ORIGINAL PAPER

APPLICATION OF ADOMIAN DECOMPOSITION, VARIATIONAL ITERATION, AND SERIES SOLUTION METHODS TO ANALYSIS OF INTEGRAL DIFFERENTIAL EQUATIONS

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Abstract. *In this paper, the analytical solution of integral equations is presented by using various advance analytical techniques. The comparison between the prososed methods: variational iteration method (VIM), and series solution method (SSM) with the Adomian decomposition equations is given to show the effeficency of these methods. From the Mathematical point of view, the variational iteration method (VIM) is effective, appropriate and easily using to solve the problems. Particularly, the langrange multiplier in variational iteration method plays very importnant role to reduce the computational work of integration. At the end, numerical and graphical results are obtained by using Maple programing.*

Keywords: *variational iteration method; Lagrange multiplier; differential equation; series solution method; adomian decomposition method.*

1. INTRODUCTION

In real-life problems mathematical modeling was generally results in these types of equations, like ordinary and partial differential equations, Integro-differential equation. Many Mathematical problems in physical sight contains integro differential equations, these equations used in numerous areas say Biological models, Fluid dynamic, Economics [1-20]. Further, analytically to solve the intrgro-diffrential equations are very difficult, so it is necessitating to acquire effective approximate solution to determining from the nonlinear problems.

The Adomian decomposition method (ADM) is very fomous analytical technique which plays very significant role to deal the nonlinear phenomina which arrising almost every field of science and this method suggested by Adomian [1-5]. This method has given very effective conclusion in analytical approximations that connected for nonlinear problems. But sometime, when the complicated nonlinearity occur, then this method may be difficult to use. However, in the literature are some other advance techniques which are more effective and easy to compute results.

The main aim of this work is to introduce the recent analytical techniques like, the variational iteration method (VIM) [5-7] and series solution method (SSM) [8-12]. These are very simple and effective methods and become more significant in the field of nonlinear scinces. In the beginning, firstly VIM proposed by J.H. He [9] and this method was

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successfully applied on many problems. For example, He used the VIM to finding the classically Blasius equation. In the recent time, many mathematicians are using these techniques. For example Ehsan et al. [9] used this techniques to find the thermo migrated radiative nanofluid flow towards a moving wedge with combined magnetic force and porous medium. He solved the autonomous ordinary differential system also by using VIM [10]. Using VIM, Soliman et al. [17] solved KdV equations. Similarly, Riaz et al. [11] was discussed nonlinear system analytically tackled with implementation of variational iteration method (VIM).

The main purpose of this paper is the introduction of the variational iteration method (VIM) and series solution method (SSM) for the integro-differential equations. Then, the comparision of the proposed methods is done with the Adomian decomposition method (ADM) in tabular and graphical form. The numarical and graphical results are obtained with the help of Maple software.

2. VARIATIONAL ITERATION METHOD

The main purpose of this method to solve the problems with linear conclusion was used as trial function. Therefore, a more precise or accurate approximation at some points can be attained. So this approximation quickly converges to accurate solution.

Further, we explain the principle of VIM, and then considered nonlinear differential equation:

$$Lu + Nu = g(x) \tag{1}$$

Since $g(x)$ nonhomogeneous term. From VIM,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{Lu_n(\tau)+Nu_n(\tau), u'(\tau)\}d\tau \quad n \geq 0 \tag{2}$$

λ Is a general Lagrangian multiplier and n denotes the nth-order approximation, Let $\tilde{u}_n$ is the restricted variation i.e. $\delta\tilde{u}_n = 0$.

Now, we solving the integro-differential equation

$$\frac{du}{dx} = f(x) + \int_0^x \psi\big(t, u(t), u'(t)\big)\, dt \tag{3}$$

$f(x)$ Is the source term. Firstly, we construct a correction function,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[(u_n)_s - f(s) - \int_0^s \psi(t, \tilde{u}(t), \tilde{u}'(t)\,)dt\right]ds \tag{4}$$

$\tilde{u}_n$ Is the restricted variation, $\tilde{u}_n = 0$. then find $\lambda(s)$,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \lambda(s)\left[(u_n)_s - f(s) - \int_0^s \psi(t, u(t), u'(t)\,)dt\right]ds \tag{5}$$

