ORIGINAL PAPER

COMPUTATIONAL SOLUTION OF FRACTIONAL REACTION DIFFUSION EQUATIONS VIA AN ANALYTICAL METHOD

KANZA NOOR[1], JAMSHAD AHMAD[1]

Manuscript received: 26.12.2021; Accepted paper: 18.08.2022;
Published online: 30.09.2022.

Abstract. *In science and technology, the phenomena of transportation are crucial. Advection and diffusion can occur in a wide range of applications. Distinct types of decay rates are feasible for different non-equilibrium systems over lengthy periods of time when it comes to diffusion. In engineering, biology, and ecology, the problems under study are used to represent spatial impacts. The fast Adomian decomposition method (FADM) is used to solve time fractional reaction diffusion equations, which are models of physical phenomena, in the current study. Caputo fractional derivative meaning applies to the specified time derivative. The results are in series form and correspond to the proposed fractional order problem. These models have a strong physical foundation, and their numerical treatments have significant theoretical and practical applications. The leaning of the rapid convergence of method-formulated sequences towards the appropriate solution is also graphically depicted. With less computational cost, this solution quickly converged to the exact solution.*

Keywords: *Fractional Taylor series; fast Adomian decomposition method; fractional reaction diffusion equation.*

1. INTRODUCTION

Spreading is a relatively regular and frequent occurrence. Diffusion is a term used to describe a process that is sufficiently slow and has particular characteristics. Self-diffusion is the process through which a particle diffuses into its own surroundings. Different parameters in different areas of space are commonly used to describe this type of diffusion. Some important problems in dynamical systems, diffusion wave, heat conduction, cellular systems, oil industries, signal processing, control theory, fluid mechanics, and other fields of science and engineering would be more precise if fractional differential equations were used; for example, see Podlubny [1]. Various researchers have investigated computational methods for achieving an approximate solution of fractional differential equations [2, 3]. Partial differential equations [4-7] such as the heat flow equation and wave propagation equation, as well as equations in plasma, and electricity, explain the majority of mathematical, technical, and physical issues. All of these phenomena are described in physics by linear or non-linear equations, which can be solved analytically. We may not be able to find an analysis, so the researchers based their efforts on solving this form of equation and introduced the Adomian Decomposition Method, a new method for solving it. In the 1980s, the Adomian proposed and developed an empirical approach for solving linear and non-linear equations. This approach approximates accepted differential equations, as well as random linear and nonlinear

[1] University of Gujrat, Department of Mathematics, 50700 Punjab, Pakistan.
E-mail: jamshadahmadm@gmail.com; kanzanoor4@gmail.com.

https://doi.org/10.46939/J.Sci.Arts-22.3-a08

equations, in an accurate and convergent manner. Differential equations can be solved to a certain degree of approximation [8].

Fractional calculus is a very important area of study for most researchers and scientists because of its broad application. Many researchers have concentrated their efforts in the field of fractional calculus on the study of fractional order partial differential equations. Linear and non-linear FODEs have been solved using different methods in this regard. In [9], for example, has a discussion of the modified Bernoulli sub-equation and non-linear time fractional Burgers equations. In [10, 11] was carried out a computational simulation of space fractional diffusion equations. The four variants of time-fractional nonlinear Caputo-type Newell– Whitehead–Segel (NWS) problem are solved by fractional variational iteration method in [9]. The solution of Burgers-Huxley models [13] has been computed using OHAM. In [14], different types of fuzzy integro-differential equations Via Laplace homotopy perturbation method. In the same line, [15] discusses the approximate solution to the generalised Mittag-Leffler law via exponential decay. In addition, [16] discusses various applications of derivatives and integrals of arbitrary order. In order to advance this area, some researchers provided numerical schemes and stability for two groups of FOPDEs in [17, 18].

