t))\right\|_{C_\phi^\mu} < \varepsilon, \quad \forall t \in J. \tag{20}$$

Let $v \in C_\varphi^\mu$ be a unique solution of the following equation

$${}^{H}\mathbf{D}_{a^+}^{\alpha_1,\beta_1;\varphi}\left({}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\varphi}v\right)(t) = h(t,v(t),{}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}v(t)), \quad \forall t \in J$$

and

$$\begin{cases} u(a)=v(a),\ u(b)=v(b) \\ \text{and} \\ {}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(a) = {}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\varphi}v(a),\ {}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(b) = {}^{H}\mathbf{D}_{a^+}^{\alpha_2,\beta_2;\varphi}v(b), \end{cases}$$

By using Proof of Lemma 2.3

$$\begin{aligned} v(t) &= \mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}h_v(t) \\ &+\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)} - \frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\mathbf{I}_{a^+}^{\rho;\varphi}u(\zeta) \\ &-\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)}\left[\frac{\varphi_{\gamma_1+\alpha_2-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)} - \frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\right]\mathbf{I}_{a^+}^{\alpha_1;\varphi}h_v(b) \\ &+\left[\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)\right] - \left[\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}h_v(b)\right]. \end{aligned}$$

By integration of inequality (20), for any $t \in J$, we have

$$\begin{aligned}
&\Bigg\| u(t) - \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} h_u(t) \\
&- \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)} \left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)} - \frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \right] \mathtt{I}_{a^+}^{\rho;\phi} u(\zeta) \\
&+ \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)} \left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)} - \frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \right] \mathtt{I}_{a^+}^{\alpha_1;\phi} h_u(b) \\
&- \left[\frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \sum_{i=1}^{n} \lambda_i u(\zeta_i) \right] + \left[\frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} h_u(b) \right] \Bigg\|_C \\
&\leq \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} \varepsilon = \frac{\varepsilon t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}.
\end{aligned} \tag{21}$$

On the other hand, for any $u \in C_{\varphi}^{\mu}$, we have the following estimate

$$\begin{aligned}
\| u(t) - v(t) \|_C = &\Bigg\| u(t) - \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} h_u(t) \\
&- \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)} \left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)} - \frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \right] \mathtt{I}_{a^+}^{\rho;\phi} u(\zeta) \\
&+ \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)} \left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)} - \frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \right] \mathtt{I}_{a^+}^{\alpha_1;\phi} h_u(b) \\
&- \left[\frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \sum_{i=1}^{n} \lambda_i u(\zeta_i) \right] + \left[\frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} h_u(b) \right] \Bigg\|_C \\
&+ \Bigg\| \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} \left(h_u(t) - h_v(t) \right) \\
&- \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)} \left[\frac{\phi_{\gamma_1+\alpha_2-1}(t,a)}{\phi_{\gamma_1-1}(b,a)} - \frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \right] \mathtt{I}_{a^+}^{\alpha_1;\phi} \left(h_u(b) - h_v(b) \right) \\
&- \frac{\phi_{\gamma_2-1}(t,a)}{\phi_{\gamma_2-1}(b,a)} \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} \left(h_u(b) - h_v(b) \right) \Bigg\|_C .
\end{aligned}$$

So, we have

$$\begin{aligned}
&\| u(t) - v(t) \|_C \\
&\leq \frac{\varepsilon t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \\
&+2 \left\| \mathtt{I}_{a^+}^{\alpha_1+\alpha_2;\phi} \left(h_u(t) - h_v(t) \right) \right\|_C + \frac{\left(M^{\alpha_2} + M^{\alpha_2-1} \right) \Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2)} \left\| \mathtt{I}_{a^+}^{\alpha_1;\phi} \left(h_u(b) - h_v(b) \right) \right\|_C
\end{aligned} \tag{22}$$

$$\leq \frac{\varepsilon t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \Upsilon M^{\alpha_1+\alpha_2}\left(\frac{2}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(1+M)\Gamma(\gamma_1)}{M\Gamma(\alpha_1+1)\Gamma(\gamma_1+\alpha_2)}\right)\|u-v\|_{C_\phi^\mu}$$
$$\leq \frac{\varepsilon b^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + N_1\|u-v\|_{C_\phi^\mu}.$$

Also, for any $t \in J$, we have

$$\Bigg\| {}^{RL}\mathbf{D}_{a^+}^{\mu;\varphi}u(t) - \mathbf{I}_{a^+}^{\alpha_1+\alpha_2-\mu;\varphi}h_u(t) - \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}h_u(b)$$
$$+\left[\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2-\mu)}\frac{\varphi_{\gamma_1+\alpha_2-\mu-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\right]\mathbf{I}_{a^+}^{\alpha_1;\varphi}h_u(b)$$
$$+\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_1+\alpha_2-1}(b,a)\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_1;\varphi}h_u(b)\Bigg\|_C < \frac{\varepsilon t^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}$$