Therefore

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)(u_n)_s\,(s)ds \tag{6}$$

which result in

$$\delta u_{n+1}(x) = \delta u_n(x) + \lambda(s)\delta u_n(x) - \int_0^x \delta u_n(x)\lambda'\,(s)ds \tag{7}$$

$$\lambda'(s) = 0 \; and \; 1 + \lambda(s)|_{s=t} = 0 \tag{8}$$

$$\lambda(s) = -1 \tag{9}$$

Finally, the iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s - f(s) - \int_0^s \psi\big(t, u(t), u'(t)\big)dt\right]ds \tag{10}$$

3. SERIES SOLUTION METHOD

$u(x)$ is called analytic if it is the real function. If the derivatives of any order, then the Taylor series at any point ‘b’

$$u(x) = \sum_{k=0}^{n} \frac{f^k(b)}{k!}(x-b)^k \tag{11}$$

If $f(x)$ converges to a neighborhood point ‘b’. Further, at $x = 0$ the Taylor series in general form

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{12}$$

In this part we will prove a very effective method, which find from the Taylor series for analytic functions, then we will solve a Voltera integral equations.
So we suppose that the $u(x)$

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\,u(t)dt \tag{13}$$

Further, we will solve the Eq. (13) And also find the coefficients a_n of x.So we use the Eq. (12).
In Eq. (13) on both sides

$$\sum_{n=0}^{\infty} a_n x^n = T\big(f(x)\big) + \lambda \int_0^x K(x,t)\,\left(\sum_{n=0}^{\infty} a_n x^n\right)dt \tag{14}$$

Generally, we used

$$a_0 + a_1 x + a_2 x^2 + \cdots = T\big(f(x)\big) + \lambda \int_0^x K(x,t)\,(a_0 + a_1 t + a_2 t^2 + \cdots)dt \tag{15}$$

The Taylor series of $f(x)$ is $T\big(f(x)\big)$. Then we solved the equation $f(x)$.

But first we solved integral in the equation (15) so we integrate the unknown function $u(x)$, terms of the form $t^n, n \geq 0$ will be

Integrated. As we are finding series solution. Further, if $f(x)$ including are elementary functions like that logarithmic, exponential and trigonometric functions etc., so the $f(x)$ also expand through Taylor series. After integrating and expanding the Taylor series in Eq. $f(x)$. So we compared the both sides of the equation. Therefore we collect the coefficients of like powers of x. So we solved the coefficients and then we find a recurrence relation in $a_j, j \geq 0$. So we solved this relation.

After we finding the coefficients $a_j, j \geq 0$ then these Coefficients put in the Eq. (15) So we obtained a may be exact solution if there is exist. But if it is not existing exact solution then we obtained a series which can be used for numerical solution.

4. NUMERICAL EXPERIMENTS

We solved the two nonlinear integro-differential equations by using VIM and SSM. The main purpose here is to solve two problems using the VIM and SSM are above given.

4.1. PROBLEM #1

4.1.1. Variational Iteration Method (VIM)

We consider the nonlinear integro-differential equation

$$u'(x) = -1 + \int_0^x u^2(t)dt \tag{16}$$

For $x \in [0,1]$ and boundary conditions $u(0) = 0$
By using VIM, Put Eq.(16) In Eq.(10)
Therefore

$$\begin{gathered} u_{n+1}(x) = u_n(x) - \int_0^x \left[(u_n)_s(s) + 1 - \int_0^s u^2(t)dt \right] ds \\ u_0(x) = -x \end{gathered} \tag{17}$$

Then we obtained first three iteration

$$u_1(x) = -x + \frac{1}{12}x^4 \tag{18}$$

$$u_2(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10} \tag{19}$$

$$\begin{gathered} u_3(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10} - \frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16} - \\ \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{20} \end{gathered} \tag{20}$$

4.1.2. Series Solution Method (SSM)

$$u'(x) = -1 + \int_0^x u^2(t)dt \tag{21}$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{22}$$