Natural phenomena have significant implications for applied mathematics, physics, and engineering; many of these physical phenomena are expressed as fractional order linear and nonlinear PDEs. Obtaining approximate or exact solutions to those PDEs is a constant problem that necessitates the development of novel methods for obtaining approximate or exact solutions [19-24]. Many areas of interest, such as chemical processes, chemistry, thermodynamics, bioinformatics, coral reefs, and engineering, use reaction-diffusion equations as a model for many evolution phenomena. Many scholars and researchers have clarified the significance of reaction-diffusion equations and some important physical issues. To solve fractional reaction diffusions, an effective algorithm with finite difference method and domain decomposition approach was recently applied in [25, 26]. Consider the general time fractional RDE is as follow

$$ {}^{c}D_{t}^{\gamma} v_i + D v_{ixx} = R(v_i), \quad 0 < \gamma \leq 1, \tag{1} $$

with condition

$$ v_i\,(x,0) = f_i(x),\ i = 1 \leq n. \tag{2} $$

2. MATERIALS AND METHODS

2.1. BASIC PRELIMINARIES

Definition 2.1. The Caputo fractional derivative of function v(t) is

$$ {}^{c}D_{t}^{\gamma} v(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{n-\gamma-1}\, v^n(t)dt, \tag{3} $$

where $n-1 < \gamma < n,\ n \epsilon N, t > 0, v \in C_{-1}^{n}$.

Definition 2.2. The RLF operator of a is

$$\mathrm{I}_t^{\gamma} v(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1}\, v(\tau) d\tau,\ 0 < \gamma, \tag{4}$$

where $\Gamma(.)$ is the renowned Gamma function.

Property 2.3. The Leibniz law integral

$$\mathrm{I}_t^{\gamma}\, {}^{c}\mathrm{D}_t^{\gamma} v(t) = v(t) - v(0), 0 < \gamma \le 1,\ 0 < t. \tag{5}$$

Property 2.4. The Adomian polynomial for non-linear terms $N(v)$,

$$A_k = \frac{1}{k} \sum_{j=0}^{k-1} (j+1) \varphi_k \frac{dA_{k-1-j}}{d\varphi_0}. \tag{6}$$

2.2. METHOD

Taking the R-L integral of order γ of equation (1) on both sides, we get

$$\mathrm{I}_t^{\gamma}[\ {}^{c}\mathrm{D}_t^{\beta} v_i] = \mathrm{I}_t^{\beta}[R(v_i) - Dv_{ixx}], \tag{7}$$

by applying Leibniz integral law

$$v_i(x,t) = v_i(x,0) + \mathrm{I}_t^{\gamma}[R(v_i) - Dv_{ixx}], \tag{8}$$

by using initial conditions

$$v_i(x,t) = f_i(x) + \mathrm{I}_t^{\gamma}[R(v_i)] - \mathrm{I}_t^{\gamma}[Dv_{ixx}]. \tag{9}$$

Consider the solution in fractional Taylor series

$$v_i(x,t) = \sum_{n=0}^{\infty} v_{i,n} = \sum_{n=0}^{\infty} \varphi_{i,k} t^{\gamma k}, \varphi_{i,0} = f_{i,0}(x). \tag{10}$$

The recurrence relation for non-linear source term is

$$\begin{cases} \varphi_{i,0} = f_{i,0}(x)\ , \\ \varphi_{i,k+1} = \frac{\Gamma(k\gamma+1)}{\Gamma(k\gamma+\gamma+1)} \left[D\varphi_{ixx,k} + R\left(A_{i,k}\right)\right].\ \ k \ge 1 \end{cases} \tag{11}$$

where $A_{i,k}$ represents Adomian polynomial defined by (6)

Thus, the required solution is obtained by

$$v_i(x,t) = \sum_{n=0}^{\infty} v_{i,n} = \sum_{n=0}^{\infty} \varphi_{i,k} t^{\gamma k}. \tag{12}$$

3. RESULTS AND DISCUSSION

Consider time fractional reaction diffusion equation having linear source term [27]

$$ {}^{c}D_t^{\gamma} v = v_{xx} + v,\ 0 < \gamma \le 1. \tag{13}$$

with attached condition

$$v(x,0) = e^{-x} + x.$$

Applying the RL integration of Eq. (13)

$$v(x,t) = v(x,0) + I_t^{\gamma}[v_{xx} + v], \tag{14}$$

by using initial condition

$$v(x,t) = e^{-x} + x + I_t^{\gamma}[v_{xx} + v]. \tag{15}$$

Considering the solution in fractional Taylor series

$$v(x,t) = \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} \varphi_k t^{\gamma k}, \varphi_0 = e^{-x} + x. \tag{16}$$