Also, one can prove that

$$\left\|{}^{RL}\mathbf{D}_{a^+}^{\mu;\phi}\left(u(t)-v(t)\right)\right\|_C$$
$$<\Bigg\|\mathbf{I}_{a^+}^{\alpha_1+\alpha_2-\mu;\phi}\left(h_u(t)-h_v(t)\right) - \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\phi_{\gamma_2-\mu-1}(t,a)}{\phi_{\gamma_2-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\phi}\left(h_u(t)-h_v(t)\right)$$
$$+\frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1+\alpha_2-\mu)}\frac{\phi_{\gamma_1+\alpha_2-\mu-1}(t,a)}{\phi_{\gamma_1-1}(b,a)}\mathbf{I}_{a^+}^{\alpha_1;\phi}\left(h_u(t)-h_v(t)\right)$$
$$+\left[\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\phi_{\gamma_1+\alpha_2-1}(b,a)\phi_{\gamma_2-\mu-1}(t,a)}{\phi_{\gamma_2-1}(b,a)}\right]\mathbf{I}_{a^+}^{\alpha_1;\phi}\left(h_u(t)-h_v(t)\right)\Bigg\|_C \tag{23}$$
$$+\frac{\varepsilon t^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}$$
$$<\Upsilon M^{\alpha_1+\alpha_2-\mu}\left(\frac{1}{\Gamma(\alpha_1+\alpha_2-\mu+1)}+\frac{\Gamma(\gamma_2)}{\Gamma(\alpha_1+\alpha_2+1)\Gamma(\gamma_2-\mu)}\right)\|u-v\|_{C_\phi^\mu}$$
$$+\left(\frac{\Gamma(\gamma_1)M^{\alpha_2-\mu}}{\Gamma(\gamma_1+\alpha_2-\mu)}+\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)M^{\gamma_1+\alpha_2-\mu-1}}{\Gamma(\gamma_1+\alpha_2)\Gamma(\gamma_2-\mu)}\right)\frac{\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)}\|u-v\|_{C_\phi^\mu}$$
$$+\frac{\varepsilon t^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}$$
$$<\frac{\varepsilon b^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}+N_2\|u-v\|_{C_\phi^\mu}.$$

So, by (22) and (23) we have

$$\|u-v\|_{C_\varphi^\mu} \leq \varepsilon\left(\frac{b^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{b^{\alpha_1+\alpha_2-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\right)+(N_1+N_2)\|u-v\|_{C_\varphi^\mu}.$$

Therefore, we get

$$\|u(t)-v(t)\|_{C_{\varphi}^{\mu}} \leq \lambda\varepsilon.$$

where

$$\lambda = \frac{b^{\alpha_1+\alpha_2}}{1-(N_1+N_2)}\left(\frac{1}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{b^{-\mu}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\right),$$

for any $t \in J$. This implies that the Ulam-Hyers stability condition is satisfied.

4. EXAMPLE

Consider the following problem

$$\begin{cases} {}^{H}\mathbf{D}_{0^+}^{\frac{3}{2},0;t^2}\left({}^{H}\mathbf{D}_{0^+}^{\frac{3}{2},0;t^2}u\right)(t) = h(t,u(t),{}^{RL}\mathbf{D}_{a^+}^{1;t^2}u(t)), t\in J = \left[0,\frac{1}{5}\right] \\ u(0)=0, u(\frac{1}{5}) = \sum_{i=1}^{2}\frac{1}{40}u\left(\zeta_i\right), \\ {}^{H}\mathbf{D}_{0^+}^{\frac{3}{2},0;t^2}u(0)=0, {}^{H}\mathbf{D}_{0^+}^{\frac{3}{2},0;t^2}u(\frac{1}{5}) = \mathbf{I}_{a^+}^{\frac{3}{2};t^2}u\left(\zeta\right) \\ f(t,u(t),v(t)) = \dfrac{u(t)}{70\left(1+t^2\right)}+\dfrac{v(t)}{70\left(1+e^t\right)}, \end{cases} \tag{24}$$

then assumptions ($\mathbf{H}_1$) and ($\mathbf{H}_2$) are satisfied with $\Upsilon = \frac{1}{70}$, and $M = \frac{1}{25}$.

By according to a little calculation, we get $\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4 \leq 1,\quad \Upsilon_1+\Upsilon_2<1$ and $N_1+N_2 < 1$. We conclude that (24) has at least one solution and stable.

5. CONCLUSION

In this article, we have demonstrated the existence of solutions and stability of a nonlinear differential problem via phi - Hilfer's derivative using the *fixed point theorem of Krasnoselskii's*, we have completed our work by example illustrative.

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