Taking the derivative of the Eq. 22

$$u'(x) = n\sum_{n=1}^{\infty} a_n x^{n-1} \tag{23}$$

Again taking the derivative of Eq.23

$$\begin{gathered} u^2(x) = n(n-1)\sum_{n=2}^{\infty} a_n x^{n-2} \\ n\sum_{n=1}^{\infty} a_n x^{n-1} = -1 + \int_0^x n(n-1)\sum_{n=2}^{\infty} a_n x^{n-2}\, dt \end{gathered} \tag{24}$$

$$n\sum_{n=1}^{\infty} a_n x^{n-1} = -1 + \sum_{n=2}^{\infty} \frac{n(n-1)}{(n-1)} a_n x^{n-1} \tag{25}$$

Both sides are open the series and comparing the coefficients of x. Then we obtained the final results

$$u(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{12960}x^{10} \dots \tag{26}$$

Both the results are approximately same obtained by VIM and SSM

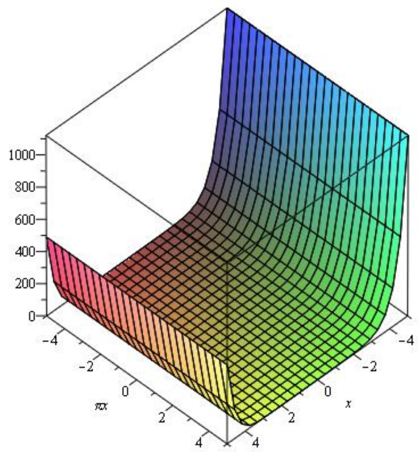

Figure 1. Behaviour of Solution obtained by VIM

Table 1. Comparison of ADM, VIM AND SSM

X	ADM	VIM	SSM
0.0000	0.000000000	-0.000000000	-0.000000000
0.0938	-0.0937935	-0.0937935	-0.0937936
0.2188	-0.2186090	-0.2186091	-0.2186092
0.3125	-0.3117060	-0.3117064	-0.3117065
0.4062	-0.4039390	-0.4039385	-0.4039386
0.5000	-0.4948230	-0.4948226	-0.4948227
0.6250	-0.6124310	-0.6124315	-0.6124316
0.7188	-0.6969410	-0.6969446	-0.6969447
0.8125	-0.7770900	-0.7771007	-0.7771008
0.9062	-0.8519340	-0.8519654	-0.8519655
1.0000	-0.9204760	-0.9205578	-0.9205579

4.2. PROBLEM #2

4.2.1. Variational Iteration Method (VIM)

Consider the integro-differential equation

$$u'(x) = 1 + \int_0^x u(t)\frac{du(t)}{dt}\,dt \tag{27}$$

For $x \in [0,1]$ and boundary conditions $u(0) = 0$
By using VIM, the iteration formula, the first two iterations are easily find

$$u_2(x) = x + \frac{1}{12}x^4 \tag{28}$$

$$u_2(x) = x + \frac{1}{12}x^4 + \frac{1}{87}x^7 + \frac{1}{1440}x^{10} + \frac{1}{67392}x^{13} \tag{29}$$

4.2.2. Series Solution Method (SSM)

Consider the Iintegro-differential equation

$$u'(x) = 1 + \int_0^x u(t)\frac{du(t)}{dt}\,dt \tag{30}$$

Given that $u(0) = 0 = a_0$
Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{31}$$

Taking the derivative of the Eq. 31

$$u'(x) = n\sum_{n=1}^{\infty} a_n x^{n-1} \tag{32}$$

$$n\sum_{n=1}^{\infty} a_n x^{n-1} = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(n\sum_{n=1}^{\infty} a_n x^{n-1}\right)dt \tag{33}$$

Both sides are open the series, solve the integral and comparing the coefficients of x. Then we obtained the final results

$$u(x) = x + \frac{1}{12}x^4 + \frac{1}{87}x^7 + \frac{1}{1440}x^{10} + \frac{1}{67392}x^{13} \ldots \tag{34}$$