The iteration formula is

$$\begin{cases} \varphi_0 = e^{-x} + x, \\ \varphi_{k+1} = \frac{\Gamma(k\gamma+1)}{\Gamma(k\gamma+\gamma+1)}[K(\varphi_k)].\ \ k \ge 1 \end{cases} \tag{17}$$

by the iteration, the following results are obtained

$$\varphi_0 = e^{-x} + x, \varphi_1 = \frac{-x}{\Gamma(\gamma+1)},\ \varphi_2 = \frac{x}{\Gamma(2\gamma+1)},\ \varphi_3 = \frac{-x}{\Gamma(3\gamma+1)},$$

and so on. Thus,

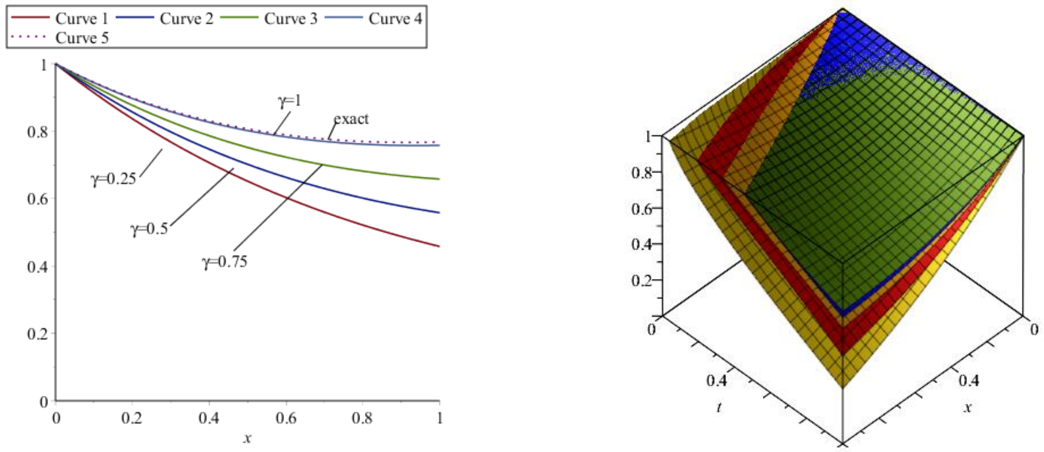
$$v(x,t) = e^{-x} + x - \frac{xt^{\gamma}}{\Gamma(\gamma+1)} + \frac{xt^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots, \tag{18}$$

If $\gamma = 1$, one have

$$v(x,t) = e^{x} + x - \frac{xt^{1}}{\Gamma(2)} + \frac{xt^{2}}{\Gamma(3)} - \frac{xt^{3}}{\Gamma(4)} + \cdots, \tag{19}$$

and finally we have

$$v(x,t) = e^{x} + xe^{-t}. \tag{20}$$



(a) **(b)**

Figure 3.1. (a) 2-D and (b) 3-D multiple plots of the solution of Eq. (14) show the numerical simulation of two term approximate solution respectively. In these graphs we can see that by changing the value of γ, these surfaces show the small change. For $\gamma = 1$, the graph of the solution is nearly same as for exact solution. The accuracy can be enhanced by presenting more terms in solutions.

Consider time fractional reaction diffusion equation [27

$$ {}^{c}D_t^{\gamma} v = v_{xx} - v_x + v_{xx} v - v^2 + v, \qquad 0 < \gamma \leq 1. \tag{21}$$

subject to condition

$$ v(x,0) = e^x. \tag{22}$$

Taking the RL integration of Eq. (21) and using given condition (22), we have

$$ v(x,t) = e^x + I_t^{\gamma}[v_{xx} - v_x + v_{xx}v - v^2 + v], \tag{23}$$

$$ v(x,t) = e^x + I_t^{\gamma}[v_{xx} - v_x + v] + I_t^{\gamma}[v_{xx}v - v^2] \tag{24}$$

Taking

$$ K\big(v(x,t)\big) = [v_{xx} - v_x + v], \text{ and } N(v(x,t)) = [v_{xx}v - v^2]. $$

Eq. (24) becomes

$$ v\,(x,t) = e^x + I_t^{\gamma} K\big(v(x,t)\big) + I_t^{\gamma} N\big(v(x,t)\big). \tag{25}$$

Consider the solution in fractional Taylor series

$$ v(x,t) = \sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty} \varphi_k t^{\gamma k}, \varphi_0 = e^x. \tag{26}$$

The iteration formula is

$$ \begin{cases} \varphi_0 = e^x, \\ \varphi_{k+1} = \frac{\Gamma(k\gamma+1)}{\Gamma(k\gamma+\gamma+1)}[K(\varphi_k) + N(A_k)]. \quad k \geq 1. \end{cases} \tag{27}$$

By the iteration, the following results are obtained

$$\varphi_0 = e^x,\ \varphi_1 = \frac{e^x}{\Gamma(\gamma+1)},\ \varphi_2 = \frac{e^x}{\Gamma(2\gamma+1)},\ \varphi_3 = \frac{e^x}{\Gamma(3\gamma+1)}, \ldots.$$

Thus,

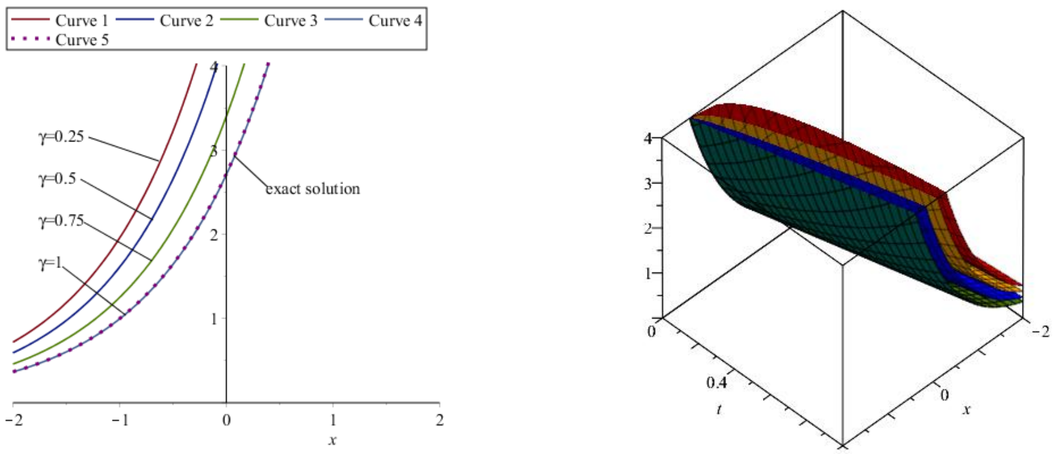
$$v(x,t) = e^x + \frac{e^x t^{\gamma}}{\Gamma(\gamma+1)} + \frac{e^x t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{e^x t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots, \tag{28}$$

If $\gamma = 1$, from Eq. (28) we have

$$v(x,t) = e^x + \frac{e^x t^1}{\Gamma(2)} + \frac{e^x t^2}{\Gamma(3)} + \frac{e^x t^3}{\Gamma(4)} + \cdots, \tag{29}$$

and the solution is

$$v(x,t) = e^{x+t}. \tag{30}$$



(a) **(b)**

Figure 3.2. (a) 2-D and (b) 3-D multiple plots of the solution of Eq. (21) shows the numerical simulation of two term approximate solution respectively In these graphs we can see that by changing the value of *γ*, these surfaces show the small change. For *γ* =1, the graph of the solution is nearly same as for exact solution. The accuracy can be enhanced by presenting more terms in the considering approximate solutions.

Taking a system of TFRDEs [28]

$$D_t^{\gamma} v_1 = v_1^2 v_2 - 2v_1 + \frac{1}{4\{v_{1xx} + v_{1yy}\}}, \tag{31}$$

$$D_t^{\gamma} v_2 = v_1 - v_1^2 v_2 + 1/4\{v_{2xx} + v_{2yy}\}, \tag{32}$$

attached to conditions

$$v_1(x,y,0) = e^{-x-y}, \quad v_2(x,y,0) = e^{x+y}.$$

With the RL integration to Eq. (31) and Eq. (32)

$$v_1(x,y,t) = v_1(x,y,0) + I_t^{\gamma}[v_1^2 v_2 - 2v_1 + 1/4\{v_{1xx} + v_{1yy}\}], \tag{33}$$

$$v_2(x,y,t) = v_2(x,y,0) + I_t^{\gamma}\left[v_1 - v_1^2 v_2 + \frac{1}{4\{v_{2xx} + v_{2yy}\}}\right]. \tag{34}$$