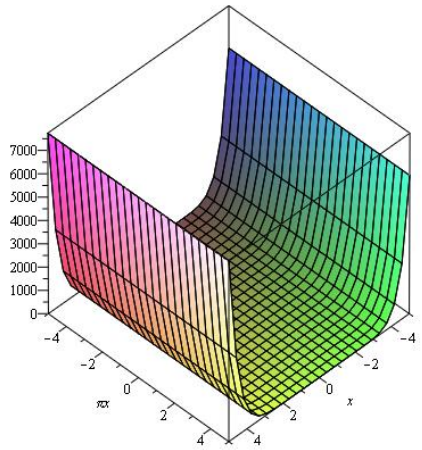

Figure 2. Behaviour of Solution obtained by SSM

Table 2. Comparison of ADM, VIM AND SSM

X	ADM	VIM	SSM
0.0000	0.000000000	0.000000000	0.000000000
0.0938	0.0938065	0.0938065	0.0938065
0.2188	0.2189910	0.0938065	0.0938065
0.3125	0.3132980	0.3132982	0.3132983
0.4062	0.4084910	0.4084907	0.4084905
0.5000	0.5053030	0.5053032	0.5053028
0.6250	0.6381770	0.6381768	0.6381762
0.7188	0.7422990	0.7422988	0.7422982
0.8125	0.8518530	0.8518520	0.8518510
0.9062	0.9691440	0.9691418	0.9691402
1.0000	1.0973700	1.0973681	1.0973671

4. CONCLUSION

In this article, the variation iteration method (VIM) and series solution method (SSM) have been prosperously applied to find the approximate solution of the integro-differential for nonlinear problems. These methods without using any limiting conclusion to apply directly on the problem. Especially, variation iteration method is the more effective technique to solve integro-differential for nonlinear problems. The involvement of Langrange Multiplier produce more accuracy toward the exact solution. The comparison of techniques mentioned above with the Adomian decomposition method was made by using Maple software. The graphical and tabular results were shown the efficiency of the proposed method.

REFERENCES

[1] Adomian. G., *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic, Boston, 1994.
[2] Chen, W., Lu, Z., *Applied Mathematics and Computation*, **159**, 221, 2004.
[3] Evans, D.J., Raslan, K.R., *International Journal of Computer Mathematics*, **82**, 49, 2005.
[4] Keskin, A.Ü., *Boundary Value Problems for Engineers with MATLAB Solutions*, chapter Adomian Decomposition Method (ADM), Springer Cham, pp. 311-359, 2019.
[5] Qiu, K., Zeng, T., Liao, Y., Min, J., Zhang, N., Peng, M., Chen, L.L., *Frontiers in Immunology*, **12**, 798719, 2021.
[6] Haq, E.U., Hassan, Q.M.U., Ahmad, J., Ehsan, K., *Alexandria Engineering Journal*, **61**(4), 3051, 2022.
[7] Sweilam, N.H., *Computers & Mathematics with Applications*, **54**, 1086, 2007.
[8] Khan, N., Hassan, Q.M.U., Haq, E.U., Khan, M.Y., Ayub, K., Ayub, J., *Journal of Science and Arts*, **21**, 5, 2021.
[9] Haq, E.U., Khan, S.U., Abbas, T., Smida, K., Hassan, Q.M.U., Ahmad, B., Galal, A. M., *Scientific Reports*, **12**(1), 1, 2022.
[10] Riaz, A., Hassan, Q.M.U., Abbas, T., Ghachem, K., Majeed, A., Noori, F.M., Kolsi, L., *Alexandria Engineering Journal*, **61**(12), 10779, 2022.
[11] Haq, E., Abbas, T., Hassan, Q.M.U., Ahmad, B., *Journal of Science and Arts*, **21**(4), 1143, 2021.
[12] Anjum, N., He, J.H., *Applied Mathematics Letters*, **92**, 134, 2019.
[13] Wang, X.Y., *Physics Letters A*, **131**, 277, 1988.
[14] Wadati, M., *Journal of the Physical Society of Japan*, **32,** 1681, 1972.
[15] Ayub, K., Khan, M.Y., Hassan, Q.M.U., *Journal of Science and Arts*, **17**, 183, 2017.
[16] Rani, A., Saeed, M., Ul-Hassan, Q.M., Ashraf, M., Khan, M.Y., Ayub, K., *Journal of Science and Arts*, **17**(3), 457, 2017.
[17] Jeffrey, A., Mohamad, M.N.B., *Wave Motion*, **14,** 369, 1991.
[18] Soliman, A.A., *Chaos, Solitons & Fractals*, **41**, 1034, 2009.
[19] Senthilvelan, M., *Applied Mathematics and Computation*, **123**, 381, 2001.
[20] Li, D.S., Zhang, H.Q., Applied *Mathematics and Computation*, **145**, 351, 2003.