Further, we have

$$v_1(x,y,t) = e^{-x-y} + I_t^{\gamma}[v_1^2 v_2 - \ 2v_1 + 1/4\{v_{1xx} + v_{1yy}\}], \tag{35}$$

$$v_2(x,y,t) = e^{x+y} + I_t^{\gamma}[v_1 - v_1^2\ v_2 + 1/4\{v_{2xx} + v_{2yy}\}]. \tag{36}$$

Let K_1, K_2 be linear terms and Let N_1, N_2 be non-linear terms of the Eq. (35) and Eq. (36)

$$v_1(x,y,t) = e^{-x-y} + I_t^{\gamma}[N_1(v_1 v_2(x,y,t)) - K_1(v_1 v_2(x,y,t))], \tag{37}$$

$$v_2(x,y,t) = \ e^{x+y} + I_t^{\gamma}[K_2(v_2 v_1(x,y,t)) - N_2(v_1 v_2(x,y,t))]. \tag{38}$$

Consider the solution in fractional Taylor series

$$v_1(x,y,t) = \sum_{n=0}^{\infty} v_{1,n} = \sum_{n=0}^{\infty} \varphi_{1,k} t^{\gamma k}, \varphi_{1,0} = f_{1,0}(x,y)$$

$$v_2(x,y,t) = \sum_{n=0}^{\infty} v_{2,n} = \sum_{n=0}^{\infty} \varphi_{2,k} t^{\gamma k}, \varphi_{2,0} = f_{2,0}(x,y)$$

According to described steps, we have

$$\varphi_{1,0} = e^{-x-y}, \varphi_{2,0} = e^{x+y},\ \varphi_{1,1} = \frac{-e^{-x-y}}{2\Gamma(\gamma+1)},\ \ \varphi_{2,1} = \frac{e^{x+y}}{2\Gamma(\gamma+1)},$$

$$\varphi_{1,2} = \frac{e^{-x-y}}{\Gamma(2\gamma+1)\Gamma(\gamma+1)}\left[\frac{-1}{2\Gamma(\gamma+1)} + \frac{3}{4}\right],$$

$$\varphi_{2,2} = \frac{1}{\Gamma(2\gamma+1)}\left[\left(\frac{e^{x+y}}{4\Gamma(\gamma+1)}\right) + \frac{e^{-x-y}}{2\Gamma(\gamma+1)}\left(-1 + \frac{1}{\Gamma(\gamma+1)}\right)\right],$$

and so on.

Finally, we have

$$v_1(x,y,t) = e^{-x-y} - \frac{e^{-x-y} t^{\gamma}}{2\Gamma(\gamma+1)} + \frac{e^{-x-y} t^{2\gamma}}{\Gamma(2\gamma+1)\Gamma(\gamma+1)}\left[\frac{-1}{2\Gamma(\gamma+1)} + \frac{3}{4}\right] + \cdots, \tag{39}$$

$$v_2(x,y,t) = e^{x+y} + \frac{e^{x+y} t^{\gamma}}{2\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)}\left[\left(\frac{e^{x+y}}{4\Gamma(\gamma+1)}\right) + \frac{e^{-x-y}}{2\Gamma(\gamma+1)}\left(-1 + \frac{1}{\Gamma(\gamma+1)}\right)\right] + \ldots \tag{40}$$

If $\gamma = 1$, then Eq. (39) and (40) reduces to

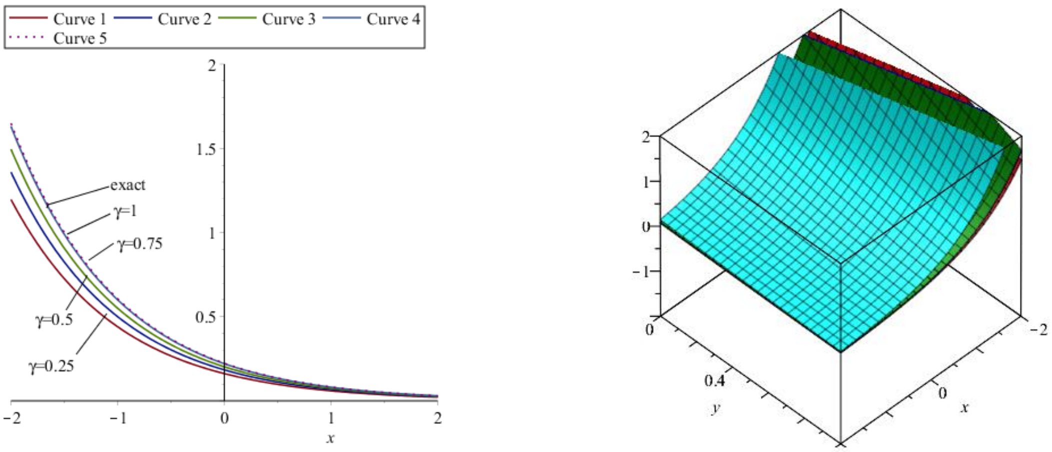
$$v_1(x,y,t) = e^{-x-y} - \frac{e^{-x-y}t}{2} + \frac{e^{-x-y}t^2}{8} + \cdots, \tag{41}$$

$$v_2(x,y,t) = e^{x+y} + \frac{e^{x+y}t}{2} + \frac{e^{x+y}t^2}{8} + \cdots. \tag{42}$$

and these lead to

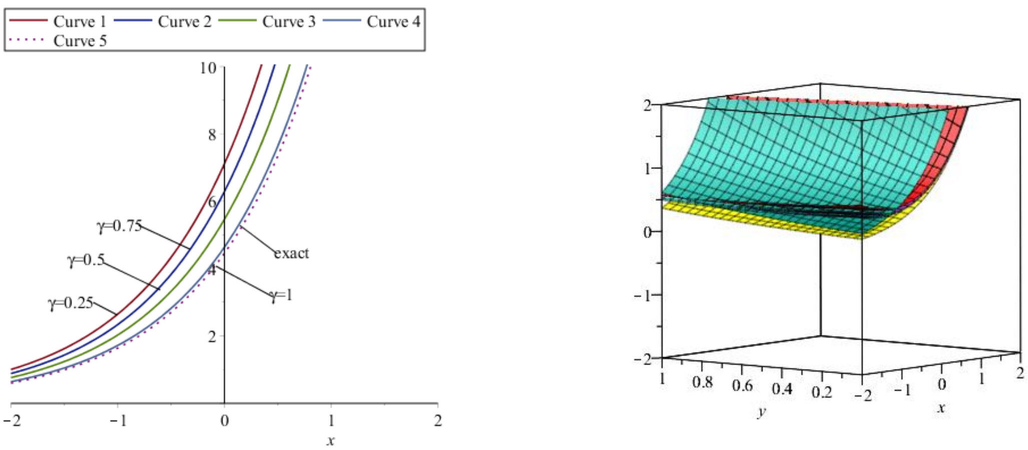
$$v_1(x,y,t) = e^{-x-y-\frac{t}{2}}, \tag{43}$$

$$v_2(x,y,t) = e^{x+y+\frac{t}{2}}. \tag{44}$$



(a) (b)

Figure 3.3. (a) 2-D and (b) 3-D multiple plots of the solution of Eq. (31) show the numerical simulation of two term approximate solution respectively. In these graphs, we can see that by changing the value of *γ*, these surfaces show the small change. For *γ* =1, the graph of the solution is nearly same as for exact solution. The accuracy can be enhanced by taking more terms in the approximate solutions.



(a) (b)

Figure 3.4. (a) 2-D and (b) 3-D multiple plots of the solution of Eq. (32) show the numerical simulation of two term approximate solution respectively. In these surface graphs, we can see that by changing the value of *γ*, these surfaces show the small change. For *γ* =1, the graph of the solution is nearly same as for exact solution. The better results can be enhanced by taking more terms in the approximate solutions.

4. CONCLUSIONS

In this paper, the fast ADM solutions compare well with the exact solution to the time fractional reaction diffusion equation in the three cases as shown in figures. The fundamental objective of the work is to implement the fast ADM to find the approximate solution. The reliability and efficacy of this method are shown by solving different TFRDEs and swift to the approximate solutions are also revealed analytically and graphically. The graphical comparison shows the diffusion behavior influence of fractional orders of the solutions indicating the proposed method yields very accurate and convergent solutions. It is observed that the approximate series solutions up to two terms are very precise. The required solutions specify that, the FADM contains a satisfactory less amount of work to solve the TFRDEs.

REFERENCES

[1] Podlubny, I., *IEEE Transactions on Automatic Control*, **44**, 208, 1999.
[2] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and applications of fractional differential equations,* Amsterdam: Elsevier Science, 2006.
[3] Lakshmikantham, V., Leela, S., Devi, J.V., *Theory of fractional dynamic systems*, Cambridge, England: Cambridge Scientific Publishers, 2009.
[4] Abdulwahed, H,G.,El-Shewy, E,K., Alghanim,S., Abdulrehman, M,A,E., *Fractal and Fractional*, **6**, 1, 2022.
[5] Islam, S.M.R., Khan, S., Arafat, S.M.Y., Akbar, M.A., *Results in Physics*, **40**, 1, 2022.
[6] Zulfiqar, A., Ahmad, J., Hassan, Q.M., *Optical and Quantum Electronics*, **54**, 1, 2022.
[7] Deniz, S., Bildik, N., *International Journal of Modeling and Optimization*, **4**, 292, 2014.
[8] Mustafa, I., *Journal of Zhejiang University-Science A*: *Applied Physics & Engineering*, **6**, 1058, 2005.
[9] Zulfiqar, A., Ahmad, J., *Journal of Science and Arts*, **55**, 487, 2021.
[10] Hashim, I., Noorani, M.S.M., Al-Hadidi, M.R.S., *Mathematical and Computer Modelling*, **43**, 1404, 2006.
[11] Supriya, Y., Devendra, K.,Kottakkaran, S.N., *Journal of King Saud University–Science,* **33**(2), 101320, 2021.
[12] Zulfiqar, A., Ahmad, J., Hassan, Q,M., *Journal of Science and Arts,* **49**, 839, 2019.
[13] Batiha, B., Noorani, Hashim, M.S.M., *Chaos, Solitons & Fractals*, **36**, 660, (2008).
[14] Ahmad, J., Nosher, H., *Journal of Science and Arts*, **38**, 5, 2017.
[15] Atangana, A., Gmez-Aguilar, J.F., *Numerical Methods for Partial Differential Equations*, **34**, 1502, 2018.
[16] Abdeljawad, T., *Computers & Mathematics with Applications*, **62**, 1602, 2011.
[17] Li, Y., Haq, F., Shah, K., Shahzad, M., Rahman, *Journal of Mathematical and Computational Science*, **17**, 420, 2017.
[18] Shaikh, A., Tassaddiq, A., Nisar, K.S., Baleanu, D., *Advances in Difference Equations*, **2019**, 178, 2019.
[19] Zulfiqar, A., Ahmad, J., *Ain Shams Engineering Journal*, **12**, 2021, https://doi.org/10.1016/j.asej.2021.06.014.
[20] Rehman, S.U., Ahmad, J., *Alexandria Engineering Journal*, **60**, 1339, 2021.
[21] Aksoy, Y., Pakdemirli, M., *Computers & Mathematics with Applications*, **59**, 2802, 201.
[22] Ahmad, J., Mohyud-Din, S.T., Yang, X.J., *Journal of Science and Arts*, **14**(1), 73, 2014.

[23] Ahmad, J., Mustaq, M., Sajjad, N., *Journal of Arts and Science*, **15**(1), 5, 2015.
[24] Olayiwola, M.O., Adisa, S.A., Adegoke, A., Usman, M.A., Ozoh, P., *Journal of Science and Arts*, **21**(3), 699, 2021.
[25] Gong, C. Bao, W., Tang, G., Yang, B., *Journal of Supercomputing*, **68**,1521, 2014.
[26] Gong, C. Bao, W., Tang, G., Jiang, Y., Liu, J., *Scientific World Journal*, **2014**, 681707, 2014.
[27] Rajaraman, R., Hariharan, G., Kannan, K., *Asian Journal of Current Engineering and Mathematics*, **2**, 24, 2013.
[28] Shidfar, L., Paivarinta, L., Molabahrami, A., *Proceedings of International Symposium on Intelligent Signal Processing and Communications Systems, ISPACS 2011*, 2011.